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ASSOCIATED SEQUENCES OF GENERAL ORDER

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1. INTRODUCTION

The purpose of this article is to extend an idea in [1] to the generalized sequence $\{W_n\}$ defined [2] for all integers n by the recurrence relation

(1.1) $$W_{n+2} = pW_{n+1} + qW_n$$

in which

(1.2) $$W_0 = a,\ W_1 = b$$

where a, b, p, q are arbitrary integers.

The explicit *Binet form* is

(1.3) $$W_n = \frac{(b-a\beta)\alpha^n - (b-a\alpha)\beta^n}{\alpha-\beta}$$

where

(1.4) $$\alpha = \frac{p+\sqrt{p^2+4q}}{2},\quad \beta = \frac{p-\sqrt{p^2+4q}}{2}$$

are the roots of

(1.5) $$x^2 - px - q = 0.$$

From (1.4), or (1.5), we deduce that

(1.6) $$\alpha+\beta = p,\ \alpha\beta = -q,\ \alpha-\beta = \sqrt{p^2+4q} = \Delta$$

and

(1.7) $$\begin{cases} \frac{p\alpha}{2}+q = \frac{\Delta\alpha}{2}, & \frac{p\beta}{2}+q = -\frac{\Delta\beta}{2} \\ \alpha^2+q = \Delta\alpha, & \beta^2+q = -\Delta\beta \\ \alpha^2-q = p\alpha, & \beta^2-q = p\beta. \end{cases}$$

Special cases of $\{W_n\}$ which interest us are

(1.8) $$\begin{cases} \text{the Fibonacci sequence } \{F_n\}: & p=1, q=1, a=0, b=1 \\ \text{the Lucas sequence } \{L_n\}: & p=1, q=1, a=2, b=1 \\ \text{the Pell sequence } \{P_n\}: & p=2, q=1, a=0, b=1 \\ \text{the Pell-Lucas sequence } \{Q_n\}: & p=2, q=1, a=2, b=2. \end{cases}$$

2. THE ASSOCIATED SEQUENCE $\{W_n^{(k)}\}$

Define the *first associated sequence* $\{W_n^{(1)}\}$ of $\{W_n\}$ from (1.1) by

(2.1) $$W_n^{(1)} = pW_{n+1} + qW_{n-1}.$$

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Repeat the operation in (2.1) to obtain

(2.2) $$W_n^{(2)} = pW_{n+1}^{(1)} + qW_{n-1}^{(1)} = \frac{(b-a\beta)(p\alpha^2+q)^2\alpha^{n-2} - (b-a\alpha)(p\beta^2+q)^2\beta^{n-2}}{\Delta}$$

on using (1.3) and (2.1).

Generally,

(2.3) $$W_n^{(k)} = pW_{n+1}^{(k-1)} + qW_{n-1}^{(k-1)}$$

where $W_n^{(k)}$ $(k = 1, 2, 3, \ldots)$ represents the k^{th} repetition of the association process (2.1). We may call $\{W_n^{(k)}\}$ the *associated sequence of* $\{W_n\}$ *of order k.*

(2.3)′ **Convention:** $W_n^{(0)} = W_n.$

Induction, with appeal to (2.3), may be used to establish the Binet form

(2.4) $$W_n^{(k)} = \frac{A\alpha^{n-k} - B\beta^{n-k}}{\Delta}$$

in which

(2.5) $$\begin{cases} A = (b-a\beta)(p\alpha^2+q)^k \\ B = (b-a\alpha)(p\beta^2+q)^k \end{cases}$$

As we expect, $\{W_n^{(k)}\}$ is a recurrence sequence like $\{W_n\}$, for, by (2.4),

(2.6) $$\begin{aligned} pW_{n+1}^{(k)} + qW_n^{(k)} &= \frac{1}{\Delta}\{p(A\alpha^{n+1-k} - B\beta^{n+1-k}) + q(A\alpha^{n-k} - B\beta^{n-k})\} \\ &= \frac{1}{\Delta}\{(p\alpha+q)A\alpha^{n-k} - (p\beta+q)B\beta^{n-k}\} \\ &= W_{n+2}^{(k)} \end{aligned}$$

on putting $x = \alpha, x = \beta$ in turn in (1.5). Thus, (1.1) is valid for $\{W_n^{(k)}\}$. Consequently, (1.5) is also true for $\{W_n^{(k)}\}$.

