

# RECURSION RULES FOR THE HYPERGEOMETRIC ZETA FUNCTION

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ABSTRACT. The hypergeometric zeta function is defined in terms of the zeros of the Kummer function  $M(a, a + b; z)$ . It is established that this function is an entire function of order 1. The classical factorization theorem of Hadamard gives an expression as an infinite product. This provides linear and quadratic recurrences for the hypergeometric zeta function. A family of associated polynomials is characterized as Appell polynomials and the underlying distribution is given explicitly in terms of the zeros of the associated hypergeometric function. These properties are also given a probabilistic interpretation in the framework of Beta distributions.

## 1. INTRODUCTION

The zeta function attached to a collection of non-zero complex numbers  $\mathbb{A} = \{a_n \neq 0 : n \in \mathbb{N}\}$ , is defined by

$$(1.1) \quad \zeta_{\mathbb{A}}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s}, \text{ for } \operatorname{Re} s > c.$$

The most common choice of sequences  $\mathbb{A}$  includes those coming from the zeros of a given function  $f$ :

$$(1.2) \quad \mathbb{A}(f) = \{z \in \mathbb{C} : f(z) = 0\} = \{z_n \in \mathbb{C} : f(z_n) = 0, \quad n \in \mathbb{N}\},$$

to produce the associated zeta function

$$(1.3) \quad \zeta_f(s) = \sum_{n=1}^{\infty} \frac{1}{z_n^s}.$$

The prototypical example is the classical Riemann zeta function

$$(1.4) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

coming from (half of) the zeros  $\mathbb{A} = \{z_n = n > 0\}$  of the function  $f(z) = \frac{\sin \pi z}{\pi z}$ .

The literature contains a variety of zeta functions  $\zeta_{\mathbb{A}}$  and their study is concentrated in reproducing the basic properties of (1.4). For example  $\zeta(s)$ , originally

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defined in the half-plane  $\operatorname{Re} s > 1$ , admits a meromorphic extension to the complex plane, with a single pole at  $s = 1$ . Moreover, the function  $\zeta(s)$  admits special values

$$(1.5) \quad \zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}$$

where the Bernoulli numbers  $B_n$  are defined by the generating function

$$(1.6) \quad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi.$$

Carlitz introduced in [5] coefficients  $\beta_n$  by

$$(1.7) \quad \frac{x^2}{e^x - 1 - x} = \sum_{n=0}^{\infty} \beta_n \frac{x^n}{n!}$$

and stated that *nothing is known about them*. Howard [15] used the notation  $A_s = \frac{1}{2}\beta_s$ , and in [16] he introduced the generalization  $A_{k,r}$  by

$$(1.8) \quad \frac{x^k}{k!} \left( e^x - \sum_{s=0}^{k-1} \frac{x^s}{s!} \right)^{-1} = \sum_{r=0}^{\infty} A_{k,r} \frac{x^r}{r!}.$$

These numbers satisfy the recurrence

$$(1.9) \quad \sum_{r=0}^n \binom{n+k}{r} A_{k,r} = 0, \quad \text{for } n > 0$$

with  $A_{k,0} = 1$ . It follows that  $A_{k,n}$  is a rational number and some of their arithmetical properties are reviewed in Section 5.

The work presented here considers a zeta function constructed in terms of the *Kummer function*

$$(1.10) \quad M(a, b; z) = {}_1F_1 \left( \begin{matrix} a \\ b \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}.$$

The next definition introduces the main function considered here. The notation

$$(1.11) \quad \Phi_{a,b}(z) = {}_1F_1 \left( \begin{matrix} a \\ a+b \end{matrix} \middle| z \right) = M(a, a+b; z), \quad \text{for } a, b \in \mathbb{R}$$

is employed throughout.

**Definition 1.1.** Let  $a, b$  be positive real numbers. The *hypergeometric zeta function* is defined by

$$(1.12) \quad \zeta_{a,b}^H(s) = \sum_{k=1}^{\infty} \frac{1}{z_{k;a,b}^s} \quad \text{for } \operatorname{Re} s > 1,$$

where  $z_{k;a,b}$  is the sequence of complex zeros of the function  $\Phi_{a,b}(z)$ .

The special case  $\Phi_{1,1}(z) = (e^z - 1)/z$  is the reciprocal of the generating function for the Bernoulli numbers (1.6). The coefficients  $B_n^{(b)}$  are defined by

$$(1.13) \quad \frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}.$$

In the case  $b \in \mathbb{N}$ , these numbers are the coefficients  $A_{k,r}$  defined by Howard in (1.8) (with  $k = b$  and  $r = n$ ). These numbers are discussed in Section 4. The

function  $\Phi_{1,2}(z)$  recently appeared in [7] in the asymptotic expansion of  $n!$ . Indeed, it can be shown that the coefficients  $a_k$  in the expansion

$$(1.14) \quad n! \sim \frac{n^n \sqrt{2\pi n}}{e^n} \sum_{k=0}^{\infty} \frac{a_k}{n^k} \text{ as } n \rightarrow \infty,$$

are given by

$$(1.15) \quad a_k = \frac{1}{2^k k!} \left( \frac{d}{dz} \right)^{2k} \Phi_{1,2}^{-(k+1/2)} \Big|_{z=0}.$$

K. Dilcher [10, 9] considered the zeta function  $\zeta_{a,b}^H$ . In particular, he established an expression for  $\zeta_{a,b}^H(m)$ , for  $a, b, m \in \mathbb{N}$ , in terms of the hypergeometric Bernoulli numbers  $B_{a,b}^n$  introduced in Section 7.

