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Research Article

On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2

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Here we present a new method to construct the explicit formula of a sequence of numbers and polynomials generated by a linear recurrence relation of order 2. The applications of the method to the Fibonacci and Lucas numbers, Chebyshev polynomials, the generalized Gegenbauer-Humbert polynomials are also discussed. The derived idea provides a general method to construct identities of number or polynomial sequences defined by linear recurrence relations. The applications using the method to solve some algebraic and ordinary differential equations are presented.

1. Introduction

Many number and polynomial sequences can be defined, characterized, evaluated, and classified by linear recurrence relations with certain orders. A number sequence $\{a_n\}$ is called sequence of order 2 if it satisfies the linear recurrence relation of order 2:

$$a_n = pa_{n-1} + qa_{n-2}, \quad n \geq 2, \tag{1.1}$$

for some nonzero constants p and q and initial conditions a_0 and a_1. In Mansour [1], the sequence $\{a_n\}_{n\geq 0}$ defined by (1.1) is called Horadam's sequence, which was introduced in 1965 by Horadam [2]. The work in [1] also obtained the generating functions for powers of Horadam's sequence. To construct an explicit formula of its general term, one may use a generating function, characteristic equation, or a matrix method (see Comtet [3], Hsu [4], Strang [5], Wilf [6], etc.) In [7], Benjamin and Quinn presented many elegant combinatorial meanings of the sequence defined by recurrence relation (1.1). For instance, a_n counts the number of ways to tile an n-board (i.e., board of length n) with squares (representing 1ss) and dominoes (representing 2s) where each tile, except the initial one has a color. In addition, there are p colors for squares and q colors for dominoes. In this paper, we will present a new method to construct an explicit formula of $\{a_n\}$ generated by (1.1). The key idea of our method is to reduce the relation (1.1) of order 2 to a linear recurrence relation of order 1:

$$a_n = ca_{n-1} + d, \quad n \geq 1, \tag{1.2}$$

for some constants $c \neq 0$ and d and initial condition a_0 via geometric sequence. Then, the expression of the general term of the sequence of order 2 can be obtained from the formula of the general term of the sequence of order 1:

$$a_n = \begin{cases} a_0 c^n + d\dfrac{c^n - 1}{c - 1}, & \text{if } c \neq 1; \\ a_0 + nd, & \text{if } c = 1. \end{cases} \tag{1.3}$$

The method and some related results on the generalized Gegenbauer-Humbert polynomial sequence of order 2 as well as a few examples will be given in Section 2. Section 3 will discuss the application of the method to the construction of the identities of sequences of order 2. There is an extension of the above results to higher order cases. In Section 4, we will discuss the applications of the method to the solution of algebraic equations and initial value problems of second-order ordinary differential equations.

2. Main Results and Examples

Let α and β be two roots of of quadratic equation $x^2 - px - q = 0$. We may write (1.1) as

$$a_n = (\alpha + \beta)a_{n-1} - \alpha\beta a_{n-2}, \tag{2.1}$$

where α and β satisfy $\alpha + \beta = p$ and $\alpha\beta = -q$. Therefore, from (2.1), we have

$$a_n - \alpha a_{n-1} = \beta(a_{n-1} - \alpha a_{n-2}), \tag{2.2}$$

which implies that $\{a_n - \alpha a_{n-1}\}_{n\geq 1}$ is a geometric sequence with common ratio β. Hence,

$$\begin{aligned} a_n - \alpha a_{n-1} &= (a_1 - \alpha a_0)\beta^{n-1}, \\ a_n &= \alpha a_{n-1} + (a_1 - \alpha a_0)\beta^{n-1}. \end{aligned} \tag{2.3}$$

Consequently,

$$\frac{a_n}{\beta^n} = \frac{\alpha}{\beta}\left(\frac{a_{n-1}}{\beta^{n-1}}\right) + \frac{a_1 - \alpha a_0}{\beta}. \tag{2.4}$$

Let $b_n := a_n/\beta^n$. We may write (2.4) as

$$b_n = \frac{\alpha}{\beta}b_{n-1} + \frac{a_1 - \alpha a_0}{\beta}. \tag{2.5}$$

If $\alpha \neq \beta$, by using (1.3), we immediately obtain

$$\begin{aligned}\frac{a_n}{\beta^n} &= a_0\left(\frac{\alpha}{\beta}\right)^n + \frac{a_1 - \alpha a_0}{\beta}\frac{(\alpha/\beta)^n - 1}{(\alpha/\beta) - 1}\\ &= \frac{1}{\beta^n}\left[\alpha^n a_0 + \frac{a_1 - \alpha a_0}{\alpha - \beta}(\alpha^n - \beta^n)\right],\end{aligned} \tag{2.6}$$

which yields

$$a_n = \left(\frac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\frac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n. \tag{2.7}$$

Similarly, if $\alpha = \beta$, then (1.3) implies

$$a_n = a_0\alpha^n + n\alpha^{n-1}(a_1 - \alpha a_0) = na_1\alpha^{n-1} - (n-1)a_0\alpha^n. \tag{2.8}$$

We may summarize the above result as follows.

Proposition 2.1. *Let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (2.1). Then*

$$a_n = \begin{cases}\left(\dfrac{a_1 - \beta a_0}{\alpha - \beta}\right)\alpha^n - \left(\dfrac{a_1 - \alpha a_0}{\alpha - \beta}\right)\beta^n, & \text{if } \alpha \neq \beta;\\[2ex] na_1\alpha^{n-1} - (n-1)a_0\alpha^n, & \text{if } \alpha = \beta.\end{cases} \tag{2.9}$$

In particular, if $\{a_n\}$ satisfies the linear recurrence relation (1.1) with $q = 1$, namely,

$$a_n = pa_{n-1} + a_{n-2}, \tag{2.10}$$

then the equation $x^2 - px - 1 = 0$ has two solutions:

$$\alpha = \frac{p + \sqrt{p^2 + 4}}{2}, \qquad \beta = -\frac{1}{\alpha} = \frac{p - \sqrt{p^2 + 4}}{2}. \tag{2.11}$$

From Proposition 2.1, we have the following corollary.

Corollary 2.2. *Let* $\{a_n\}$ *be a sequence of order* 2 *satisfying the linear recurrence relation* $a_n = pa_{n-1} + a_{n-2}$*. Then*

$$a_n = \frac{2a_1 - \left(p - \sqrt{p^2+4}\right)a_0}{2\sqrt{p^2+4}}\alpha^n - \frac{2a_1 - \left(p + \sqrt{p^2+4}\right)a_0}{2\sqrt{p^2+4}}\left(-\frac{1}{\alpha}\right)^n, \tag{2.12}$$

where α *is defined by* (2.11).

