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Matrices determined by a linear recurrence relation among entries☆

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Abstract

A matrix $A = [a_{ij}]$ is called a 7-matrix if its entries satisfy the recurrence relation $\alpha a_{i-1,j-1} + \beta a_{i-1,j} = a_{ij}$ where α, β are fixed numbers. A 7-matrix is completely determined by its first row and first column. In this paper we determine the structure of 7-matrices and investigate the sequences represented by columns of infinite 7-matrices.

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1. Introduction

Suppose that a particle at the origin of the coordinate plane starts to move and that, at each time step, it jumps from the current position to right or upward by one unit length with the probability α, β respectively, not staying trapped.

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At each time step the particle either moves right or up, but not both. Thus in particular $\alpha + \beta = 1$.

Notice that since the particle changes its position at each step, it passes through the point (k, l) if and only if it reaches (k, l) at the clock time $k + l$. Let $p_{k,l}$ denote the probability that this particle passes through the point (k, l). Then, since the particle starts at (0, 0) it does pass through (0, 0) and hence $p_{0,0} = 1$. Clearly, $p_{0,1} = \alpha$ and $p_{1,0} = \beta$. To reach the point (k, l), the particle either goes to the right from $(k - 1, l)$ to (k, l) or jumps up from $(k, l - 1)$ to (k, l) and no other ways. Thus we have the relation

$$\alpha p_{k-1,l} + \beta p_{k,l-1} = p_{k,l}.$$

Let $q_{k+l,k} = p_{k,l}$. Then $q_{0,0} = 1$, $q_{0,1} = 0$, $q_{1,0} = \beta$, and the above equation may be rewritten as

$$\alpha q_{n-1,k-1} + \beta q_{n-1,k} = q_{n,k}, \tag{1}$$

where $n = k + l$. It is easily seen that

$$q_{ij} = \alpha^j \beta^{i-j} \binom{i}{j} \quad (i, j = 0, 1, 2, \ldots).$$

Motivated by this example, we are interested in matrices whose entries satisfy a recurrence relation like (1).

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let α, β be a pair of nonzero real numbers. Suppose that the entries of A satisfy the recurrence relation

$$\alpha a_{i-1,j-1} + \beta a_{i-1,j} = a_{ij} \quad (i = 2, 3, \ldots, m; \quad j = 2, 3, \ldots, n). \tag{2}$$

We see that once the entries in the first row and first column are determined then all the entries are determined by the relation (2). The relative positions of the entries $a_{i-1,j-1}$, $a_{i-1,j}$ and a_{ij} in (2) form the 'mirrored gamma' or 'figure 7' shape. So, we call the relation (2) the $7_{\alpha,\beta}$-*law*, and we call a matrix a $7_{\alpha,\beta}$-*matrix* if its entries satisfies the $7_{\alpha,\beta}$-law. A matrix is called a 7-*matrix* if it is a $7_{\alpha,\beta}$-matrix for some α, β. $7_{1,1}$-matrices are called *simple* 7-*matrices*. The $(n + 1) \times (n + 1)$ Pascal matrix

$$P_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix}$$

is a typical example of a simple 7-matrix.

In the past couple of decades, the Pascal triangle has been recognized as a matrix with nice properties and various extensions of a Pascal matrix have been investigated by several authors [2–5]. All the extensions of P_n that have appeared so far are 'Pascal functional matrices', namely those that are obtained from P_n by replacing each of the entries $\binom{i}{j}$ with a monomial in one or two indeterminates whose coefficient is $\binom{i}{j}$, for example, $P_n[x] = [x^{i-j}\binom{i}{j}]$ or $\Phi_n(x, y) = [x^{i-j}y^{i+j}\binom{i}{j}]$. These 'extended' Pascal matrices have the same zero pattern as P_n.

The notion of a 7-matrix is certainly another extension of the Pascal matrix. 7-Matrices, however, look quite different from the other extensions that have appeared so far.

In this paper we investigate the structure and properties of 7-matrices along with some applications of 7-matrices.