**Note 1.2.** The many examples of zeta functions discussed in the literature include the *Bessel zeta function*

$$(1.16) \quad \zeta_{Bes,a}(s) = \sum_{n=1}^{\infty} \frac{1}{j_{a,n}^s}$$

where  $\{j_{a,n}\}$  are the zeros of  $J_a(z)/z^a$ , with  $J_a(z)$  the Bessel function of the first kind

$$(1.17) \quad J_a(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+a+1)} \left( \frac{z}{2} \right)^{2m+a}.$$

Papers considering  $\zeta_{Bes,a}$  include [2, 11, 14, 21]. A second example is the *Airy-zeta function*, defined by

$$(1.18) \quad \zeta_{Ai}(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s},$$

where  $\{a_n\}$  are the zeros of the Airy function

$$(1.19) \quad Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3}t^3 + xt\right) dt.$$

This is considered by R. Crandall [6] in the so-called *quantum bouncer*. Special values include the remarkable

$$(1.20) \quad \zeta_{Ai}(2) = \frac{3^{5/3}}{4\pi^2} \Gamma^4\left(\frac{2}{3}\right).$$

A third example is the zeta function studied by A. Hassen and H. Nguyen [12, 13]. This is defined by the integral

$$(1.21) \quad \zeta_{HN,b}(s) = \frac{1}{\Gamma(s+b-1)} \int_0^{\infty} \frac{x^{s+b-2} dx}{e^x - 1 - z - z^2/2! - \dots - z^{b-1}/(b-1)!}.$$

The main results presented here include a relation among the Kummer function  $\Phi_{a,b}(z)$  and the hypergeometric zeta function  $\zeta_{a,b}^H$  defined in (1.12). This is

$$(1.22) \quad \frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k$$

and it appears in Proposition 3.1. It is shown that the function  $\zeta_{a,b}^H$  satisfies a couple of linear recurrence relations:

$$(1.23) \quad \sum_{\ell=1}^p B(a+p-\ell, b) \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1) = -\frac{bp}{(a+b)(a+b+p)} B(a+p, b).$$

appearing in Theorem 3.2 and

$$(1.24) \quad (n-1)! \sum_{j=2}^n \frac{B_{n-j}^{(a,b)}}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B_{n-1}^{(a,b)} + B_n^{(a,b)},$$

established in Theorem 7.12, where  $B_n^{(a,b)}$  are the so-called *hypergeometric Bernoulli numbers* introduced in Section 7. The third recurrence is quadratic

$$(1.25) \quad \sum_{k=1}^p \zeta_{a,b}^H(k+1) \zeta_{a,b}^H(p-k+1) = (a+b+p+1) \zeta_{a,b}^H(p+2) + \left(\frac{a-b}{a+b}\right) \zeta_{a,b}^H(p+1).$$

This is given in Theorem 3.4.

Theorem 4.1 expresses the rational numbers  $B_n^{(b)} := B_n^{(1,b)}$  in terms of the values  $\zeta_{1,b}^H(n)$ . This extends the classical result for the Bernoulli numbers  $B_n = B_n^{(1,1)}$ . Section 5 states some conjectures on arithmetical properties of the denominators of  $B_n^{(b)}$  extending the von Staudt-Clausen theorem for Bernoulli numbers. Section 6 introduces a probabilistic technique to approach these questions and Section 7 discusses a family of polynomials introduced by K. Dilcher and proposes a natural generalization.

This work provides linear identities linking three types of functions: the classical beta function, the hypergeometric Bernoulli polynomials and the hypergeometric zeta function. Explicitly, Theorem 3.2 gives a linear recurrence involving the beta function and the hypergeometric zeta function, Theorem 7.8 gives a linear recurrence involving the beta function (written as binomial coefficients) and the hypergeometric Bernoulli polynomials and, finally, Theorem 7.12 gives a relation between the hypergeometric Bernoulli numbers and the hypergeometric zeta function.

**Notation.** It is an unfortunate fact that many of the terms used in the present work are denoted by the letter  $B$ . The list below shows the symbols employed here.

$B_n$	Bernoulli number	(1.6)
$M(a, b; z)$	Kummer function	(1.10)
$\Phi_{a,b}(z)$	Kummer function	(1.11)
$\zeta_{a,b}^H(s)$	hypergeometric zeta function	(1.12)
$B_n^{(b)}$	hypergeometric Bernoulli number	(1.13)
$B(a, b)$	the beta function	(3.6)
$\mathfrak{B}_{a,b}$	a beta distributed random variable	(6.2)
$\mathfrak{Z}_{a,b}$	a complex random variable	(6.8)
$B_n^{(a,b)}(x)$	hypergeometric Bernoulli polynomial	(7.8)
$B_n^{(a,b)}$	hypergeometric Bernoulli number	(7.9)

2. PROPERTIES OF THE KUMMER FUNCTION  $\Phi_{a,b}(z)$ .

The function

$$(2.1) \quad \Phi_{a,b}(z) = {}_1F_1 \left( \begin{matrix} a \\ a+b \end{matrix} \middle| z \right) = M(a, a+b; z), \text{ for } a, b \in \mathbb{R}$$

defined in terms of the Kummer function  $M(a, b; z)$  is the main object considered in the present work. The function  $M(a, b; z)$  satisfies the differential equation

$$(2.2) \quad z \frac{d^2 M}{dz^2} + (b-z) \frac{dM}{dz} - aM = 0,$$

obtained from the standard hypergeometric equation

$$(2.3) \quad z \frac{dw^2}{dz^2} + [c - (a+b+1)z] \frac{dw}{dz} - abw = 0$$

by scaling  $z \mapsto z/b$ , letting  $b \rightarrow \infty$  and replacing the parameter  $c$  by  $b$ .

The first result shows that the special case  $a = 1$  gives the function considered by Howard [16].