Similarly, let $\{a_n\}$ *be a sequence of order* 2 *satisfying the linear recurrence relation* $a_n = a_{n-1} + qa_{n-2}$*. Then*

$$a_n = \begin{cases} \dfrac{2a_1 - (1-\sqrt{4q+1})a_0}{2\sqrt{4q+1}}\alpha_1^n - \dfrac{2a_1 - (1+\sqrt{4q+1})a_0}{2\sqrt{4q+1}}\alpha_2^n, & \text{if } q \neq -\dfrac{1}{4}; \\ \dfrac{1}{2^n}(2na_1 - (n-1)a_0), & \text{if } q = -\dfrac{1}{4}, \end{cases} \tag{2.13}$$

where $\alpha_1 = (1/2)(1+\sqrt{4q+1})$ *and* $\alpha_2 = (1/2)(1-\sqrt{4q+1})$ *are solutions of the equation* $x^2 - x - q = 0$.

The first special case (2.12) was studied by Falbo in [8]. If $p = 1$, the sequence is clearly the Fibonacci sequence. If $p = 2$ ($q = 1$), the corresponding sequence is the sequence of numerators (when two initial conditions are 1 and 3) or denominators (when two initial conditions are 1 and 2) of the convergent of a continued fraction to $\sqrt{2}$: $\{1/1, 3/2, 7/5, 17/12, 41/29, \ldots\}$, called the closest rational approximation sequence to $\sqrt{2}$. The second special case is also a corollary of Proposition 2.1. If $q = 2$ ($p = 1$), $\{a_n\}$ is the Jacobsthal sequence (see Bergum et al. [9]).

Remark 2.3. Proposition 2.1 can be extended to the linear recurrence relations of order 2 with more general form: $a_n = pa_{n-1} + qa_{n-2} + \ell$ for $p + q \neq 1$. It can be seen that the above recurrence relation is equivalent to the form (1.1) $b_n = pb_{n-1} + qb_{n-2}$, where $b_n = a_n - k$ and $k = \ell/(1-p-q)$.

We now show some examples of the applications of our method including the presentation of much easier proofs of some well-known formulas of the sequences of order 2.

Remark 2.4. Denote

$$u_n = \begin{bmatrix} a_{n+1} \\ \alpha_n \end{bmatrix}, \qquad A = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}. \tag{2.14}$$

We may write relation $a_n = pa_{n-1} + qa_{n-2}$ and $a_{n-1} = a_{n-1}$ into a matrix form $u_{n-1} = Au_{n-2}$ with respect to 2×2 matrix A defined above. Thus $u_{n-1} = A^{n-1}u_0$. To find explicit expression of u_{n-1}, the real problem is to calculate A^{n-1}. The key lies in the eigenvalues and eigenvectors. The eigenvalues of A are precisely α and β, which are two roots of the characteristic equation $x^2 - px - q = 0$ for the matrix A. However, an obvious identity can be obtained from $(u_n, u_{n-1}) = A^{n-1}(u_1, u_0)$ by taking determinants on the both sides: $a_{n+1}a_{n-1} - a_n^2 = (-q)^{n-1}(a_2a_0 - a_1^2)$ (see, e.g., [5] for more details).

Example 2.5. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence with the linear recurrence relation $F_n = F_{n-1} + F_{n-2}$, where F_0 and F_1 are assumed to be 0 and 1, respectively. Thus, the recurrence relation is a special case of (1.1) with $p = q = 1$ and the special case of the sequence in Corollary 2.2, which can be written as (2.1) with

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}. \tag{2.15}$$

Since $\alpha - \beta = \sqrt{5}$, from (2.12) we have the expression of F_n as follows:

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}. \tag{2.16}$$

Example 2.6. We have mentioned above that the denominators of the closest rational approximation to $\sqrt{2}$ form a sequence satisfying the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$. With an additional initial condition 0, the sequence becomes the Pell number sequence: $\{p_n = 0, 1, 2, 5, 12, 29, \dots\}$, which also satisfies the recurrence relation $p_n = 2p_{n-1} + p_{n-2}$. Using formula (2.23) in Corollary 2.2, we obtain the general term of the Pell number sequence:

$$p_n = \frac{1}{2\sqrt{2}}\left\{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right\}. \tag{2.17}$$

The numerators of the closest rational approximation to $\sqrt{2}$ are half the companion Pell numbers or Pell-Lucas numbers. By adding in initial condition 2, we obtain the Pell-Lucas number sequence $\{c_n = 2, 2, 6, 14, 34, 82, \dots\}$, which satisfies $c_n = 2c_{n-1} + c_{n-2}$. Similarly, Corollary 2.2 gives

$$c_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n. \tag{2.18}$$

We now consider the sequence of the sums of Pell number: $\{\sigma_n = 0, 1, 3, 8, 20, \dots\}$, which satisfies the recurrence relation

$$\sigma_n = 2\sigma_{n-1} + \sigma_{n-2} + 1. \tag{2.19}$$

From Remark 2.3, the above expression can be transfered to an equivalent form

$$b_n = 2b_{n-1} + b_{n-2}, \tag{2.20}$$

where $b_n = \sigma_n + 1/2$. Using Corollary 2.2, one easily obtain

$$b_n = \frac{1}{4}\left\{\left(1+\sqrt{2}\right)^{n+1} + \left(1-\sqrt{2}\right)^{n+1}\right\}. \tag{2.21}$$

Thus,

$$\sigma_n = \frac{1}{4}\left\{\left(1+\sqrt{2}\right)^{n+1} + \left(1-\sqrt{2}\right)^{n+1}\right\} - \frac{1}{2}. \tag{2.22}$$

If the coefficients of the linear recurrence relation of a function sequence $\{a_n(x)\}$ of order 2 are real or complex-value functions of variable x, that is,

$$a_n(x) = p(x)a_{n-1}(x) + q(x)a_{n-2}(x), \tag{2.23}$$

we obtain a function sequence of order 2 with initial conditions $a_0(x)$ and $a_1(x)$. In particular, if all of $p(x)$, $q(x)$, $a_0(x)$, and $a_1(x)$ are polynomials, then the corresponding sequence $\{a_n(x)\}$ is a polynomial sequence of order 2. Denote the solutions of

$$t^2 - p(x)t - q(x) = 0 \tag{2.24}$$

by $\alpha(x)$ and $\beta(x)$. Then

$$\alpha(x) = \frac{1}{2}\left(p(x) + \sqrt{p^2(x) + 4q(x)}\right), \qquad \beta(x) = \frac{1}{2}\left(p(x) - \sqrt{p^2(x) + 4q(x)}\right). \tag{2.25}$$