Throughout this paper, for a pair α, β of real numbers, let $P_n(\alpha, \beta)$ denote the $(n+1) \times (n+1)$ matrix whose rows and columns are indexed by $0, 1, 2, \ldots, n$ and whose (i, j)-entry q_{ij} is defined by

$$q_{ij} = \alpha^j \beta^{i-j} \binom{i}{j} \quad (i, j = 0, 1, 2, \ldots, n).$$

For a number x, let

$$D_x = \operatorname{diag}(1, x, x^2, \ldots, x^n).$$

Then it is easily seen that

$$P_n(\alpha, \beta) = D_\beta P_n D_{\alpha\beta^{-1}}.$$

Note that $P_n = P_n(1, 1)$.

We close this section with the following simple property of $P_n(\alpha, \beta)$.

Theorem 1. *If* $\alpha, \beta \neq 0$, *then* $P_n(\alpha, \beta)$ *is invertible and* $P_n(\alpha, \beta)^{-1} = P_n(1/\alpha, -\beta/\alpha)$.

Proof. That $P_n(\alpha, \beta)$ is invertible for $\alpha, \beta \neq 0$ is trivial.

Since $P_n^{-1} = [(-1)^{i+j}\binom{i}{j}]$, we have $P_n^{-1} = D_{-1} P_n D_{-1}$. Thus it follows that

$$\begin{aligned} P_n(\alpha, \beta)^{-1} &= D_{\alpha\beta^{-1}}^{-1} P_n^{-1} D_\beta^{-1} = D_{\beta\alpha^{-1}} D_{-1} P_n D_{-1} D_{\beta^{-1}} \\ &= D_{-\beta\alpha^{-1}} P_n D_{-\beta^{-1}} = P_n\left(\frac{1}{\alpha}, -\frac{\beta}{\alpha}\right). \qquad \square \end{aligned}$$

2. The structure of 7-matrices

In this section, we determine the structure of 7-matrices. 7-Matrices are strongly related to Pascal matrices. Let $V_n(\alpha, \beta)$ denote the set of all $(n+1) \times (n+1) 7_{\alpha,\beta}$-matrices. Observe that the matrices in $V_n(\alpha, \beta)$ are completely determined by their first row and their first column and the recurrence relation.

In the sequel, we assume that the rows and columns of matrices in $V_n(\alpha, \beta)$ are indexed by $0, 1, \ldots, n$, and we let $\mathbf{e}_i$ $(i = 1, \ldots, n)$, denote the ith column of I_n, the identity matrix of order n, and let $\mathbf{f}_i$ $(i = 0, 1, \ldots, n)$, denote the $(i + 1)$th column of I_{n+1} so that $[\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_n] = I_{n+1}$.

Theorem 2. *For $n \geqslant 2$, $V_n(\alpha, \beta)$ is a $(2n + 1)$-dimensional vector space.*

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}] \in V_n(\alpha, \beta)$ and let λ, μ be real numbers. Let $C = \lambda A + \mu B = [c_{ij}]$. Then the entries c_{ij} of C certainly satisfy the $7_{\alpha,\beta}$-law because both a_{ij}'s and b_{ij}'s do, so that $V_n(\alpha, \beta)$ is a subspace of the vector space of all $(n + 1) \times (n + 1)$ matrices.

For $i, j = 1, 2, \ldots, n$, let B_{i0} and B_{0j} denote the $(n + 1) \times (n + 1)$ $7_{\alpha,\beta}$-matrices of the form

$$B_{i0} = \begin{bmatrix} 0 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{e}_i & X_{i0} \end{bmatrix}, \quad B_{0j} = \begin{bmatrix} 0 & \mathbf{e}_j^{\mathrm{T}} \\ \mathbf{0} & X_{0j} \end{bmatrix},$$

and let

$$B_{00} = \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & X_{00} \end{bmatrix} \in V_n(\alpha, \beta),$$

where each of the matrices X_{i0}, X_{0j} and X_{00} is an $n \times n$ matrix determined by the row 0 and the column 0 of B_{i0}, B_{0j} and B_{00} respectively and the recurrence relation. It is now readily seen that the $2n + 1$ matrices $B_{n0}, B_{n-1,0}, \ldots, B_{10}, B_{00}, B_{01}, B_{02}, \ldots, B_{0n}$ form a basis for $V_n(\alpha, \beta)$. $\square$

The notion of 7-matrices can be extended to rectangular matrices with infinite number of rows or columns. By the same argument as above we can show that the set of all $7_{\alpha,\beta}$-matrices of the same size, infinite or finite, forms a vector space. If the size is $m \times n$, then the dimension of the vector space is $m + n - 1$.