**Theorem 2.1.** For  $b \in \mathbb{N}$ , the function  $\Phi_{1,b}(z)$  is given by

$$(2.4) \quad \Phi_{1,b}(z) = \frac{b!}{z^b} \left( e^z - \sum_{k=0}^{b-1} \frac{z^k}{k!} \right).$$

*Proof.* This follows directly from the expansion

$${}_1F_1 \left( \begin{matrix} 1 \\ 1+b \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(1)_k z^k}{(1+b)_k k!} = \sum_{k=0}^{\infty} \frac{b!}{(b+k)!} z^k = \frac{b!}{z^b} \sum_{k=b}^{\infty} \frac{z^k}{k!}.$$

□

**Corollary 2.2.** The zeta function  $\zeta_{HA}(s)$  in (1.21) is given by

$$(2.5) \quad \zeta_{HA}(s) = \frac{b!}{\Gamma(s+b-1)} \int_0^{\infty} \frac{x^{s-2} dx}{\Phi_{1,b}(x)}.$$

The next property of  $\Phi_{a,b}(z)$  is a representation as an infinite product. The result comes from the classical Hadamard factorization theorem for entire functions. A preliminary lemma is given first.

**Lemma 2.3.** The Kummer function  $\Phi_{a,b}(z)$  satisfies

$$(2.6) \quad \frac{d}{dz} \Phi_{a,b}(z) = \frac{a}{a+b} \Phi_{a+1,b}(z).$$

*Proof.* This comes directly from formula

$$(2.7) \quad \frac{d}{dz} {}_1F_1 \left( \begin{matrix} a \\ b \end{matrix} \middle| z \right) = \frac{a}{b} {}_1F_1 \left( \begin{matrix} a+1 \\ b+1 \end{matrix} \middle| z \right),$$

which is entry 13.3.15 on page 325 of [19].

□

The next step is to analyze the factorization of the function  $\Phi_{a,b}(z)$ . Recall that the *order* of an entire function  $h(z)$  is defined as the infimum of  $\alpha \in \mathbb{R}$  for which there exists a radius  $r_0 > 0$  such that

$$(2.8) \quad |h(z)| < e^{|z|^\alpha} \text{ for } |z| > r_0.$$

The order of  $h(z)$  is denoted by  $\rho(h)$ . See [4] for information on the order of entire functions. The main result used here is Hadamard's theorem stated below.

**Theorem 2.4.** For  $p \in \mathbb{N}$ , define the elementary factors

$$(2.9) \quad E_p(z) = \begin{cases} 1 - z & \text{if } p = 0, \\ (1 - z)\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right) & \text{otherwise.} \end{cases}$$

Assume  $h$  is an entire function of finite order  $\rho = \rho(h)$ . Let  $\{a_n\}$  be the collection of zeros of  $h$  repeated according to multiplicity. Then  $h$  admits the factorization

$$(2.10) \quad h(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right).$$

where  $g(z)$  is a polynomial of degree  $q \leq \rho$ ,  $p = \lfloor \rho \rfloor$  and  $m \geq 0$  is the order of the zero of  $h$  at the origin.

The next result establishes the order of  $\Phi_{a,b}(z)$ .

**Theorem 2.5.** Let  $a, b > 0$ . Then  $\Phi_{a,b}$  is an entire function of order 1.

*Proof.* The ratio test shows that the function  $\Phi_{a,b}(z)$  is entire. Moreover

$$(2.11) \quad \frac{(a)_\ell}{(a+b)_\ell} = \prod_{k=0}^{\ell-1} \frac{a+k}{a+b+k} < 1,$$

therefore

$$(2.12) \quad |\Phi_{a,b}(z)| \leq \sum_{\ell=0}^{\infty} \frac{(a)_\ell}{(a+b)_\ell} \left| \frac{z^\ell}{\ell!} \right| \leq \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} = e^{|z|}.$$

This proves  $\rho(h) \leq 1$ .

To establish the opposite inequality, use the asymptotic behavior

$$(2.13) \quad M(a, b; z) \sim \frac{e^{-z} z^{a-b}}{\Gamma(a)}, \text{ as } z \rightarrow \infty$$

(see [19], page 323) to see that, for every  $0 \leq \varepsilon < 1$  and  $z \in \mathbb{R}$ ,

$$(2.14) \quad \lim_{z \rightarrow \infty} \frac{\Phi_{a,b}(z)}{\exp(z^\varepsilon)} = +\infty.$$

Hence, for any given  $r_0 > 0$ , there is  $r > r_0$  such that

$$(2.15) \quad |\Phi_{a,b}(r)| = \Phi_{a,b}(r) > \exp(r^\varepsilon) = \exp(|r|^\varepsilon).$$

This proves  $\rho(h) \geq 1$  and the proof is complete.  $\square$

The factorization of  $\Phi_{a,b}(z)$  in terms of its zeros  $\{z_{a,b;k}\}$  is discussed next. Section 13.9 of [19] states that if  $a$  and  $b \neq 0, -1, -2, \dots$ , then  $\Phi_{a,b}(z)$  has infinitely many complex zeros. Moreover, if  $a, b \geq 0$ , then there are no real zeros. The growth of the large zeros of  $M(a, a+b; z)$  is given by

$$(2.16) \quad z_{a,b;n} = \pm(2n+a)\pi i + \ln\left(-\frac{\Gamma(a)}{\Gamma(b)}(\pm 2n\pi i)^{b-a}\right) + O(n^{-1} \ln n),$$

where  $n$  is a large positive integer, and the logarithm takes its principal value.