Similar to Proposition 2.1, we have

Proposition 2.7. *Let* $\{a_n\}$ *be a sequence of order 2 satisfying the linear recurrence relation* (2.23)*. Then*

$$a_n(x) = \begin{cases} \left(\dfrac{a_1(x) - \beta(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\alpha^n(x) - \left(\dfrac{a_1(x) - \alpha(x)a_0(x)}{\alpha(x) - \beta(x)}\right)\beta^n(x), & \text{if } \alpha(x) \neq \beta(x); \\ n a_1(x)\alpha^{n-1}(x) - (n-1)a_0(x)\alpha^n(x), & \text{if } \alpha(x) = \beta(x), \end{cases} \tag{2.26}$$

where $\alpha(x)$ *and* $\beta(x)$ *are shown in* (2.25)*.*

Example 2.8. Consider the Chebyshev polynomials of the first kind, $T_n(x)$, defined by

$$T_n(x) = \cos(n \text{ arc } \cos\ x), \tag{2.27}$$

which satisfies the recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \tag{2.28}$$

with $T_0(x) = 1$ and $T_1(x) = x$. Thus the corresponding p, q, α, and β are, respectively, $2x$, -1, $x+\sqrt{x^2-1}$, and $x-\sqrt{x^2-1}$, which yields $T_1(x)-\beta T_0(x) = \sqrt{x^2-1}$, $T_1(x)-\alpha T_0(x) = -\sqrt{x^2-1}$, and $\alpha-\beta = 2\sqrt{x^2-1}$. Substituting the quantities into (2.7) yields

$$T_n(x) = \frac{1}{2}\left\{\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right\}. \tag{2.29}$$

All the Chebyshev polynomials of the second kind, third kind, and fourth kind satisfy the same recurrence relationship as the Chebyshev polynomials of the first kind with the same constant initial term 1. However, they possess different linear initial terms, which are $2x$, $2x-1$, and $2x+1$, respectively (see, e.g., Mason and Handscomb [10] and Rivlin [11]). We will give the expression of the Chebyshev polynomials of the second kind later by sorting them into the class of the generalized Gegenbauer-Humbert polynomials. As for the Chebyshev polynomials of the third kind, $T_n^{(3)}(x)$, and the Chebyshev polynomials of the fourth kind, $T_n^{(4)}(x)$, when $x^2 \neq 1$, we clearly have the following expressions using a similar argument presented for the Chebyshev polynomials of the first kind:

$$\begin{aligned} T_n^{(3)}(x) &= \frac{\sqrt{x^2-1}+x-1}{2\sqrt{x^2-1}}\left(x+\sqrt{x^2-1}\right)^n + \frac{\sqrt{x^2-1}-x+1}{2\sqrt{x^2-1}}\left(x-\sqrt{x^2-1}\right)^n, \\ T_n^{(4)}(x) &= \frac{\sqrt{x^2-1}+x+1}{2\sqrt{x^2-1}}\left(x+\sqrt{x^2-1}\right)^n + \frac{\sqrt{x^2-1}-x-1}{2\sqrt{x^2-1}}\left(x-\sqrt{x^2-1}\right)^n. \end{aligned} \tag{2.30}$$

Example 2.9. In [12], André-Jeannin studied the generalized Fibonacci and Lucas polynomials defined, respectively, by

$$\begin{aligned} U_n(x;a,b) &= U_n = (x+a)U_{n-1} - bU_{n-2}, \qquad U_0(x) = 0, \qquad U_1(x) = 1, \\ V_n(x;a,b) &= V_n = (x+a)V_{n-1} - bV_{n-2}, \qquad V_0(x) = 2, \qquad V_1(x) = x+a, \end{aligned} \tag{2.31}$$

where a and b are real parameters. Clearly, $V_n(2x;0,1) = 2T_n(x)$. Using Proposition 2.7, we obtain

$$\begin{aligned} U_n(x;a,b) &= \frac{1}{2^n\sqrt{(x+a)^2-4b}}\left\{\left(x+a+\sqrt{(x+a)^2-4b}\right)^n - \left(x+a-\sqrt{(x+a)^2-4b}\right)^n\right\}, \\ V_n(x;a,b) &= \frac{1}{2^n}\left\{\left(x+a+\sqrt{(x+a)^2-4b}\right)^n - \left(x+a-\sqrt{(x+a)^2-4b}\right)^n\right\}. \end{aligned} \tag{2.32}$$

From the last expression, we also see $V_n(2x;0,1) = 2T_n(x)$.

A sequence of the generalized Gegenbauer-Humbert polynomials $\{P_n^{\lambda,y,C}(x)\}_{n\geq 0}$ is defined by the expansion (see, e.g., Gould [13] and He et al. [14]):

$$\Phi(t) \equiv \left(C-2xt+yt^2\right)^{-\lambda} = \sum_{n\geq 0} P_n^{\lambda,y,C}(x)t^n, \tag{2.33}$$

where $\lambda > 0$, y and $C \neq 0$ are real numbers. As special cases of (2.33), we consider $P_n^{\lambda,y,C}(x)$ as follows (see [14]):

$$\begin{aligned}
P_n^{1,1,1}(x) &= U_n(x), \text{ Chebyshev polynomial of the second kind,}\\
P_n^{1/2,1,1}(x) &= \psi_n(x), \text{ Legendre polynomial,}\\
P_n^{1,-1,1}(x) &= P_{n+1}(x), \text{ Pell polynomial,}\\
P_n^{1,-1,1}\left(\frac{x}{2}\right) &= F_{n+1}(x), \text{ Fibonacci polynomial,}\\
P_n^{1,2,1}\left(\frac{x}{2}\right) &= \Phi_{n+1}(x), \text{ Fermat polynomial of the first kind,}\\
P_n^{1,2a,2}(x) &= D_n(x,a), \text{ Dickson polynomial,}
\end{aligned} \tag{2.34}$$

where a is a real parameter and $F_n = F_n(1)$ is the Fibonacci number.