For a row vector $\mathbf{v} = (v_1, v_2, \ldots, v_k)$, let $\overleftarrow{\mathbf{v}}$ denote the column vector obtained from $\mathbf{v}$ by reading the components in reverse order, namely, $\overleftarrow{\mathbf{v}} = (v_k, v_{k-1}, \ldots, v_1)^{\mathrm{T}}$.

Lemma 3. *Let $A \in V_n(\alpha, \beta)$ and let $\mathbf{x}$ and $\mathbf{y}$ be the topmost row and the rightmost column of A respectively. Then $P_n(\alpha, \beta)\overleftarrow{\mathbf{x}} = \mathbf{y}$.*

Proof. Let $A = [a_{ij}]$. Then $\mathbf{x} = (a_{00}, a_{01}, \ldots, a_{0n})$. Let B_{i0}, B_{0j} $(i, j = 0, 1, \ldots, n)$, be the basis for $V_n(\alpha, \beta)$ defined in the proof of Theorem 2 and let $Q_n = P_n(\alpha, \beta)$. Then $B_{i0}\mathbf{f}_n = \mathbf{0}$ $(i = 1, 2, \ldots, n)$. Notice that, for each $j = 1, 2, \ldots, n$, $B_{0j} = [O_j, Q_{n,j}]$, where O_j denotes the $(n + 1) \times j$ zero matrix and $Q_{n,j}$ denotes the $(n + 1) \times (n + 1 - j)$ matrix obtained from Q_n by deleting all columns except those numbered $0, 1, \ldots, n - j$, and hence that the column n of B_{0j} equals the column $n - j$ of Q_n, i.e., that $B_{0j}\mathbf{f}_n = Q_n\mathbf{f}_{n-j}$ $(j = 0, 1, 2, \ldots, n)$. Since $A = \sum_{i=1}^{n} a_{i0}B_{i0} + \sum_{j=0}^{n} a_{0j}B_{0j}$, we have

$$\mathbf{y} = A\mathbf{f}_n = \sum_{j=0}^{n} a_{0j} B_{0j} \mathbf{f}_n = \sum_{j=0}^{n} a_{0j} Q_n \mathbf{f}_{n-j} = Q_n \sum_{j=0}^{n} a_{0,n-j} \mathbf{f}_j$$
$$= Q_n (a_{0,n}, a_{0,n-1}, \ldots, a_{0,1}, a_{0,0})^{\mathrm{T}} = Q_n \overleftarrow{\mathbf{x}}. \qquad \square$$

For the row vector $\mathbf{c} = (c_{-n}, c_{-n+1}, \ldots, c_{-1}, c_0, c_1, c_2, \ldots, c_n)$ with $2n+1$ components, let $T(\mathbf{c})$ denote the $(n+1) \times (n+1)$ Toeplitz matrix

$$\begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} & c_n \\ c_{-1} & c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{-2} & c_{-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-n+1} & c_{-n+2} & c_{-n+3} & \cdots & c_0 & c_1 \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & c_{-1} & c_0 \end{bmatrix}.$$

Associated with the above row vector $\mathbf{c}$, let the $(n+1)$-vectors $\mathbf{c}_k$ $(k = 0, 1, 2, \ldots, n)$, be defined by

$$\mathbf{c}_k = (c_{k-n}, c_{k-n+1}, \ldots, c_{k-1}, c_k). \tag{3}$$

Then $T(\mathbf{c}) = [\overleftarrow{\mathbf{c}}_0, \overleftarrow{\mathbf{c}}_1, \ldots, \overleftarrow{\mathbf{c}}_n]$.

Theorem 4. *An $(n+1) \times (n+1)$ matrix A is a $7_{\alpha,\beta}$-matrix if and only if $A = P_n(\alpha, \beta)T$ for some $(n+1) \times (n+1)$ Toeplitz matrix T.*