**Theorem 2.6.** The function  $\Phi_{a,b}(z)$  admits the factorization

$$(2.17) \quad \Phi_{a,b}(z) = e^{az/(a+b)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}.$$

*Proof.* Hadamard's theorem shows the existence of two constants  $A, B$  such that

$$(2.18) \quad \Phi_{a,b}(z) = e^{Az+B} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_{a,b;k}}\right) e^{z/z_{a,b;k}}.$$

Evaluating at  $z = 0$ , using  $\Phi_{a,b}(0) = 1$ , gives  $B = 0$ . To obtain the value of the parameter  $A$ , take the logarithmic derivative of (2.18) and use Lemma 2.3 to produce

$$(2.19) \quad \frac{a}{a+b} \frac{{}_1F_1\left(\begin{matrix} a+1 \\ a+b+1 \end{matrix} \middle| z\right)}{{}_1F_1\left(\begin{matrix} a \\ a+b \end{matrix} \middle| z\right)} = A + \sum_{k=0}^{\infty} \left[ \frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right].$$

Expanding both sides near  $z = 0$  gives

$$(2.20) \quad \frac{a}{a+b} + \frac{ab}{(a+b)^2(1+a+b)}z + \mathcal{O}(z^2) = A - \sum_{k=1}^{\infty} \frac{z}{z_{a,b;k}^2} + \mathcal{O}(z^2).$$

This gives  $A = a/(a+b)$ , completing the proof.  $\square$

**Corollary 2.7.** The hypergeometric zeta function has the special value

$$(2.21) \quad \zeta_{a,b}^H(2) = -\frac{ab}{(a+b)^2(1+a+b)}.$$

*Proof.* Compare the coefficients of  $z$  in (2.20).  $\square$

The next statement presents additional properties of the Kummer function which will be useful in the next section. It appears as entries 13.4.12 and 13.4.13 in [1].

**Lemma 2.8.** The Kummer function satisfies

$$(2.22) \quad \frac{d}{dz} \Phi_{a,b}(z) = -\frac{b}{a+b} \Phi_{a,b+1}(z) + \Phi_{a,b}(z)$$

and

$$(2.23) \quad \frac{d}{dz} \Phi_{a,b+1}(z) = \frac{a+b}{z} (\Phi_{a,b}(z) - \Phi_{a,b+1}(z)).$$

### 3. RECURRENCE FOR THE HYPERGEOMETRIC ZETA FUNCTION

This section describes some recurrences for the values  $\zeta_{a,b}^H(k)$ . The proofs are based on a relation between the Kummer function  $\Phi_{a,b}(z)$  and these values. It is the analog of standard result for the usual zeta function

$$(3.1) \quad \sum_{k=1}^{\infty} \zeta(k+1)z^k = -\gamma - \psi(1-z)$$

where  $\psi(z) = \frac{d}{dz} \log \Gamma(z)$  is the digamma function and  $\gamma$  is the Euler constant. This relation is obtained directly from the product representation of  $\Gamma(z)$ . See entry 6.3.14 in [1].

**Proposition 3.1.** The Kummer function  $\Phi_{a,b}(z)$  and the hypergeometric zeta function are related by

$$(3.2) \quad \frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = 1 + \frac{a+b}{b} \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1)z^k.$$

*Proof.* The relation (2.22) gives

$$(3.3) \quad \frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left[ \frac{\Phi'_{a,b}(z)}{\Phi_{a,b}(z)} - 1 \right].$$

The fraction on the right-hand side is the logarithmic derivative of the product in Theorem 2.6. This yields

$$(3.4) \quad \frac{\Phi_{a,b+1}(z)}{\Phi_{a,b}(z)} = -\frac{a+b}{b} \left( \frac{a}{a+b} + \sum_{k=0}^{\infty} \left[ \frac{1}{z_{a,b;k}} - \frac{1}{z_{a,b;k} - z} \right] - 1 \right).$$

To establish the result, use the expansion

$$(3.5) \quad \frac{1}{z_{a,b;k} - z} = \frac{1}{z_{a,b;k}} \frac{1}{1 - z/z_{a,b;k}}$$

and expand the last term as the sum of a geometric series. Since  $\min_k \{|z_{a,b;k}|\} \neq 0$ , this series has a positive radius of convergence.  $\square$

The next result gives a linear recurrence for the hypergeometric zeta function. This involves the beta function

$$(3.6) \quad B(u, v) = \int_0^1 x^{u-1} (1-x)^{v-1} dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

with values

$$(3.7) \quad B(u, v) = \frac{u+v}{uv} \binom{u+v}{u}^{-1} \text{ for } u, v \in \mathbb{N}.$$

**Theorem 3.2.** The hypergeometric zeta function satisfies the linear recurrence

$$(3.8) \quad \sum_{\ell=1}^p B(a+p-\ell, b) \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1) = -\frac{bp}{(a+b)(a+b+p)} B(a+p, b).$$

*Proof.* Proposition 3.1 gives

$$(3.9) \quad \Phi_{a,b+1}(z) = \Phi_{a,b}(z) + \frac{a+b}{b} \Phi_{a,b}(z) \sum_{k=1}^{\infty} \zeta_{a,b}^H(k+1) z^k$$

Matching the coefficient of  $z^p$  gives the identity

$$(3.10) \quad \frac{(a)_p}{(a+b+1)_p} = \frac{(a)_p}{(a+b)_p} + \frac{a+b}{b} \sum_{\ell=1}^p \frac{(a)_{p-\ell}}{(a+b)_{p-\ell}} \frac{p!}{(p-\ell)!} \zeta_{a,b}^H(\ell+1).$$

This simplifies to produce the result.  $\square$

**Note 3.3.** The recurrence (3.8) is written as

$$\zeta_{a,b}^H(p) = -\frac{bB(a+p-1, b)}{(a+b)(a+b+p-1)(p-2)!B(a, b)} - \sum_{r=1}^{p-2} \frac{B(a+r, b)}{B(a, b)r!} \zeta_{a,b}^H(p-r).$$



The initial condition given in (2.21) shows that, for  $p \in \mathbb{N}$ , the value  $\zeta_{a,b}^H(p)$  is a rational function of  $a, b$ . The first few values are

$$\begin{aligned}\zeta_{a,b}^H(2) &= -\frac{ab}{(a+b)^2(1+a+b)} \\ \zeta_{a,b}^H(3) &= \frac{ab(a-b)}{(a+b)^3(a+b+1)(a+b+2)} \\ \zeta_{a,b}^H(4) &= -\frac{abP_4(a,b)}{(a+b)^4(a+b+1)^2(a+b+2)(a+b+3)}\end{aligned}$$

where

$$P_4(a,b) = a^2 + a^3 - 4ab - 2a^2b + b^2 - 2ab^2 + b^3.$$

The authors were unable to discern the patterns in  $\zeta_{a,b}^H(p)$ .