Theorem 2.10. *Let $x \neq \pm\sqrt{Cy}$. The generalized Gegenbauer-Humbert polynomials $\{P_n^{1,y,C}(x)\}_{n\geq 0}$ defined by expansion* (2.33) *can be expressed as*

$$P_n^{1,y,C}(x) = C^{-n-2}\left[\frac{x+\sqrt{x^2-Cy}}{2\sqrt{x^2-Cy}}\left(x+\sqrt{x^2-Cy}\right)^n - \frac{x-\sqrt{x^2-Cy}}{2\sqrt{x^2-Cy}}\left(x-\sqrt{x^2-Cy}\right)^n\right]. \tag{2.35}$$

Proof. Taking derivative with respect to x to the two sides of (2.33) yields

$$2\lambda(x-yt)\left(C-2xt+yt^2\right)^{-\lambda-1} = \sum_{n\geq 1} nP_n^{\lambda,y,C}(x)t^{n-1}. \tag{2.36}$$

Then, substituting the expansion of $(C-2xt-yt^2)^{-\lambda}$ of (2.33) into the left-hand side of (2.36) and comparing the coefficients of term t^n on both sides, we obtain

$$C(n+1)P_{n+1}^{\lambda,y,C}(x) = 2x(\lambda+n)P_n^{\lambda,y,C}(x) - y(2\lambda+n-1)P_{n-1}^{\lambda,y,C}(x). \tag{2.37}$$

By transferring $n+1 \mapsto n$, we have

$$P_n^{\lambda,y,C}(x) = 2x\frac{\lambda+n-1}{Cn}P_{n-1}^{\lambda,y,C}(x) - y\frac{2\lambda+n-2}{Cn}P_{n-2}^{\lambda,y,C}(x) \tag{2.38}$$

for all $n \geq 2$ with

$$\begin{aligned}
P_0^{\lambda,y,C}(x) &= \Phi(0) = C^{-\lambda},\\
P_1^{\lambda,y,C}(x) &= \Phi'(0) = 2\lambda x C^{-\lambda-1}.
\end{aligned} \tag{2.39}$$

Thus, if $\lambda = 1$, $P_n^{1,y,C}(x)$ satisfies linear recurrence relation

$$P_n^{1,y,C}(x) = \frac{2x}{C} P_{n-1}^{1,y,C}(x) - \frac{y}{C} P_{n-2}^{1,y,C}(x), \quad n \geq 2,$$
$$P_0^{1,y,C}(x) = C^{-1}, \qquad P_1^{1,y,C}(x) = 2xC^{-2}. \tag{2.40}$$

Therefore, we solve $t^2 - pt - q = 0$, where $p = 2x/C$ and $q = -y/C$, for t, and obtain solutions:

$$\alpha = \frac{1}{C}\left\{x + \sqrt{x^2 - Cy}\right\},$$
$$\beta = \frac{1}{C}\left\{x - \sqrt{x^2 - Cy}\right\}, \tag{2.41}$$

where $x \neq \pm\sqrt{Cy}$. Hence, Proposition 2.7 gives the formula of $P_n^{1,y,C}(x)$ $(n \geq 2)$ as

$$P_n^{1,y,C}(x) = C^{-2}\left[\frac{x + \sqrt{x^2 - Cy}}{2\sqrt{x^2 - Cy}}\alpha^n - \frac{x - \sqrt{x^2 - Cy}}{2\sqrt{x^2 - Cy}}\beta^n\right], \tag{2.42}$$

where α and β are shown as (2.41). This completes the proof. □

Remark 2.11. We may use recurrence relation (2.40) to define various polynomials that were defined using different techniques. Comparing recurrence relation (2.40) with the relations of the generalized Fibonacci and Lucas polynomials shown in Example 2.9, with the assumption of $P_0^{1,y,C} = 0$ and $P_1^{1,y,C} = 1$, we immediately know that

$$P_n^{1,1,1}(x) = 2xP_{n-1}^{1,1,1}(x) - P_{n-2}^{1,1,1}(x) = U_n(2x; 0, 1) \tag{2.43}$$

defines the Chebyshev polynomials of the second kind,

$$P_n^{1,-1,1}(x) = 2xP_{n-1}^{1,-1,1}(x) - P_{n-2}^{1,-1,1}(x) = U_n(2x; 0, -1) \tag{2.44}$$

defines the Pell polynomials, and

$$P_n^{1,-1,1}\left(\frac{x}{2}\right) = xP_{n-1}^{1,-1,1}\left(\frac{x}{2}\right) + P_{n-2}^{1,-1,1}\left(\frac{x}{2}\right) = U_n(x; 0, -1) \tag{2.45}$$

defines the Fibonacci polynomials.

In addition, in [15], Lidl et al. defined the Dickson polynomials are also the special case of the generalized Gegenbauer-Humbert polynomials, which can be defined uniformly using recurrence relation (2.40), namely,

$$D_n(x; a) = xD_{n-1}(x; a) - aD_{n-2}(x; a) = P_n^{1,2a,2}(x) \tag{2.46}$$

with $D_0(x;a) = 2$ and $D_1(x;a) = x$. Thus, the general terms of all of above polynomials can be expressed using (2.35).

Example 2.12. For $\lambda = y = C = 1$, using (2.35), we obtain the expression of the Chebyshev polynomials of the second kind:

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}}, \tag{2.47}$$

where $x^2 \neq 1$. Thus, $U_2(x) = 4x^2 - 1$.

For $\lambda = C = 1$ and $y = -1$, formula (2.35) gives the expression of a Pell polynomial of degree $n+1$:

$$P_{n+1}(x) = \frac{\left(x+\sqrt{x^2+1}\right)^{n+1} - \left(x-\sqrt{x^2+1}\right)^{n+1}}{2\sqrt{x^2+1}}. \tag{2.48}$$

Thus, $P_2(x) = 2x$.

Similarly, let $\lambda = C = 1$ and $y = -1$, the Fibonacci polynomials are

$$F_{n+1}(x) = \frac{\left(x+\sqrt{x^2+4}\right)^{n+1} - \left(x-\sqrt{x^2+4}\right)^{n+1}}{2^{n+1}\sqrt{x^2+4}}, \tag{2.49}$$

and the Fibonacci numbers are

$$F_n = F_n(1) = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}, \tag{2.50}$$

which has been presented in Example 2.5.

Finally, for $\lambda = C = 1$ and $y = 2$, we have Fermat polynomials of the first kind:

$$\Phi_{n+1}(x) = \frac{\left(x+\sqrt{x^2-2}\right)^{n+1} - \left(x-\sqrt{x^2-2}\right)^{n+1}}{2\sqrt{x^2-2}}, \tag{2.51}$$

where $x^2 \neq 2$. From the expressions of Chebyshev polynomials of the second kind, Pell polynomials, and Fermat polynomials of the first kind, we may get a class of the generalized Gegenbauer-Humbert polynomials with respect to y defined as follows.

