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# A SHIFT FORMULA FOR RECURRENCE RELATIONS OF ORDER m

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It is well known that if $F_i$ is the $i^{th}$ Fibonacci number, then

$$F_{n+k+1} = F_{n+1}F_{k+1} + F_nF_k$$

for all integers n, k. A generalization of this identity to recurrence relations of any order m is given here.

Let m be a positive integer and let $p_1, p_2, \cdots, p_m$ ($p_m \neq 0$) be m elements of a field F. Furthermore, let $\{y_i\}$ and $\{U_i\}$ be two sequences in F obeying the recurrence relation whose auxiliary polynomial is

$$P(x) = x^m - \sum_{j=0}^{m-1} p_{m-j}x^j ,$$

and let $\{U_i\}$ have the initial values

$$U_0 = U_1 = \cdots = U_{m-2} = 0$$

and

$$U_{m-1} = 1 .$$

Then,

$$\text{(1)} \qquad y_{n+k} = \sum_{j=0}^{m-1}\sum_{i=0}^{j} p_{m-i}U_{k+i-j-1}y_{n+j}$$

for all integers n and k.

The proof of (1) is by induction on $k$. Let $n$ be fixed. For $0 \le k < m$ it is clear that

$$(2) \qquad \sum_{i=0}^{j} p_{m-i} U_{k+i-j-1} = \begin{cases} 0 & \text{if } j < k \\ p_m U_{-1} = 1 & \text{if } j = k \\ \sum_{i=0}^{m-1} p_{m-i} U_{k+i-j-1} = U_{k+m-j-1} = 0 & \text{if } k < j < m. \end{cases}$$

From (2) it immediately follows that (1) holds for $k = 0, 1, \cdots, m-1$. From here, applications of the recurrence relation (corresponding to $P(x)$) for $\{y_i\}$ and $\{U_i\}$, in both the forward and backward directions, easily prove that if (1) holds for $k = h, h+1, \cdots, h+m-1$, then (1) holds for $k = h-1, h, \cdots, h+m$. By application of finite induction, it follows that (1) holds for all integers $n, k$.

Let $P(x) = (x - r_1)(x - r_2) \cdots (x - r_m)$ in an extension $G$ of $F$ and suppose that $G$ is of characteristic zero. Further suppose that the $r_j$ are pairwise distinct. Define $D_k$ as the determinant produced by the process of substituting the vector $(r_1^k, r_2^k, \cdots, r_m^k)$ for the $m^{th}$ row $(r_1^{m-1}, r_2^{m-1}, \cdots, r_m^{m-1})$ in the Vandermonde determinant of $r_1, r_2, \cdots, r_m$. It is proven in [1] that for every integer $k$,

$$(3) \qquad U_k = \frac{D_k}{D_{m-1}} .$$

The case for repetitions among the $r_j$ is handled in the following way: Start with the form for $U_k$ in (3) and, pretending that the $r_j$ are real, apply L'Hospital's Rule successively as $r_I \to r_J$ for all repetitions $r_I = r_J$ among the $r_j$.

A combination of (1) and (3) now comes with ease. Still taking the $r_j$ to be pairwise distinct, define $E_k$ as the determinant produced by the process of replacing the element $r_h^k$ of the $m^{th}$ row of $D_k$ by

$$\sum_{j=0}^{m-1} \sum_{i=0}^{j} p_m y_j r_h^{k+i-j-1} ,$$

and this for $h = 1, 2, \cdots, m$. Then combination of (1) with (3) yields: For every integer $k$,

$$y_k = \frac{E_k}{D_{m-1}} \quad . \tag{4}$$

The case for repeated roots is handled as with (3). In [2] identities akin to (4) are developed.

## REFERENCES


1. Arkin, Joseph, "Recurring Series," to appear in the Fibonacci Quarterly.
2. Styles, C. C., "On Evaluating Certain Coefficients," The Fibonacci Quarterly, Vol. 4, No. 2, April, 1966.


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