The next result presents a different type of recurrence for  $\zeta_{a,b}^H(s)$ . It is the analogue of the classical identity

$$(3.11) \quad \left(n + \frac{1}{2}\right) \zeta(2n) = \sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k), \text{ for } n \geq 2.$$

See 25.6.16 in [19].

**Theorem 3.4.** The hypergeometric zeta function satisfies the quadratic recurrence

$$(3.12) \quad \sum_{k=1}^p \zeta_{a,b}^H(k+1)\zeta_{a,b}^H(p-k+1) = (a+b+p+1)\zeta_{a,b}^H(p+2) + \left(\frac{a-b}{a+b}\right)\zeta_{a,b}^H(p+1).$$

*Proof.* The function  $f(z) = \Phi_{a,b+1}(z)/\Phi_{a,b}(z)$  satisfies the differential equation

$$(3.13) \quad f'(z) = \frac{a+b}{z}(1-f(z)) - f(z) + \frac{b}{a+b}f^2(z).$$

This can be verified directly using the results of Lemma 2.8. Now use Theorem 3.1 to match the coefficients of  $z^p$  and produce

$$\begin{aligned}\frac{a+b}{b}(p+1)\zeta_{a,b}^H(p+2) &= -\frac{(a+b)^2}{b}\zeta_{a,b}^H(p+2) - \frac{a+b}{b}\zeta_{a,b}^H(p+1) \\ &\quad + \frac{a+b}{b}\sum_{k=1}^p \zeta_{a,b}^H(k+1)\zeta_{a,b}^H(p-k+1) + 2\zeta_{a,b}^H(p+1),\end{aligned}$$

which, after simplification, yields the result.  $\square$

**Note 3.5.** Matching the constant terms recovers the value of  $\zeta_{a,b}^H(2)$  in (2.21).

#### 4. THE HYPERGEOMETRIC BERNOULLI NUMBERS

This section considers properties of the *hypergeometric Bernoulli numbers*  $B_n^{(b)}$ , defined by the relation

$$(4.1) \quad \frac{1}{\Phi_{1,b}(z)} = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}.$$

These are precisely the numbers  $A_{b,n}$  studied by Howard [16]. This follows from Theorem 2.1 and (1.8). The special case  $b = 1$  corresponds to the Bernoulli numbers.

The next result appears in [16] in the case  $b = 2$ .

**Theorem 4.1.** Let  $b \in \mathbb{N}$ . The hypergeometric Bernoulli numbers  $B_n^{(b)}$  are expressed in terms of the hypergeometric zeta function as

$$(4.2) \quad B_n^{(b)} = \begin{cases} 1 & \text{for } n = 0 \\ -1/(1+b) & \text{for } n = 1 \\ -n! \zeta_{1,b}^H(n)/b & \text{for } n \geq 2. \end{cases}$$

*Proof.* The product representation of  $\Phi_{1,b}(z)$  given in Theorem 2.6 is

$$(4.3) \quad \Phi_{1,b}(z) = e^{z/(1+b)} \prod_{k=1}^{\infty} \left( 1 - \frac{z}{z_{1,b;k}} \right) e^{z/z_{1,b;k}}.$$

Logarithmic differentiation yields

$$\begin{aligned} \frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \left[ \frac{1}{z_{1,b;k}} - \frac{1}{z_{1,b;k} - z} \right] \\ &= \frac{1}{1+b} + \sum_{k=1}^{\infty} \frac{1}{z_{1,b;k}} \left( 1 - \sum_{\ell=0}^{\infty} \left( \frac{z}{z_{1,b;k}} \right)^{\ell} \right) \\ &= \frac{1}{1+b} - \sum_{\ell=1}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell+1). \end{aligned}$$

On the other hand, Theorem 2.1 gives

$$(4.4) \quad \Phi_{1,b}(z) = \frac{b!}{z^b} \left( e^z - \sum_{j=0}^{b-1} \frac{z^j}{j!} \right)$$

and logarithmic differentiation produces

$$(4.5) \quad \frac{\Phi'_{1,b}(z)}{\Phi_{1,b}(z)} = 1 + \frac{b}{z} \cdot \frac{1}{\Phi_{1,b}(z)} - \frac{b}{z}.$$

Therefore,

$$(4.6) \quad \frac{1}{\Phi_{1,b}(z)} = 1 - \frac{z}{1+b} - \frac{1}{b} \sum_{\ell=2}^{\infty} z^{\ell} \zeta_{1,b}^H(\ell) = \sum_{n=0}^{\infty} B_n^{(b)} \frac{z^n}{n!}.$$

The conclusion follows by comparing coefficients of powers of  $z$ .  $\square$

**Note 4.2.** The previous theorem suggests the definition

$$(4.7) \quad \zeta_{1,b}^H(1) = \frac{b}{1+b}.$$

## 5. SOME ARITHMETICAL CONJECTURES

Theorem 3.2 and the relation (4.2) show that the coefficients  $B_n^{(b)}$  are rational numbers. For example:

$$\begin{aligned} B_n^{(1)} &= \{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots\}, \\ B_n^{(2)} &= \{1, -\frac{1}{3}, \frac{1}{18}, \frac{1}{90}, -\frac{1}{270}, -\frac{5}{1134}, -\frac{1}{5670}, \frac{7}{2430}, \dots\}, \\ B_n^{(3)} &= \{1, -\frac{1}{4}, \frac{1}{40}, \frac{1}{160}, \frac{1}{5600}, -\frac{1}{896}, -\frac{13}{19200}, \frac{7}{76800}, \dots\}. \end{aligned}$$

The arithmetic properties of the Bernoulli numbers  $B_n = B_n^{(1)}$  are very intriguing. It is well-known that the Bernoulli numbers of odd index vanish (except

$B_1 = -1/2$ ), leaving  $B_{2n}$  for consideration. Write  $B_{2n} = N_{2n}/D_{2n}$  in reduced form. The arithmetic properties of  $D_{2n}$  include the von Staudt-Clausen theorem stated below. The numerators  $N_{2n}$  are much more difficult to analyze. Chapter 15 of [17] contains information about their relation to Wiles theorem (previously known as Fermat's last theorem).