Proof. Suppose that $A = [a_{ij}] = [\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n]$ is an $(n+1)$-square $7_{\alpha,\beta}$-matrix. Extend A to an $(n+1) \times (2n+1)$ matrix

$$\tilde{A} = \begin{bmatrix} a_{0,-n} & a_{0,-n+1} & \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots & a_{0,n} \\ a_{1,-n} & a_{1,-n+1} & \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,-n} & a_{n,-n+1} & \cdots & a_{n,-1} & a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{bmatrix},$$

where the entries a_{ij} with $0 \leqslant i \leqslant n-1$, $-(n-i) \leqslant j \leqslant -1$ are chosen in such a way that

$$a_{ij} = \frac{a_{i+1,j+1} - \beta a_{i,j+1}}{\alpha},$$

while those a_{ij} with $1 \leqslant i \leqslant n$, $-n \leqslant j < -(n-i)$ are taken so that the entries of $\tilde{A}$ satisfy the $7_{\alpha,\beta}$-law. Let $\mathbf{c} = (c_{-n}, c_{-n+1}, \ldots, c_{-1}, c_0, c_1, c_2, \ldots, c_n)$ be the topmost row of $\tilde{A}$. Then $\mathbf{c}$ is uniquely determined by the topmost row and leftmost column of A.

For each $k = 0, 1, \ldots, n$, let $\tilde{A}_k$ denote the $(n+1) \times (n+1)$ matrix obtained from $\tilde{A}$ by deleting all columns except those numbered $k-n, k-n+1, \ldots,$

$k-1, k$, and let $\mathbf{c}_k$ be the row vector defined in (3) associated with $\mathbf{c}$. Then $\mathbf{c}_k$ is the topmost row of $\tilde{A}_k$ so that $\mathbf{a}_k = Q_n \overleftarrow{\mathbf{c}}_k$, $k = 0, 1, \ldots, n$, by Lemma 3, where $Q_n = P_n(\alpha, \beta)$. Thus $A = [Q_n \overleftarrow{\mathbf{c}}_0, Q_n \overleftarrow{\mathbf{c}}_1, \ldots, Q_n \overleftarrow{\mathbf{c}}_n] = Q_n T(\mathbf{c})$.

To prove the converse, let $A = [a_{ij}]$ be an $(n+1) \times (n+1)$ matrix where $0 \leqslant i, j \leqslant n$, and suppose that $A = P_n(\alpha, \beta)T$, where T is an $(n+1) \times (n+1)$ Toeplitz matrix so that there exists an $(2n+1)$-vector $\mathbf{c} = (c_{-n}, c_{-n+1}, \ldots, c_{-1}, c_0, c_1, c_2, \ldots, c_n)$ such that $T = T(\mathbf{c})$. Let $Q_n = P_n(\alpha, \beta) = [q_{ij}]$, $(0 \leqslant i, j \leqslant n)$, and $\mathbf{c}_r = (c_{r-n}, c_{r-n+1}, \ldots, c_{r-1}, c_r)$, $(r = 0, 1, \ldots, n)$. Then $T = [\overleftarrow{\mathbf{c}}_0, \overleftarrow{\mathbf{c}}_1, \ldots, \overleftarrow{\mathbf{c}}_n]$. Since $T\mathbf{f}_r = \overleftarrow{\mathbf{c}}_r$ for each $r = 0, 1, \ldots, n$, we see, for i, j with $0 \leqslant i, j \leqslant n-1$, that

$$\begin{aligned}
\alpha a_{ij} + \beta a_{i,j+1} &= \alpha \mathbf{f}_i Q_n T \mathbf{f}_j + \beta \mathbf{f}_i Q_n T \mathbf{f}_{j+1} \\
&= \sum_{k=0}^{i} \alpha q_{ik} c_{j-k} + \sum_{k=0}^{i} \beta q_{ik} c_{j+1-k} \\
&= \beta q_{i0} c_{j+1} + \sum_{k=0}^{i-1} (\alpha q_{ik} + \beta q_{i,k+1}) c_{j-k} + \alpha q_{ii} c_{j-i} \\
&= q_{i+1,0} c_{j+1} + \sum_{k=0}^{i-1} q_{i+1,k+1} c_{j+1-(k+1)} + q_{i+1,i+1} c_{j+1-(i+1)} \\
&= \mathbf{f}_{i+1} Q_n T \mathbf{f}_{j+1} = a_{i+1,j+1},
\end{aligned}$$

since $\beta q_{i0} = q_{i+1,0}$ and $\alpha q_{ii} = q_{i+1,i+1}$. Thus it is proved that A is a $7_{\alpha,\beta}$-matrix. □