Next, we define $\{{}^{c}W_n^{(k)}\}$, the *conjugate sequence of* $\{W_n^{(k)}\}$, by

(2.7) $${}^{c}W_n^{(k)} = A\alpha^{n-k} + B\beta^{n-k}.$$

It readily follows, on using (1.7), that

(2.8) $${}^{c}W_n^{(k)} = p\,{}^{c}W_{n+1}^{(k-1)} + q\,{}^{c}W_{n-1}^{(k-1)}$$

and on using (1.5), that

(2.9) $${}^{c}W_{n+2}^{(k)} = p\,{}^{c}W_{n+1}^{(k)} + q\,{}^{c}W_n^{(k)}.$$

That is, both the association and recurrence properties which are features of $\{W_n^{(k)}\}$ apply equally well to $\{{}^{c}W_n^{(k)}\}$.

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3. PROPERTIES OF $\{W_n^{(k)}\}$

Some consequences of our definitions and ideas now follow. Proofs of these results, obtainable from the preceding information, are left for the pleasure of the reader [employing (1.7) and (2.4)].

Firstly, we have the *Simson formula*

Theorem 1:

(3.1) $$W_{n+1}^{(k)}W_{n-1}^{(k)}-\left[W_n^{(k)}\right]^2=-(-q)^{n-1-k}AB.$$

More generally,

(3.1)′ $$W_{n+r}^{(k)}W_{n-r}^{(k)}-[W_n^{(k)}]^2=\frac{-(-q)^{n-r-k}AB(\alpha^r-\beta^r)^2}{\Delta^2}.$$

Theorem 2:

(3.2) $$\sum_{i=0}^{n}\binom{n-k}{i}q^i\left(\frac{p}{2}\right)^{n-k-i}W_{i-1}^{(k)}=\begin{cases}\left(\frac{\Delta}{2}\right)^{n-k}W_n^{(k)} & (n-k\text{ even})\\ \frac{\Delta^{n-k-1}}{2^{n-k}}\mathcal{W}_n^{(k)} & (n-k\text{ odd}).\end{cases}$$

Theorem 3:

(3.3) $$[W_{n+1}^{(k)}]^2+q[W_n^{(k)}]^2=bW_{2n+1}^{(2k)}+aqW_{2n}^{(2k)}.$$

Theorem 4:

(3.4) $$[W_{n+1}^{(k)}]^2-q[W_n^{(k)}]^2=\frac{p}{\Delta^2}\left(b\mathcal{W}_{2n+1}^{(2k)}+aq\mathcal{W}_{2n}^{(2k)}\right)+(-1)^{n-k}q^{n+1-k}4\frac{AB}{\Delta^2}.$$

(Not a pretty sight!)

Theorem 5:

(3.5) $$[W_{n+2}^{(k)}]^3-p^3[W_{n+1}^{(k)}]^3-q^3[W_n^{(k)}]^3=3pqW_{n+1}^{(k)}W_n^{(k)}W_{n-1}^{(k)}.$$

This neat cubic property is derivable directly from (2.6), or, with more effort, from (2.4).

Theorem 6:

(3.6) $$[W_{n+2}^{(k)}]^2-p^2[W_{n+1}^{(k)}]^2-q^2[W_n^{(k)}]^2=2pqW_{n+1}^{(k)}W_n^{(k)}.$$

This quadratic property which is easily deducible from (2.6) may be employed to produce a somewhat unattractive expression for $2pq\sum_{i=0}^{n}W_i^{(k)}W_{i+1}^{(k)}$.

All the above results (linear and nonlinear) for $W_n^{(k)}$ may be echoed for $\mathcal{W}_n^{(k)}$. Just one illustration (namely, the corresponding Simson formula) should suffice. Remaining results could be paralleled by the interested reader.

Theorem 1a:

(3.1a) $$\mathcal{W}_{n+1}^{(k)}\mathcal{W}_{n-1}^{(k)}-[\mathcal{W}_n^{(k)}]^2=(-q)^{n-1-k}AB\Delta^2.$$

This theorem can be extended as in (3.1)′ for $W_n^{(k)}$.

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Some hybrid results involving mathematical cross-fertilization of $W_n^{(k)}$ and ${}^{\circ}W_n^{(k)}$ are worth mentioning. For example,

Theorem 7:

(3.7) $$W_n^{(k)}\,{}^{\circ}W_n^{(k)} = bW_{2n}^{(2k)} + aqW_{2n-1}^{(2k)}.$$

When $a=0, b=1, q=1$, we see that (3.7) leads to $F_n^{(k)}L_n^{(k)} = F_{2n}^{(2k)}$ and $P_n^{(k)}Q_n^{(k)} = P_{2n}^{(2k)}$ [cf. (1.8)] which are outgrowths of well-known results occurring for $k=0$. [See (4.18).] In particular, $P_3^{(2)}Q_3^{(2)} = 137 \times 386 = 52,882 = P_6^{(4)}$ [see (4.9) and (4.10)].