Definition 2.13. The generalized Gegenbauer-Humbert polynomials with respect to y, denoted by $P_n^{(y)}(x)$, are defined by the expansion

$$\left(1-2xt+yt^2\right)^{-1} = \sum_{n\geq 0} P_n^{(y)}(x)t^n, \tag{2.52}$$

by

$$P_n^{(y)}(x) = 2xP_{n-1}^{(y)}(x) - yP_{n-2}^{(y)}(x), \tag{2.53}$$

or equivalently, by

$$P_n^{(y)}(x) = \frac{\left(x+\sqrt{x^2-y}\right)^{n+1} - \left(x-\sqrt{x^2-y}\right)^{n+1}}{2\sqrt{x^2-y}} \tag{2.54}$$

with $P_0^{(y)}(x) = 1$ and $P_1^{(y)}(x) = 2x$, where $x^2 \neq y$. In particular, $P_n^{(-1)}(x)$, $P_n^{(1)}(x)$, and $P_n^{(2)}(x)$ are, respectively, Pell polynomials, Chebyshev polynomials of the second kind, and Fermat polynomials of the first kind.

3. Identities Constructed from Recurrence Relations

From (2.2) we have the following result.

Proposition 3.1. *A sequence $\{a_n\}_{n\geq 0}$ of order 2 satisfies linear recurrence relation (2.1) if and only if it satisfies the nonhomogeneous linear recurrence relation of order 1 with the form*

$$a_n = \alpha a_{n-1} + d\beta^{n-1}, \tag{3.1}$$

where d is uniquely determined. In particular, if $\beta = 1$, then $a_n = \alpha a_{n-1} + d$ is equivalent to $a_n = (\alpha+1)a_{n-1} - \alpha a_{n-2}$, where $d = a_1 - \alpha a_0$.

Proof. The necessity is clearly from (2.1). We now prove sufficiency. If sequence $\{a_n\}$ satisfies the nonhomogeneous recurrence relation of order 1 shown in (3.1), then by substituting $n = 1$ into the above equation, we obtain $d = a_1 - \alpha a_0$. Thus, (3.1) can be written as

$$a_n - \alpha a_{n-1} = (a_1 - \alpha a_0)\beta^{n-1}, \tag{3.2}$$

which implies that $\{a_n\}$ satisfies the linear recurrence relation of order 2: $a_n = pa_{n-1} + qa_{n-2}$ with $p = \alpha + \beta$ and $q = -\alpha\beta$. In particular, if $\beta = 1$, then $p = \alpha + 1$ and $q = -\alpha$, which yields the special case of the proposition. □

An obvious example of the special case of Proposition 3.1 is the Mersenne number $a_n = 2^n - 1$ $(n \geq 0)$, which satisfies the linear recurrence relation of order 2: $a_n = 3a_{n-1} - 2a_{n-2}$ (with $a_0 = 0$ and $a_1 = 1$) and the nonhomogeneous recurrence relation of order 1: $a_n = 2a_{n-1} + 1$ (with $a_0 = 0$). It is easy to check that sequence $a_n = (k^n - 1)/(k-1)$ satisfies both the homogeneous recurrence relation of order 2, $a_n = (k+1)a_{n-1} - ka_{n-2}$, and the nonhomogeneous recurrence relation of order 1, $a_n = ka_{n-1} + 1$, where $a_0 = 0$ and $a_1 = 1$.

We now use (3.2) to prove some identities of Fibonacci and Lucas numbers and generalized Gegenbauer-Humbert polynomials. Let $\{F_n\}$ be the Fibonacci sequence. From (3.2),

$$F_n - \alpha F_{n-1} = (F_1 - \alpha F_0)\beta^{n-1} = \beta^{n-1}, \tag{3.3}$$

where the last step is due to $\alpha + \beta = 1$. Therefore, we give a simple identity

$$F_n = \alpha F_{n-1} + \beta^{n-1} = \frac{1+\sqrt{5}}{2}F_{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}, \tag{3.4}$$

which is shown in [16, (8.2) page 122] by Koshy. Similarly, we have

$$\beta F_n = \beta\alpha F_{n-1} + \beta^n = \beta^n - F_{n-1}, \tag{3.5}$$

where the last step is due to $\alpha\beta = -1$. The above identity can be written as

$$\left(\frac{1-\sqrt{5}}{2}\right)^n = \frac{1-\sqrt{5}}{2}F_n + F_{n-1}. \tag{3.6}$$

The same argument yields $\alpha^n = \alpha F_n + F_{n-1}$, or equivalently,

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{1+\sqrt{5}}{2}F_n + F_{n-1}. \tag{3.7}$$

Identities (3.6) and (3.7) were proved by using different method in [16, page 78].

Let $\{L_n\}$ be the Lucas number sequence with $L_0 = 2$ and $L_1 = 1$, which satisfies recurrence relation (2.2) with the same α and β for the Fibonacci number sequence. Then, using the same argument, we have

$$L_n - \alpha L_{n-1} = (L_1 - \alpha L_0)\beta^{n-1} = (1 - 2\alpha)\beta^{n-1} = -\sqrt{5}\beta^{n-1}. \tag{3.8}$$

Thus

$$L_n - \left(\frac{1+\sqrt{5}}{2}\right)L_{n-1} = -\sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}, \tag{3.9}$$

or equivalently,

$$-L_{n+1} + \left(\frac{1+\sqrt{5}}{2}\right)L_n = \sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{3.10}$$

(see [16, page 129]).

We now extend the above results regarding Fibonacci and Lucas numbers to more general sequences presented by Niven et al. in [17]. Let $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ be two sequences defined, respectively, by the linear recurrence relations of order 2:

$$G_n = pG_{n-1} + qG_{n-2}, \qquad H_n = pH_{n-1} + qH_{n-2} \tag{3.11}$$

with initial conditions $G_0 = 0$ and $G_1 = 1$ and $H_0 = 2$ and $H_1 = p$, respectively. Clearly, if $p = q = 1$, then G_n and H_n are, respectively, Fibonacci and Lucas numbers. From (3.2), we immediately have

$$G_n - \frac{p+\sqrt{p^2+4q}}{2}G_{n-1} = \left(\frac{p-\sqrt{p^2+4q}}{2}\right)^{n-1}. \tag{3.12}$$

Multiplying $(p-\sqrt{p^2+4q})/2$ to both sides of the above equation yields

$$\frac{p-\sqrt{p^2+4q}}{2}G_n + qG_{n-1} = \left(\frac{p-\sqrt{p^2+4q}}{2}\right)^{n}. \tag{3.13}$$

Similarly, we obtain

$$\frac{p+\sqrt{p^2+4q}}{2}G_n + qG_{n-1} = \left(\frac{p+\sqrt{p^2+4q}}{2}\right)^{n}. \tag{3.14}$$

When $p = q = 1$, the last two identities are (3.6) and (3.7), respectively.