Corollary. *Let A be an $(n+1) \times (n+1) 7_{\alpha,\beta}$-matrix and let T be a Toeplitz matrix such that $A = P_n(\alpha, \beta)T$. Then A is invertible if and only if T is. If T is invertible, then*

$$A^{-1} = T^{-1} P_n\left(\frac{1}{\alpha}, -\frac{\beta}{\alpha}\right).$$

3. The sequences of columns of 7-matrices

In this section we observe the relationship between the sequences $\overleftarrow{\mathbf{x}}$ and $\mathbf{y}$ where $\mathbf{x}, \mathbf{y}$ are the topmost row and the rightmost column of an infinite $7_{\alpha,\beta}$-matrix

$$A = \begin{bmatrix} \cdots & a_{0,-2} & a_{0,-1} & a_{0,0} \\ \cdots & a_{1,-2} & a_{1,-1} & a_{1,0} \\ \cdots & a_{2,-2} & a_{2,-1} & a_{2,0} \\ & \vdots & \vdots & \vdots \end{bmatrix}. \tag{4}$$

Theorem 5. *Let A be an infinite* $7_{\alpha,\beta}$*-matrix in* (4) *and let* $\mathbf{x}$, $\mathbf{y}$ *be the topmost row and the rightmost column of A. If* $f(x)$, $g(x)$ *are the generating functions of the sequences* $\overleftarrow{\mathbf{x}}$ *and* $\mathbf{y}$ *respectively, then*

(a) $$g(x) = \frac{1}{1-\beta x} f\Big(\frac{\alpha x}{1-\beta x}\Big),$$

(b) $$f(x) = \frac{1}{1+\beta x/\alpha} g\left(\frac{x/\alpha}{1+\beta x/\alpha}\right).$$

Proof. (a) Let $q_{nk} = \alpha^k \beta^{n-k}\binom{n}{k}$, $(n, k = 0, 1, 2, \ldots)$. For each $k = 0, 1, 2, \ldots,$ we have

$$\frac{(\alpha x)^k}{(1-\beta x)^{k+1}} = \sum_{j=0}^{\infty} \alpha^k x^k \binom{k+j}{j} \beta^j x^j = \sum_{j=0}^{\infty} \alpha^k \beta^j \binom{k+j}{k} x^{k+j}$$
$$= \sum_{n=k}^{\infty} \alpha^k \beta^{n-k} \binom{n}{k} x^n = \sum_{n=0}^{\infty} \alpha^k \beta^{n-k} \binom{n}{k} x^n = \sum_{n=0}^{\infty} q_{nk} x^n.$$

Since $q_{nk} = 0$ for $k > n$, we also have

$$a_{n0} = (q_{n0}, q_{n1}, \ldots, q_{nn})(a_{0,0}, a_{0,-1}, \ldots, a_{0,-n})^{\mathrm{T}} = \sum_{k=0}^{\infty} q_{nk} a_{0,-k}.$$

Thus we get that

$$\frac{1}{1-\beta x} f\left(\frac{\alpha x}{1-\beta x}\right) = \frac{1}{1-\beta x} \sum_{k=0}^{\infty} a_{0,-k} \left(\frac{\alpha x}{1-\beta x}\right)^k$$
$$= \sum_{k=0}^{\infty} \frac{a_{0,-k}(\alpha x)^k}{(1-\beta x)^{k+1}} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} q_{nk} a_{0,-k} x^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q_{nk} a_{0,-k} x^n = \sum_{n=0}^{\infty} a_{n0} x^n = g(x),$$

and the proof of (a) is complete.

(b) The matrix

$$\tilde{A} = \begin{bmatrix} \cdots & a_{2,0} & a_{1,0} & a_{0,0} \\ \cdots & a_{2,-1} & a_{1,-1} & a_{0,-1} \\ \cdots & a_{2,-2} & a_{1,-2} & a_{0,-2} \\ & \vdots & \vdots & \vdots \end{bmatrix}$$

obtained form A by flipping along the back diagonal is a $7_{\lambda,\mu}$-matrix with $\lambda = 1/\alpha$, $\mu = -\beta/\alpha$. Thus (b) follows from (a). $\square$

As an example of Theorem 4, let us look at the following infinite matrix

$$A = \begin{bmatrix} \cdots & 0 & \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \cdots & 0 & \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \cdots & 0 & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \cdots & 0 & \binom{p}{0} & \binom{p}{1} & \binom{p}{2} & \cdots & \binom{p}{p} \\ \cdots & 0 & \binom{p+1}{0} & \binom{p+1}{1} & \binom{p+1}{2} & \cdots & \binom{p+1}{p} \\ & \vdots & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}.$$