Another interesting result is

Theorem 8:

(3.8) $${}^{\circ}W_{n+1}^{(k)} + q\,{}^{\circ}W_{n-1}^{(k)} = \Delta^2 W_n^{(k)}.$$

Thus, when $k=0, q=1$ we have (1.8)

$$L_{n+1} + L_{n-1} = 5F_n, \;\; Q_{n+1} + Q_{n-1} = 8P_n.$$

Observe that

$$F_n^{(1)} = L_n, \;\; F_n^{(2)} = 5F_n = L_n^{(1)}, \;\; F_n^{(3)} = 5F_n^{(1)} = 5L_n, \;\; F_n^{(4)} = 5L_n^{(1)} = 5^2 F_n, \ldots.$$

Generally,

(3.9) $$\begin{cases} F_n^{(2m)} = 5^m F_n \\ F_n^{(2m+1)} = 5^m L_n \\ L_n^{(2m)} = 5^m L_n \\ L_n^{(2m-1)} = 5^m F_n. \end{cases}$$

Consequently, the association process effectively stops after two operations on F_n or L_n. However, for the Pell and Pell-Lucas numbers, for which $p=2$, the association process is never-ending. [For P_n' to equal Q_n, we would require $P_n' = P_{n+1} + P_{n-1}$, which is contrary to (4.1).]

4. ASSOCIATED PELL SEQUENCE

As our special application of the general theory, we consider the Pell sequence $\{P_n\}$ defined in (1.8) by the recurrence relation

(4.1) $$P_{n+2} = 2P_{n+1} + P_n$$

in which

(4.2) $$P_0 = 0, \;\; P_1 = 1$$

with Binet form

(4.3) $$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$$

where

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(4.4) $$\alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2}$$

are the roots of

(4.5) $$x^2 - 2x - 1 = 0.$$

From (4.4)

(4.6) $$\alpha+\beta = 2,\ \alpha\beta = -1,\ \alpha-\beta = 2\sqrt{2}.$$

The *associated Pell sequence of order k*, $\{P_n^{(k)}\}$, is given by

(4.7) $$P_n^{(k)} = 2P_{n+1}^{(k-1)} + P_{n-1}^{(k-1)} \quad (k \ge 1)$$

for which the Binet form is

(4.8) $$P_n^{(k)} = \frac{(7+4\sqrt{2})^k \alpha^{n-k} - (7-4\sqrt{2})^k \beta^{n-k}}{2\sqrt{2}}.$$

Some elements of the first few sequences $\{P_n^{(k)}\}$ are:

(4.9)

$k \backslash n$	−1	0	1	2	3	4	5	6	⋯
0	1	0	1	2	5	12	29	70	⋯
1	−2	3	4	11	26	63	152	367	⋯
2	13	6	25	56	137	330	797	1924	⋯
3	−8	63	118	299	716	1731	4178	10087	⋯
4	205	228	661	1550	3761	9072	21905	52882	⋯

The conjugate sequence $\{Q_n^{(k)}\}$ for which

(4.10) $$Q_n^{(k)} = P_{n+1}^{(k)} + P_{n-1}^{(k)}$$

has the Binet form

(4.11) $$Q_n^{(k)} = (7+4\sqrt{2})^k \alpha^{n-k} + (7-4\sqrt{2})^k \beta^{n-k}.$$

One has also

(4.12) $$Q_n^{(k)} = 2Q_{n+1}^{(k-1)} + Q_{n-1}^{(k-1)}.$$

Interested readers might wish to list the first few members of some of the $\{Q_n^{(k)}\}$, as was done for $\{P_n^{(k)}\}$ in (4.9). For example

$$Q_3^{(2)} = 386 = 56+330 = P_2^{(2)} + P_4^{(2)}.$$

Paralleling results in section 3, we have, for instance

Theorem 1′:

(4.13) $$P_{n+1}^{(k)}P_{n-1}^{(k)} - [P_n^{(k)}]^2 = (-1)^{n+k}17^k$$

since $AB = 17^k$ in the case of Pell numbers, by (1.4), (1.8), and (2.5). There is an obvious extension analogous to that in (3.1)′.

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Theorem 2':

(4.14) $$\sum_{i=0}^{n}\binom{n-k}{i}P_{n-i}^{(k)}=\begin{cases}(\sqrt{2})^{n-k}P_n^{(k)} & (n-k\text{ even})\\ \frac{(\sqrt{2})^{n-k}}{2\sqrt{2}}Q_n^{(k)} & (n-k\text{ odd}).\end{cases}$$

Theorem 3':

(4.15) $$[P_{n+1}^{(k)}]^2+[P_n^k]^2=P_{2n+1}^{(2k)}.$$

Theorem 4':

(4.16) $$[P_{n+1}^{(k)}]^2-[P_n^k]^2=\frac{Q_{2n+1}^{(2k)}+(-1)^{n-k}\cdot 2\cdot 17^k}{4}.$$

Other special cases for Pell numbers, as in Theorems 5 and 6 and (2.6), follow.