Using (3.2) we can also obtain the identity

$$\frac{p+\sqrt{p^2+4q}}{2}H_n - H_{n+1} = \sqrt{p^2+4q}\left(\frac{p-\sqrt{p^2+4q}}{2}\right)^{n}, \tag{3.15}$$

which implies (3.10) when $p = q = 1$.

Aharonov et al. (see [18]) have proved that the solution of any sequence of numbers that satisfies a recurrence relation of order 2 with constant coefficients and initial conditions $a_0 = 0$ and $a_1 = 1$ can be expressed in terms of Chebyshev polynomials. For instance, the authors show $F_n = i^{-n}U_n(i/2)$ and $L_n = 2i^{-n}T_n(i/2)$. Thus, we have identities

$$\begin{gathered}\left(\frac{1+\sqrt{5}}{2}\right)^n = \frac{1+\sqrt{5}}{2i^n}U_n\left(\frac{i}{2}\right) + i^{-(n-1)}U_n\left(\frac{i}{2}\right), \\ \sqrt{5}\left(\frac{1-\sqrt{5}}{2}\right)^n = 2i^{-n}T_n\left(\frac{i}{2}\right) - \left(\frac{1+\sqrt{5}}{i^{n-1}}\right)T_{n-1}\left(\frac{i}{2}\right).\end{gathered} \tag{3.16}$$

In [19], Chen and Louck obtained $F_{n+1} = i^n U_n(-i/2)$. Thus we have identity

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = i^{n-1}\frac{1+\sqrt{5}}{2}U_{n-1}\left(\frac{-i}{2}\right) + i^{n-2}U_{n-2}\left(\frac{-i}{2}\right). \tag{3.17}$$

Identities (3.6) and (3.7) can be used to prove the following radical identity given by Sofo in [20]:

$$(\alpha F_n + F_{n-1})^{1/n} + (-1)^{n+1}(F_{n+1} - \alpha F_n)^{1/n} = 1. \tag{3.18}$$

Identity (3.7) shows that the first term on the left-hand side of (3.18) is simply α. Assume the sum in the third parenthesis on the left-hand side of (3.18) is c, then

$$\beta c = \beta F_{n+1} - \alpha\beta F_n = F_n + \beta F_{n+1} = \beta^{n+1}, \tag{3.19}$$

where the last step is from (3.6) with transform $n \mapsto n+1$. Thus, we have $c = \beta^n$. If n is odd, the left-hand side of (3.18) is $\alpha + \beta = 1$. If n is even, the left-hand side of (3.18) becomes $\alpha - |\beta| = \alpha + \beta = 1$, which completes the proof of Sofo's identity.

In general, let $\{a_n\}$ be a sequence of order 2 satisfying linear recurrence relation (1.1) or equivalently (2.1). Then we sum up our results as follows.

Theorem 3.2. *Let $\{a_n\}_{n\geq 0}$ be a sequence of numbers or polynomials defined by the linear recurrence relation $a_n = pa_{n-1} + qa_{n-2}$ $(n \geq 0)$ with initial conditions a_0 and a_1, and let $p = \alpha + \beta$ and $q = -\alpha\beta$. Then we have identity*

$$a_n = \alpha a_{n-1} + (a_1 - \alpha a_0)\beta^{n-1}. \tag{3.20}$$

In particularly, if $\{a_n = P_n^{\lambda,y,C}(x)\}$, the sequence of the generalized Gegenbauer-Humbert polynomial is defined by (2.33)*, then we obtain the polynomial identity:*

$$P_n^{1,y,C}(x) = \alpha P_{n-1}^{1,y,C}(x) + C^{-2}(2x - \alpha C)\beta^{n-1}, \tag{3.21}$$

where α and β are shown in (2.41)*. For $C = 1$ (i.e., the generalized Gegenbauer-Humbert polynomials with respect to y), we denote $P_n^{(y)}(x) \equiv P_n^{1,y,1}(x)$ and have*

$$P_n^{(y)}(x) = \left(x + \sqrt{x^2 - y}\right)P_{n-1}^{(y)}(x) + \left(x - \sqrt{x^2 - y}\right)^n. \tag{3.22}$$

Actually, (3.22) can also be proved directly. Similarly, for the Chebyshev polynomials of the first kind $T_n(x)$, we have the identity

$$\left(x + \sqrt{x^2 - 1}\right)T_{n-1}(x) - T_n(x) = \sqrt{x^2 - 1}\left(x - \sqrt{x^2 - 1}\right)^{n-1}. \tag{3.23}$$

Let $x = \cos\theta$, the above identity becomes

$$e^{i\theta}\cos((n-1)\theta) - \cos(n\theta) = i\sin\theta e^{-i(n-1)\theta}, \tag{3.24}$$

which is equivalent to $\cos n\theta = \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta$.

Another example is from the sequence $\{a_n\}$ shown in Corollary 2.2 with $a_0 = 1$ and $a_1 = p$. Then (3.20) gives the identity $a_n = \alpha a_{n-1} + \beta^n$, where $\alpha = (p+\sqrt{p^2+4})/2$ and $\beta = -1/\alpha$. Similar to (3.6) and (3.7), for those sequences $\{a_n\}$ with $a_0 = 1$ and $a_1 = p$, we obtain identities

$$\begin{aligned}\left(\frac{p-\sqrt{p^2+4}}{2}\right)^{n+1} &= \frac{p-\sqrt{p^2+4}}{2}a_n + a_{n-1},\\ \left(\frac{p+\sqrt{p^2+4}}{2}\right)^{n+1} &= \frac{p+\sqrt{p^2+4}}{2}a_n + a_{n-1}.\end{aligned} \tag{3.25}$$

When $p = 1$, the above identities become (3.6) and (3.7), respectively. Similarly, we can prove $a_{n+k} = \alpha^k a_n + \beta^n a_k$, where $\alpha = (p+\sqrt{p^2+4})/2$ and $\beta = -1/\alpha$.