**Theorem 5.1.** The denominator of the Bernoulli number  $B_{2n}$  is given by

$$(5.1) \quad D_{2n} = \prod_{(p-1)|2n} p.$$

In particular, the denominator of  $B_{2n}$  is even (actually always divisible by 6) and it is square-free. In the case of the hypergeometric Bernoulli numbers, computer experiments suggest an extension of these properties. Let

$$(5.2) \quad \mathfrak{D}(b) = \left\{ \text{denominator} \left( B_n^{(b)} \right) : n \geq 0 \right\}.$$

The examples

$$(5.3) \quad \begin{aligned} \mathfrak{D}(2) &= \{1, 3, 18, 90, 270, 1134, 5670, 2430, \dots\} \\ \mathfrak{D}(3) &= \{1, 4, 40, 160, 5600, 896, 19200, 76800, \dots\} \\ \mathfrak{D}(4) &= \{1, 5, 75, 875, 26250, 78750, 918750, 3093750, \dots\}, \end{aligned}$$

show that  $\mathfrak{D}(b)$  contains an initial segment of *odd* numbers.

**Conjecture 5.2.** Let  $\alpha(b)$  be the number of odd terms at the beginning of  $\mathfrak{D}(b)$ . Then

$$(5.4) \quad \alpha(b) = \begin{cases} \nu_2(b) + 1 & \text{if } b \not\equiv 0 \pmod{4} \\ 2^{\nu_2(b)} & \text{if } b \equiv 0 \pmod{4}. \end{cases}$$

The prime factorization of  $D_{2n} = \text{den}(B_{2n})$ , shows that if  $p$  is a prime dividing  $D_{2n}$ , then  $p \leq 2n + 1$ . Numerical evidence of the corresponding statement for  $B_n^{(b)}$  leads to the next conjecture.

**Conjecture 5.3.** Every prime  $p$  dividing the denominator of  $B_n^{(b)}$  satisfies  $p \leq n + b$ .

These two conjectures have been verified up to  $b = 1000$ .

**Note 5.4.** It is no longer true that the denominators are square-free. For example  $B_7^{(3)} = 7/76800$  and  $76800 = 2^{10} \cdot 3 \cdot 5^2$ .

## 6. A PROBABILISTIC APPROACH

This section presents an interpretation of the Kummer function  $\Phi_{a,b}(z)$  as the expectation of a complex random variable. For a random variable  $\mathfrak{R}$  with a continuous distribution function  $r(x)$ , the expectation operator is defined by

$$(6.1) \quad \mathbb{E}u(\mathfrak{R}) = \int_{\mathbb{R}} u(x)r(x) dx,$$

for the class of functions  $u$  for which the integral is finite.

The techniques employed here were recently used in [3] to solve a problem proposed by D. Zeilberger related to the Narayana polynomials discussed in [18].

**Definition 6.1.** Let  $a, b > 0$ . The random variable  $\mathfrak{B}_{a,b}$  is called *beta distributed*, or simply *a beta random variable*, if its distribution function is given by

$$(6.2) \quad f_{\mathfrak{B}_{a,b}}(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here  $B(a, b)$  is the beta function (3.6).

The integral representation [1, 13.2.1]

$$(6.3) \quad \Phi_{a,b}(z) = {}_1F_1 \left( \begin{matrix} a \\ a+b \end{matrix} \middle| z \right) = \frac{1}{B(a,b)} \int_0^1 e^{tz} t^{a-1} (1-t)^{b-1} dt,$$

shows that  $\Phi_{a,b}(z)$  is the moment generating function of a beta random variable  $\mathfrak{B}_{a,b}$ :

$$(6.4) \quad \mathbb{E} \left( e^{z\mathfrak{B}_{a,b}} \right) = \Phi_{a,b}(z),$$

where  $\mathbb{E}$  is the expectation operator.

This representation is used to construct a new random variable. Some preliminary discussion is given first.

A random variable  $\Gamma$  is said to be *exponentially distributed* if its distribution function is given by

$$(6.5) \quad f_{\Gamma}(x) = \begin{cases} e^{-x} & \text{if } x \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

The moment generating function of an exponentially distributed random variable  $\Gamma$  is

$$(6.6) \quad \mathbb{E} \left[ e^{z\Gamma} \right] = \frac{1}{1-z}, \text{ for } |z| < 1.$$

**Note 6.2.** The authors hope that not too much confusion will be created by using the symbol  $\Gamma$  for this kind of random variables. The choice of name is clear: if  $\Gamma$  is exponentially distributed, then

$$(6.7) \quad \mathbb{E} \left[ \Gamma^{\alpha-1} \right] = \Gamma(\alpha).$$

Consider a sequence  $\{\Gamma_k\}_{k \geq 1}$  of independent identically distributed random variables, each with the same exponential distribution (6.5).

**Definition 6.3.** Let  $\{z_{a,b;k} : k \in \mathbb{N}\}$  be the collection of zeros of the Kummer function  $\Phi_{a,b}(z)$ . The complex-valued random variable  $\mathfrak{Z}_{a,b}$  is defined by

$$(6.8) \quad \mathfrak{Z}_{a,b} = -\frac{a}{a+b} + \sum_{k=1}^{\infty} \frac{\Gamma_k - 1}{z_{a,b;k}}.$$

Some properties of  $\mathfrak{Z}_{a,b}$  are given below. The main relation between  $\mathfrak{Z}_{a,b}$  and the Kummer function  $\Phi_{a,b}(z)$  is stated first.