A is a simple infinite 7-matrix. The generating function of the sequence obtained by reading the topmost row backward is $f(x) = x^p$. We can calculate

$$\frac{1}{1-x} f\left(\frac{x}{1-x}\right) = \frac{x^p}{(1-x)^{p+1}} = x^p \sum_{k=0}^{\infty} \binom{p+k}{p} x^k = \sum_{n=0}^{\infty} \binom{n}{p} x^n,$$

which, indeed, is the generating function of a sequence appeared in the rightmost column of A.

If $g(x)$ is the generating function of a sequence, then certainly $g(x)/(1-x)$ is the generating function of the partial sums of the sequence. Now

$$\begin{aligned} \frac{1}{1-x} \frac{1}{1-x} f\left(\frac{x}{1-x}\right) &= \frac{x^p}{(1-x)^{p+2}} = x^p \sum_{k=0}^{\infty} \binom{p+k+1}{p+1} x^k \\ &= \sum_{n=0}^{\infty} \binom{n+1}{p+1} x^n \end{aligned}$$

gives rise to the well known identity

$$\binom{p}{p} + \binom{p+1}{p} + \cdots + \binom{n}{p} = \binom{n+1}{p+1}.$$

As another example, let

$$A = \begin{bmatrix} \cdots & 125 & 64 & 27 & 8 & 1 & 0 \\ \cdots & 91 & 61 & 37 & 19 & 7 & 1 \\ \cdots & 36 & 30 & 24 & 18 & 12 & 6 \\ \cdots & 6 & 6 & 6 & 6 & 6 & 6 \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where the topmost row read backward is the sequence $(0^3, 1^3, 2^3, \ldots)$ and the row i read backward is the difference sequence of the sequence in row $i-1$ read backward $(i = 1, 2, \ldots)$. Then A is a $7_{1,-1}$-matrix. Since $g(x) = x + 6x^2 + 6x^3$, we have the generating function

$$\begin{aligned} f(x) &= \frac{1}{1-x} f\Big(\frac{x}{1-x}\Big) \\ &= \frac{1}{1-x}\Big(\frac{x}{1-x} + \frac{6x^2}{(1-x)^2} + \frac{6x^3}{(1-x)^3}\Big) \\ &= \frac{x + 4x^2 + x^3}{(1-x)^4} \end{aligned}$$

of the sequence $(0^3, 1^3, 2^3, \ldots)$.

4. Some applications

Let us consider an infinite simple 7-matrix, i.e., an infinite $7_{\alpha,\beta}$-matrix with $\alpha = \beta = 1$, of the form

$$A = \begin{bmatrix} \cdots & a_{0,-2} & a_{0,-1} & a_{0,0} \\ \cdots & a_{1,-2} & a_{1,-1} & a_{1,0} \\ \cdots & a_{2,-2} & a_{2,-1} & a_{2,0} \\ & \vdots & \vdots & \vdots \end{bmatrix}.$$

Since, for a fixed j, $a_{i,-j-1} + a_{i,-j} = a_{i+1,-j}$, i.e, $a_{i+1,-j} - a_{i,-j} = a_{i,-j-1}$ $(i = 0, 1, \ldots)$, we see that for each $j = 0, 1, 2, \ldots$, the column $-j-1$ is the difference sequence of the column $-j$ so that it is the jth difference sequence of the sequence in the column 0. Let $b_j = a_{0,-j}$ $(j = 0, 1, 2, \ldots)$, for brevity, and let $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots$. Then

$$\begin{aligned} \frac{1}{1-x} f\Big(\frac{x}{1-x}\Big) &= \frac{b_0}{1-x} + \frac{b_1 x}{(1-x)^2} + \frac{b_2 x^2}{(1-x)^3} + \cdots \\ &= b_0 \sum_{n=0}^{\infty} x^n + b_1 \sum_{n=0}^{\infty} \binom{n}{1} x^n + b_2 \sum_{n=0}^{\infty} \binom{n}{2} x^n + \cdots \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} b_j \binom{n}{j} x^n. \end{aligned}$$