Results involving $Q_n^{(k)}$ include (say)

Theorem 1a':

(4.17) $$Q_{n+1}^{(k)}Q_{n-1}^{(k)}-[Q_n^{(k)}]^2=(-1)^{n+1+k}\cdot 8\cdot 17.$$

Theorem 7':

(4.18) $$P_n^{(k)}Q_n^{(k)}=P_{2n}^{(2k)}$$

[already noted after (3.7)].

Reverting momentarily to $\{W_n^{(k)}\}$, we can use previously applied techniques to demonstrate that

(4.19) $$bW_{n+2}^{(k)}+aqW_{n+1}^{(k)}=W_{n+1}W_n^{(k)}+qW_nW_1^{(k)}.$$

From (1.8), it follows that

(4.20) $$P_{n+2}^{(k)}=P_{n+1}P_2^{(k)}+P_nP_1^{(k)},$$

which expresses the $(n+2)^{\text{th}}$ term of the associated sequence in terms of the $(n+1)^{\text{th}}$ and n^{th} Pell numbers. When $k=4, n=3$, for example,

$$P_5^{(4)}=P_4P_2^{(4)}+P_3P_1^{(4)}=12\times 1550+5\times 661=21,905.$$

If $k=0$, then (4.20) leads directly to (4.1). Thus, in a pleasing way, (4.20) appears as a mathematical offspring of the definition of Pell numbers.

Equation (4.19) in conjunction with (1.8) also yields a result for Fibonacci numbers similar to (4.20), namely,

(4.21) $$F_{n+2}^{(k)}=F_{n+1}F_2^{(k)}+F_nF_1^{(k)}.$$

5. NEGATIVE VALUES OF *k* AND *n*

As $\{W_n\}$ was defined in (1.1) for all integers n, the results we have obtained for $\{W_n^{(k)}\}$ apply irrespective of whether n is positive or negative. Indeed, the tabulation in (4.9) gives a brief indication of this aspect which could be extended to other negative subscript values.

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But what happens if k is negative?

The Binet form for $\{W_n^{(-k)}\}$ is readily written down from (2.4) by replacing k by $-k$, and the theory for negative subscript k follows as for the case of k positive. Unhappily, computation does not always produce pleasing formulas. Indeed, the calculation of, say, $W_{-n}^{(k)}[W_n^{(-k)}]^{-1}$ leads to some unlovely algebra.

However, things are easier if we consider a particular instance of the general sequence, say, the Pell sequence. Calculation using (4.8) leads to

(5.1) $$P_{-n}^{(k)}[P_n^{(-k)}]^{-1} = (1)^{n+k+1} \cdot (17)^k.$$

Application of (5.1) with the assistance of (4.8) allows us to compute numerical values of $P_n^{(-k)}$ for particular values of n and k to our heart's content. A short tabulation of $P_n^{(k)}$ for $k < 0$, $= 0$, > 0 is [cf. (4.9)]:

(5.2)

$k \backslash n$	-2	-1	0	1	2	$\cdots$
-2	$\frac{-56}{17^2}$	$\frac{25}{17^2}$	$\frac{-6}{17^2}$	$\frac{13}{17^2}$	$\frac{20}{17^2}$	$\cdots$
-1	$\frac{11}{17}$	$\frac{-4}{17}$	$\frac{3}{17}$	$\frac{2}{17}$	$\frac{7}{17}$	$\cdots$
0	-2	1	0	1	2	$\cdots$
1	7	-2	3	4	11	$\cdots$
2	-20	13	6	25	56	$\cdots$

Calculation gives [see (3.1)′ and (4.13)]

(5.3) $$P_{n+r}^{(-k)}P_{n-r}^{(-k)} - \left[P_n^{(-k)}\right]^2 = (-1)^{n+r+k-1}\frac{(\alpha^r - \beta^r)^2}{8}(17)^{-k}.$$

Thus,

$$P_2^{(-1)}P_{-2}^{(-1)} - \left[P_0^{(-1)}\right]^2 = \frac{4}{17}.$$

One may readily reinterpret the earlier theory for $k \geq 0$ in terms of $k < 0$.

Similar procedures apply to $Q_n^{(-k)}$ on use of (4.11).

Likewise, the elementary properties of $F_n^{(-k)}$ and $L_n^{(-k)}$ can be established.

Discovery of other formulas pertinent to associated sequences is offered to the curiosity of the reader. While this brief exposition is only an introduction to the topic, it does allow us to savor something of the flow of ideas from definition (2.1).

Finally, it may be mentioned that, in analyzing the nature of associated sequences, the author first examined $\{P_n^{(k)}\}$ before proceeding to the generalization. This approach was helpful in investigating some of the more awkward features of the general theory.

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