**Theorem 6.4.** The complex-valued random variable  $\mathfrak{Z}_{a,b}$  satisfies

$$(6.9) \quad \mathbb{E} e^{z\mathfrak{Z}_{a,b}} = \frac{1}{\Phi_{a,b}(z)}.$$

*Proof.* The independence of the family  $\{\Gamma_k\}$  and Theorem 2.6 give

$$(6.10) \quad \mathbb{E}e^{z\mathfrak{Z}_{a,b}} = e^{-az/(a+b)} \prod_{k=1}^{\infty} \frac{1}{1 - z/z_{a,b;k}} e^{-z/z_{a,b;k}}.$$

This is the stated result.  $\square$

**Lemma 6.5.** The symmetry property

$$(6.11) \quad \mathfrak{Z}_{a,b} = -1 - \mathfrak{Z}_{b,a},$$

holds in the sense of distribution.

*Proof.* The symmetry property follows from Kummer transformation

$$(6.12) \quad {}_1F_1\left(\begin{matrix} a \\ b \end{matrix} \middle| z\right) = e^z {}_1F_1\left(\begin{matrix} b-a \\ b \end{matrix} \middle| -z\right).$$

$\square$

Now consider  $\mathfrak{B}_{a,b}$  to be a beta-distributed random variable independent of  $\mathfrak{Z}_{a,b}$ . Their moment generating functions satisfy

$$(6.13) \quad 1 = \mathbb{E}\left[e^{z\mathfrak{B}_{a,b}}\right] \times \mathbb{E}\left[e^{z\mathfrak{Z}_{a,b}}\right] = \mathbb{E}\left[e^{z(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})}\right].$$

Expanding in a power series yields the identity

$$(6.14) \quad \mathbb{E}[(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^n] = \delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

This is now used to provide a probabilistic representation of an analytic function.

**Theorem 6.6.** Let  $\mathfrak{B}_{a,b}$  and  $\mathfrak{Z}_{a,b}$  as before. Then, if  $f$  is an analytic function,

$$(6.15) \quad \mathbb{E}f(z + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b}) = f(z),$$

provided the expectation is finite.

*Proof.* It suffices to establish the result for  $f(z) = z^p$ , with  $p \in \mathbb{N}$ . This follows directly from (6.14):

$$\begin{aligned} \mathbb{E}[(z + \mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^p] &= \sum_{k=0}^p \binom{p}{k} z^{p-k} \mathbb{E}[(\mathfrak{B}_{a,b} + \mathfrak{Z}_{a,b})^k] \\ &= z^p. \end{aligned}$$

$\square$

**Note 6.7.** Hassen and Nguyen [12] show that

$$(6.16) \quad \int_0^1 x^{a-1}(1-x)^{b-1} B_n^{(a,b)}(x) dx = \begin{cases} B(a,b) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

This follows directly from Theorem 6.6 by taking  $f(z) = z^n$  and then  $z = 0$ .

The next item in this section gives a probabilistic point of view of the linear recurrence in Theorem 3.2. Some preliminary background is discussed first.

**Definition 6.8.** The independent random variables  $X$  and  $Y$  are called *conjugate* if

$$(6.17) \quad \mathbb{E}[(X + Y)^n] = \delta_n.$$

For example, (6.14) shows that  $\mathfrak{B}_{a,b}$  and  $\mathfrak{Z}_{a,b}$  are conjugate.

**Definition 6.9.** Let  $X$  be a random variable. The *moment generating function* of  $X$  is

$$(6.18) \quad \varphi_X(t) = \sum_{n=0}^{\infty} \mathbb{E}[X^n] \frac{t^n}{n!}.$$

The sequence of *cumulants*  $\kappa_X(n)$  is defined by

$$(6.19) \quad \log \varphi_X(z) = \sum_{m=1}^{\infty} \kappa_X(m) \frac{z^m}{m!}.$$

**Example 6.10.** The cumulant generating function for the  $\mathfrak{B}_{a,b}$  distribution is

$$(6.20) \quad \log \Phi_{a,b}(z) = \frac{a}{a+b}z - \sum_{p=2}^{\infty} \frac{z^p}{p} \zeta_{a,b}^H(p),$$

so the cumulants are

$$(6.21) \quad \kappa_{\mathfrak{B}_{a,b}}(1) = \frac{a}{a+b} \text{ and } \kappa_{\mathfrak{B}_{a,b}}(p) = -(p-1)! \zeta_{a,b}^H(p) \text{ for } p \geq 2.$$

For a general random variable  $X$ , the moments  $\mathbb{E}X^n$  and its cumulants  $\kappa_X(n)$  are related by

$$(6.22) \quad \kappa_X(n) = \mathbb{E}X^n - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \kappa_X(j) \mathbb{E}X^{n-j}.$$

See [20] for details. In the special case of a random variable with a beta distribution, this gives

$$(6.23) \quad (n-1)! \sum_{j=2}^n \frac{B(a+n-j, b)}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B(a+n-1, b) - B(a+n, b)$$

that is equivalent to the linear recurrence identity in Theorem 3.2.

## 7. THE GENERALIZED BERNOULLI POLYNOMIALS

K. Dilcher introduced in [10] the generalized Bernoulli polynomials by

$$(7.1) \quad \frac{e^{xz}}{\Phi_{1,b}(z)} = \sum_{k=0}^{\infty} B_k^{(b)}(x) \frac{z^k}{k!}.$$

These polynomials are now interpreted as moments.

**Theorem 7.1.** The generalized Bernoulli polynomials are given by

$$(7.2) \quad B_k^{(b)}(x) = \mathbb{E}(x + \mathfrak{Z}_{1,b})^k.$$

*Proof.* This follows directly from Theorem 6.4.  $\square$

Dilcher [10] used generating functions to provide the following recursion for these polynomials.