So, by Theorem 5(a), we see that the general term of the sequence in the rightmost column of A can be obtained from the sequence in the topmost row of A read backward as

$$a_{n,0}=\sum_{j=0}^{n}a_{0,-j}\binom{n}{j}\quad (n=0,1,2,\ldots). \tag{5}$$

Now, let s_i be the ith partial sum of the sequence in the rightmost column of A, i.e., $s_i=a_{00}+a_{10}+\cdots+a_{i0}$, $(i=0,1,2,\ldots)$. Then the generating function of the sequence $s_0,s_1,s_2,\ldots$ is

$$\begin{aligned}&\frac{1}{1-x}\frac{1}{1-x}f\left(\frac{x}{1-x}\right)\\&\quad=\frac{b_0}{(1-x)^2}+\frac{b_1x}{(1-x)^3}+\frac{b_2x^2}{(1-x)^4}+\cdots\\&\quad=b_0\sum_{n=0}^{\infty}\binom{n+1}{1}x^n+b_1\sum_{n=0}^{\infty}\binom{n+1}{2}x^n+b_2\sum_{n=0}^{\infty}\binom{n+2}{3}x^n+\cdots\\&\quad=\sum_{n=0}^{\infty}\sum_{j=0}^{n}b_j\binom{n+1}{j+1}x^n.\end{aligned}$$

Therefore

$$s_n=\sum_{j=0}^{n}a_{0,-j}\binom{n+1}{j+1}\quad (n=0,1,2,\ldots). \tag{6}$$

The formulas (5) and (6) with $(a_{0,0},a_{0,-1},a_{0,-2},\ldots)$ being a finite sequence are well known in the literature (see [1], for example).

For a pair of nonnegative integers n, k with $n\geqslant 0$, $k\geqslant 1$, let $r_{n,k}$ denote the number of regions that result from a k-dimensional Euclidean space by putting n $(k-1)$-dimensional hyperplanes in it. Then, clearly,

$$\begin{aligned}&r_{0k}=1\quad (k=1,2,\ldots),\\&r_{n1}=n+1\quad (n=0,1,2,\ldots).\end{aligned}$$

It is well known that the numbers r_{ij} satisfy the 7-law

$$r_{ij}+r_{i,j+1}=r_{i+1,j+1}\quad (i=0,1,2,\ldots;\ \ j=1,2,\ldots)$$

(see [1], for example) so that the matrix

$$A=\begin{bmatrix}r_{01}&r_{02}&r_{03}&\cdots&r_{0p}\\r_{11}&r_{12}&r_{12}&\cdots&r_{1p}\\r_{21}&r_{22}&r_{23}&\cdots&r_{2p}\\\vdots&\vdots&\vdots&&\vdots\end{bmatrix}$$

is a simple 7-matrix where p is any fixed positive integer. The matrix A can be extended to an infinite simple 7-matrix

$$
\begin{aligned}
\tilde{A} &= \begin{bmatrix} \cdots & 0 & 0 & r_{00} & r_{01} & r_{02} & \cdots & r_{0p} \\ \cdots & 0 & 0 & r_{10} & r_{11} & r_{12} & \cdots & r_{1p} \\ \cdots & 0 & 0 & r_{20} & r_{21} & r_{22} & \cdots & r_{2p} \\ & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \end{bmatrix} \\
&= \begin{bmatrix} \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\ \cdots & 0 & 0 & 1 & 2 & * & \cdots & * \\ \cdots & 0 & 0 & 1 & 3 & * & \cdots & * \\ & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}.
\end{aligned}
$$

The generating function of the sequence obtained from the topmost row read backward is

$$f(x) = 1 + x + x^2 + \cdots + x^p.$$

So, by Theorem 5 again, the generating function of the sequence $(r_{0p}, r_{1p}, r_{2p}, \ldots)$ is calculated as

$$
\begin{aligned}
\frac{1}{1-x} f\left(\frac{x}{1-x}\right) &= \frac{1}{1-x} + \frac{x}{(1-x)^2} + \cdots + \frac{x^p}{(1-x)^{p+1}} \\
&= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \binom{n}{1} x^n + \cdots + \sum_{n=0}^{\infty} \binom{n}{p} x^n \\
&= \sum_{n=0}^{\infty} \left(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p} \right) x^n,
\end{aligned}
$$

from which the well known identity

$$r_{np} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{p}$$

follows.

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