It is clear that if $1/a_n$ is bounded and $|\beta| < 1$, from (3.20) we have

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \alpha. \tag{3.26}$$

Therefore,

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \lim_{n\to\infty}\frac{L_n}{L_{n-1}} = \frac{1+\sqrt{5}}{2}. \tag{3.27}$$

The method presented in this paper cannot be extended to the higher-order setting. However, we may use the idea and a similar argument to derive some identities of sequences of order greater than 2. For instance, for a sequence $\{a_n\}$ of numbers or polynomials that satisfies the linear recurrence relation of order 3:

$$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3}, \quad n \geq 3, \tag{3.28}$$

we set the equation

$$t^3 - pt^2 - qt - r = 0. \tag{3.29}$$

Using transform $t = s + p/3$, we can change the equation to the standard form $s^3 + as + b = 0$, which can be solved by Vieta's substitution $s = u - a/(3u)$. The formulas for the three roots, denoted by α, β, γ, are sometimes known as Cardano's formula. Thus, we have

$$\alpha + \beta + \gamma = p, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -q, \qquad \alpha\beta\gamma = r. \tag{3.30}$$

Denote $y_n = a_n - \alpha a_{n-1}$. Then (3.28) can be written as

$$y_n = (\beta + \gamma) y_{n-1} - \beta\gamma y_{n-2}. \tag{3.31}$$

From Propositions 2.1 or 2.7, one may obtain

$$y_n = \begin{cases} \left(\dfrac{y_1 - \beta y_0}{\gamma - \beta}\right)\gamma^n - \left(\dfrac{y_1 - \gamma y_0}{\gamma - \beta}\right)\beta^n, & \text{if } \gamma \neq \beta; \\ n y_1 \gamma^{n-1} - (n-1) y_0 \gamma^n, & \text{if } \gamma = \beta. \end{cases} \tag{3.32}$$

Therefore, from the identity $y_n = \gamma y_{n-1} + \beta^{n-1}(y_1 - \gamma y_0)$, we obtain identity in terms of a_n:

$$\begin{aligned} \beta^{n-1}(y_1 - \gamma y_0) &= a_n - \alpha a_{n-1} - \gamma(a_{n-1} - \alpha a_{n-2}) \\ &= a_n - (\alpha + \gamma) a_{n-1} + \alpha\gamma a_{n-2}, \end{aligned} \tag{3.33}$$

or equivalently,

$$a_n = (\alpha + \gamma) a_{n-1} - \alpha\gamma a_{n-2} + \beta^{n-1}\left(a_1 - (\alpha + \gamma) a_0 + \alpha\gamma a_{-1}\right), \tag{3.34}$$

where a_{-1} can be found uniquely from $y_2 = (\beta + \gamma) y_1 - \beta\gamma y_0$ and $y_0 = a_0 - \alpha a_{-1}$, that is,

$$a_2 - \alpha a_1 = (\beta + \gamma)(a_1 - \alpha a_0) - \beta\gamma(a_0 - \alpha a_{-1}), \tag{3.35}$$

or equivalently,

$$a_{-1} = \frac{a_2 - p a_1 - q a_0}{r}, \quad r \neq 0. \tag{3.36}$$

We have seen the equivalence between the homogeneous recurrence relation of order 3, in (3.28), and the nonhomogeneous recurrence relation of order 2, in (3.34).

Remark 3.3. Similar to the particular case shown in Proposition 3.1, we may find the equivalence between the nonhomogeneous recurrence relation of order 2, $a_n = p a_{n-1} + q a_{n-2} + k$, and the homogeneous recurrence relation of order 3, $a_n = (p+1) a_{n-1} + (q - p) a_{n-2} + q a_{n-3}$, where $k = a_2 - p a_1 - q a_0$.

Example 3.4. As an example, we consider the tribonacci number sequence generated by $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ $(n \geq 3)$. Solving $t^3 - t^2 - t - 1 = 0$, we obtain

$$\begin{aligned}
\beta &= \frac{1}{3} + \frac{1}{3}\left(19 - 3\sqrt{33}\right)^{1/3} + \frac{1}{3}\left(19 + 3\sqrt{33}\right)^{1/3}, \\
\alpha &= \frac{1}{3} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\left(19 - 3\sqrt{33}\right)^{1/3} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\left(19 + 3\sqrt{33}\right)^{1/3}, \\
\gamma &= \frac{1}{3} - \frac{1}{6}\left(1 - i\sqrt{3}\right)\left(19 - 3\sqrt{33}\right)^{1/3} - \frac{1}{6}\left(1 + i\sqrt{3}\right)\left(19 + 3\sqrt{33}\right)^{1/3}.
\end{aligned} \tag{3.37}$$

Substituting α, β, γ, and $y_0 = 0$ (with the assumption $a_{-1} = 0$) and $y_1 = 1$ into (3.33), we obtain an identity regarding the tribonacci number sequence $\{a_n = 0, 1, 1, 2, 4, 7, 13, 24, \ldots\}$:

$$a_n - \frac{1}{3}(2 - a - b)a_{n-1} + \frac{1}{9}\left(-3 - a - b + a^2 + b^2\right)a_{n-2} = \frac{1}{3^{n-1}}(1 + a + b)^{n-1}, \tag{3.38}$$

where $a = (19 - 3\sqrt{33})^{1/3}$ and $b = (19 + 3\sqrt{33})^{1/3}$.

For the sequence defined by

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}, \quad n \geq 3, \tag{3.39}$$

with initial conditions $a_0 = a_1 = 1$ and $a_2 = 2$. Then, the first few numbers of the sequence are $\{1, 1, 2, 7, 26, 91, \ldots\}$. The three roots of $t^3 - 6t^2 + 11t - 6 = 0$ are $\alpha = 1$, $\beta = 2$, and $\gamma = 3$. Therefore, by assuming $a_{-1} = 7/6$, we obtain the corresponding $y_0 = -1/6$ and $y_1 = 0$ and the following identity for the above-defined sequence:

$$a_n - 4a_{n-1} + 3a_{n-2} = 2^{n-2} \tag{3.40}$$

for all $n \geq 2$. From [21] by Haye, $a_n = \left\{ {n+1 \atop 3} \right\} + 1$ $(n \geq 0)$, where $\left\{ {n \atop k} \right\}$ are Stirling numbers of the second kind. Hence, we obtain an identity of the Stirling numbers of the second kind:

$$\left\{ {n+1 \atop 3} \right\} - 4\left\{ {n \atop 3} \right\} + 3\left\{ {n-1 \atop 3} \right\} = 2^{n-2}. \tag{3.41}$$

The idea to reduce a linear recurrence relation of order 3 to order 2 can be extended to the higher order cases. In general, if we have a sequence $\{a_n\}$ satisfying the linear recurrence relation of order r:

$$a_n = \sum_{k=1}^{r} p_k a_{n-k}. \tag{3.42}$$

Assume the equation $t^r - \sum_{k=1}^{r} p_k t^{r-k} = 0$ has solutions α_k $(1 \le k \le r)$. Denote $y_n = a_n - \alpha_1 a_{n-1}$. Then the above recurrence relation can be reduced to

$$y_n = y_{n-1}\sum_{k=2}^{r}\alpha_k - y_{n-2}\sum_{2\le i\ne j\le r}\alpha_i\alpha_j + y_{n-3}\sum_{2\le i\ne j\ne k\le r}\alpha_i\alpha_j\alpha_k - \cdots + (-1)^r y_{n-r+1}\prod_{k=2}^{r}\alpha_k, \tag{3.43}$$

a linear recurrence relation of order $r-1$ for sequence $\{y_n\}$. Using this process, we may obtain the explicit formula of a_n and/or identities in terms of a_n if we know the solution of the last equation and/or the identities in terms of sequence $\{y_n\}$.

