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# ON A RECURRENCE RELATION IN TWO VARIABLES

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## 1. INTRODUCTION

Gupta [3] considers the array $\{c(n,k)\}$, which is defined by the recurrence relation

$$c(n+1,k)=c(n,k)+c(n,k-1) \tag{1}$$

with initial values $c(n,0)=a(n)$, $c(1,k)=b(k)$, $n,k\geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. In particular, if $a(n)\equiv 1$ and $b(1)=1$, $b(k)=0$ for $k\geq 2$, then $\{c(n,k)\}$ is the classical binomial array. The array $\{c(n,k)\}$ also has applications, for example, in the theory of partitions of integers ([1], [2]). The main object of Gupta [3] is to handle $\{c(n,k)\}$ with the aid of generating functions. Wilf [7, §§1.5 and 1.6] handles $\{c(n,k)\}$ and an analog of $\{c(n,k)\}$, namely, the array of the Stirling numbers of the second kind, with the aid of generating functions.

In this paper we consider a further analog of the array $\{c(n,k)\}$, namely, the array $\{L(n,k)\}$ defined by the recurrence relation

$$L(n,k)=L(n-1,k-1)-L(n,k-1) \tag{2}$$

with initial values $L(n,0)=a(n)$, $L(0,k)=b(k)$, $L(0,0)=a(0)=b(0)$, $n,k\geq 1$, where $\{a(n)\}$ and $\{b(k)\}$ are given sequences of numbers. We derive an expression for the numbers $L(n,k)$ in terms of the initial values using the method of generating functions. We motivate the study of the array $\{L(n,k)\}$ by providing a concrete example of this kind of array from the theory of stack filters.

## 2. AN EXPRESSION FOR THE NUMBERS $L(n,k)$

Let $L_k(x)$ denote the generating function of the sequence $\{L(n,k)\}_{n=0}^{\infty}$, that is,

$$L_k(x)=\sum_{n=0}^{\infty}L(n,k)x^n. \tag{3}$$

The promised expression for the numbers $L(n,k)$ comes out as follows: We use the recurrence relation (2) to obtain a recurrence relation for the generating function $L_k(x)$. This recurrence relation yields an expression for the generating function $L_k(x)$, which gives an expression for the numbers $L(n,k)$.

***Theorem 1:*** Let $\{L(n,k)\}$ be the array given in (2). Then

$$L(n,k)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a(n-j)+\sum_{j=n}^{k-1}\binom{j}{n}(-1)^{j-n}[b(k-j)-b(k-j-1)]. \tag{4}$$

***Proof:*** If $k=0$, then (4) holds. Let $k\geq 1$. Then, by (3) and (4),

$$L_k(x) = L(0,k) + \sum_{n=1}^{\infty}[L(n-1,k-1) - L(n,k-1)]x^n$$
$$= L(0,k) + [xL_{k-1}(x) - (L_{k-1}(x) - L(0,k-1))]$$

or

$$L_k(x) = (x-1)L_{k-1}(x) + b(k) + b(k-1).$$

Let $d(k) = b(k) + b(k-1)$. Proceeding by induction on $k$, we obtain

$$L_k(x) = (x-1)^k L_0(x) + \sum_{j=0}^{k-1}(x-1)^j d(k-j). \tag{5}$$

Here

$$(x-1)^k L_0(x) = \left(\sum_{n=0}^{\infty}\binom{k}{n}(-1)^{k-n}x^n\right)\left(\sum_{n=0}^{\infty}a(n)x^n\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{k}{j}(-1)^{k-j}a(n-j)\right)x^n \tag{6}$$

and

$$\sum_{j=0}^{k-1}(x-1)^j d(k-j) = \sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{j}{n}x^n(-1)^{n-j}d(k-j)$$
$$= \sum_{n=0}^{k-1}\left(\sum_{j=n}^{k-1}\binom{j}{n}(-1)^{j-n}d(k-j)\right)x^n. \tag{7}$$

Now, combining (3), (5), (6), and (7), we obtain (4).

***Remark:*** We considered above the generating function of the array $\{L(n,k)\}$ with respect to the variable $n$. We could also consider generating functions with respect to the variable $k$ and with respect to both the variables $n$ and $k$. These considerations, however, would be more laborious and are not presented here (cf. [3], [7]).

## 3. AN EXAMPLE FROM THE THEORY OF STACK FILTERS

Consider a stack filter (for definition, see [4], [6]) with continuous i.i.d. inputs having distribution function $\Phi(\cdot)$ and with window size $N$. The $\gamma$-order moment about the origin of the output can be written as

$$\alpha^\gamma = E\{Y_{\text{out}}^\gamma\} = \sum_{k=0}^{N-1}A_k M(\Phi,\gamma,N,k),$$

where

$$M(\Phi,\gamma,N,k) = \int_{-\infty}^{\infty}x^\gamma \frac{d}{dx}\left((1-\Phi(x))^k\Phi(x)^{N-k}\right)dx,\ k = 0,1,\ldots,N-1, \tag{8}$$

and where the coefficients $A_k$, $k = 0,1,\ldots,N-1$, have a certain natural interpretation (see [4], [5]). By using the output moments about the origin, we easily obtain output central moments, denoted by $\mu^\gamma = E\{(Y_{\text{out}} - E\{Y_{\text{out}}\})^\gamma\}$, for example, the second-order central output moment equals

$$\mu^2 = \sum_{k=0}^{N-1} A_k M(\Phi, 2, N, k) - \left( \sum_{k=0}^{N-1} A_k M(\Phi, 1, N, k) \right)^2. \tag{9}$$

The second-order central output moment is quite often used as a measure of the noise attenuation capability of a filter. It quantifies the spread of the input samples with respect to their mean value. Equation (9) gives an expression for the second-order central output moment. In this expression, the numbers $M(\Phi, \gamma, N, k)$ play a crucial role.

Kuosmanen [4] and Kuosmanen & Astola [5] studied the properties of the numbers $M(\Phi, \gamma, N, k)$ under certain conditions on $\Phi(\cdot)$. They showed, among other things, that the numbers $M(\Phi, \gamma, N, k)$ satisfy the recurrence relation

$$M(\Phi, \gamma, N, k) = M(\Phi, \gamma, N-1, k-1) - M(\Phi, \gamma, N, k-1), \; 1 \le k \le N,$$

with initial values

$$M(\Phi, \gamma, N, 0) = \int_{-\infty}^{\infty} x^\gamma \frac{d}{dx}\left(\Phi(x)^N\right) dx, \; N \ge 0.$$

This means that the numbers $M(\Phi, \gamma, N, k)$ satisfy recurrence relation (2). As $M(\Phi, \gamma, N, k) = 0$ if $N = 0$, that is, as $b(k) \equiv 0$ in (2), application of Theorem 1 gives

$$M(\Phi, \gamma, N, k) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} M(\Phi, \gamma, N-j, 0). \tag{10}$$

Note that Kuosmanen [4] and Kuosmanen & Astola [5] derived (10) directly from (8) using the binomial theorem.

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