The process shown in Proposition 3.1 can be applied conversely to elevate a nonhomogenous recurrence relation of order n to a homogeneous recurrence relation of order $n+1$.

4. Solutions of Algebraic Equations and Differential Equations

The results presented in Sections 2 and 3 have more applications. In this section, we will discuss the applications in the solutions of algebraic equations and initial value problems of second-order ordinary differential equations.

First, we consider roots of polynomials $p(x) = x^n - xF_n - F_{n-1}$ or the solution of $p(x) = 0$, where $\{F_n\}_{n\ge 0}$ is the Fibonacci sequence. Using the identity $\alpha^n = \alpha F_n + F_{n-1}$, we immediately know that the largest root of $p(x)$ is

$$x = \alpha = \frac{1+\sqrt{5}}{2}. \tag{4.1}$$

Indeed, $p(x)$ only changes its coefficient signs once, which implies that it has only one positive root α and all of its other roots must be negative, for example, $\beta = (1-\sqrt{5})/2$. In [22], Wall proved the largest root of $p(x)$ is α using a more complicated manner.

We may write the identity $\alpha^n = \alpha F_n + F_{n-1}$ as

$$\left(\frac{1+\sqrt{5}}{2}\right)^n = \left(\frac{1+\sqrt{5}}{2}\right)F_n + F_{n-1} = \frac{L_n + \sqrt{5}F_n}{2}, \tag{4.2}$$

where L_n is the nth Lucas number. Similarly, we have

$$\left(\frac{1-\sqrt{5}}{2}\right)^n = \left(\frac{1-\sqrt{5}}{2}\right)F_n + F_{n-1} = \frac{L_n - \sqrt{5}F_n}{2}. \tag{4.3}$$

Multiplying the last two equations side by side yields $(-1)^n = (L_n^2 - 5F_n^2)/4$, or equivalently, $L_n^2 = F_n^2 + 4(-1)^n$. The last expression means $5F_n^2 + 4(-1)^n$ is a perfect square, or equivalently, if n is a Fibonacci number, then $5n^2 \pm 4$ is a perfect square. This result is a part of Gessel's results in [23], but the method we used seems simpler. In addition, the above result also shows that the Pell's equation $x^2 - 5y^2 = \pm 4$ has a solution $(x, y) = (L_n, F_n)$.

The above results on the Fibonacci sequence can be extended to the sequence $\{G_n\}_{n\ge 0}$ shown in [17] and Section 3. Consider polynomial $p(x) = x^n - xG_n - qG_{n-1}$ with $q \ge 1$. We can see the largest root of $p(x)$ is $(p+\sqrt{p^2+4q})/2$, which implies Wall's result when $p=q=1$. In addition, because of

$$\begin{aligned}\left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n &= \frac{p+\sqrt{p^2+4q}}{2}G_n + qG_{n-1}\\ &= \frac{1}{2}\left(\sqrt{p^2+4q}G_n + H_n\right),\\ \left(\frac{p-\sqrt{p^2+4q}}{2}\right)^n &= \frac{p-\sqrt{p^2+4q}}{2}G_n + qG_{n-1}\\ &= \frac{1}{2}\left(-\sqrt{p^2+4q}G_n + H_n\right),\end{aligned} \tag{4.4}$$

we have

$$(-q)^n = \frac{1}{4}\left(H_n^2 - \left(p^2+4q\right)G_n^2\right), \tag{4.5}$$

or equivalently,

$$H_n^2 = \left(p^2+4q\right)G_n^2 + 4(-q)^n. \tag{4.6}$$

Hence, $(p^2+4q)G_n^2+4(-q)^n$ is a perfect square, which implies the special case for the Fibonacci sequence that has been presented. Therefore, Pell's equation $x^2-(p^2+4q)y^2=\pm 4q^n$ $(p^2+4q>0)$ has a solution $(x,y)=(H_n,G_n)$.

We now use the method presented in Section 2 to reduce an initial problem of a second order ordinary differential equation:

$$y'' - py' - qy = 0, \qquad y'(0) = A, \qquad y(0) = B, \tag{4.7}$$

of second-order ordinary differential equation with constant coefficients to the problem of linear equations. Let $p, q \neq 0$, and let α and β be solution(s) of $x^2 - px - q = 0$. Denote

$$v(x) := y'(x) - \alpha y(x). \tag{4.8}$$

Then $v'(x) = y''(x) - \alpha y'(x)$, and the original initial problem of the second order is split into two problems of first order by using the method shown in (2.2) for $n=2$:

$$\begin{aligned} v'(x) &= \beta v(x), \qquad v(0) = A - \alpha B;\\ y'(x) - \alpha y(x) &= v(x), \qquad y(0) = B.\end{aligned} \tag{4.9}$$

Thus, we obtain the solutions

$$v(x) = (A - \alpha B)e^{\beta x},$$
$$y(x) = \begin{cases} \dfrac{1}{\alpha - \beta}\left((A - \beta B)e^{\alpha x} - (A - \alpha B)e^{\beta x}\right), & \text{if } \alpha \neq \beta; \\ e^{\alpha x}((A - \alpha B)x + B), & \text{if } \alpha = \beta. \end{cases} \tag{4.10}$$

The above technique can be extended to the initial problems of higher-order ordinary differential equations. In this paper, we presented an elementary method for construction of the explicit formula of the sequence defined by the linear recurrence relation of order 2 and the related identities. Some other applications in solutions of algebraic and differential equations and some extensions to the higher dimensional setting are also discussed. However, besides those applications, more applications in combinatorics and the combinatorial explanations of our given formulas still remain much to be investigated.

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