**Theorem 7.2** (Dilcher). The generalized Bernoulli polynomials  $B_k^{(b)}(x)$  satisfy

$$(7.3) \quad B_k^{(b)}(x+1) = \sum_{p=0}^{b-1} \binom{k}{p} B_{k-p}^{(b)}(x) + \binom{k}{b} x^{k-b}.$$

A probabilistic proof is presented next. A preliminary result is stated first.

**Lemma 7.3.** Let  $\mathfrak{B}_{a,b}$  be a random variable with a beta distribution and  $g \in C^1[0, 1]$ . Then, for  $a, b > 1$ ,

$$(7.4) \quad \mathbb{E}g'(x + \mathfrak{B}_{a,b}) = (a + b - 1) [\mathbb{E}g(x + \mathfrak{B}_{a,b-1}) - \mathbb{E}g(x + \mathfrak{B}_{a-1,b})].$$

For  $a = 1$  and  $b > 1$

$$(7.5) \quad \mathbb{E}g'(x + \mathfrak{B}_{1,b}) = -bg(x) + b\mathbb{E}g(x + \mathfrak{B}_{1,b-1}).$$

*Proof.* A direct calculation shows

$$\begin{aligned} B(a, b)\mathbb{E}g'(x + \mathfrak{B}_{a,b}) &= \int_0^1 g'(x+t)t^{a-1}(1-t)^{b-1} dt \\ &= g(x+t)t^{a-1}(1-t)^{b-1} \Big|_0^1 \\ &\quad - (a-1) \int_0^1 g(x+t)t^{a-2}(1-t)^{b-1} dt \\ &\quad + (b-1) \int_0^1 g(x+t)t^{a-1}(1-t)^{b-2} dt. \end{aligned}$$

The result follows by simplification. The case  $a = 1$  is straightforward.  $\square$

The formula (7.5) can be extended directly to higher order derivatives.

**Lemma 7.4.** Let  $g \in C^k[0, 1]$  and  $\mathfrak{B}_{1,b}$  as before. Then, provided  $b \geq k$ ,

$$(7.6) \quad \mathbb{E}g^{(k)}(x + \mathfrak{B}_{1,b}) = - \sum_{\ell=1}^k \frac{b!}{(b-\ell)!} g^{(k-\ell)}(x) + \frac{b!}{(b-k)!} \mathbb{E}g(x + \mathfrak{B}_{1,b-k}).$$

To prove Dilcher's theorem, replace  $x$  by  $x + \mathfrak{B}_{1,b}$  and take the expectation with respect to  $\mathfrak{B}_{1,b}$  to see that Theorem 7.2 is equivalent to the identity

$$(7.7) \quad (x+1)^k = \sum_{p=0}^{b-1} \binom{k}{p} x^{k-p} + \mathbb{E}(x + \mathfrak{B}_{1,b-k})^k.$$

This is precisely the statement of Lemma 7.4 for  $g(x) = x^k$ .

The expression for the polynomials  $B_k^{(b)}(x)$  given in Theorem 7.1 provides a natural way to extend them to a two-parameter family.

**Definition 7.5.** The *hypergeometric Bernoulli polynomials* are defined by

$$(7.8) \quad B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{B}_{a,b})^n$$

and the *hypergeometric Bernoulli numbers* by

$$(7.9) \quad B_n^{(a,b)} = B_n^{(a,b)}(0) = \mathbb{E}(\mathfrak{B}_{a,b})^n.$$

**Note 7.6.** The case considered by Howard is  $B_n^{(b)} = B_n^{(1,b)}$ .

**Proposition 7.7.** The exponential generating function for the polynomials  $B_n^{(a,b)}(x)$  is given by

$$(7.10) \quad \sum_{n=0}^{\infty} B_n^{(a,b)}(x) \frac{z^n}{n!} = \frac{e^{xz}}{\Phi_{a,b}(z)}.$$

The next result appears, in the special case  $x = 0$ , as Proposition 2.1 in [10]. The result gives a change of basis formula from  $\{B_n^{(a,b)}(x) : n = 0, 1, 2, \dots\}$  to  $\{x^n : n = 0, 1, 2, \dots\}$ .

**Theorem 7.8.** The polynomial  $B_n^{(a,b)}(x)$  satisfy  $B_0^{(a,b)}(x) = 1$  and, for  $n \geq 1$ ,

$$(7.11) \quad \sum_{k=0}^n \binom{a+b+n-1}{k} \binom{a-1+n-k}{a-1} B_k^{(a,b)}(x) = (a+b)_n \frac{x^n}{n!}.$$

*Proof.* Theorem 6.6, with  $f(x) = x^n$  gives

$$(7.12) \quad \mathbb{E}(x + \mathfrak{B}_{a,b} + \mathfrak{J}_{a,b})^n = x^n.$$

The binomial theorem now gives

$$(7.13) \quad \sum_{k=0}^n \binom{n}{k} \mathbb{E}(x + \mathfrak{J}_{a,b})^k \mathbb{E}\mathfrak{B}_{a,b}^{n-k} = x^n.$$

The moments of the beta random variable  $\mathfrak{B}_{a,b}$  are

$$\begin{aligned} \mathbb{E}\mathfrak{B}_{a,b}^p &= \frac{1}{B(a,b)} \int_0^1 x^p \cdot x^{a-1} (1-x)^{b-1} dx \\ &= \frac{B(a+p,b)}{B(a,b)} = \frac{\Gamma(a+p)}{\Gamma(a+b+p)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)}. \end{aligned}$$

The expression (7.13) is now expressed as

$$(7.14) \quad \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(a+n-k)}{\Gamma(a+b+n-k)} B_k^{(a,b)}(x) = x^n,$$

and this is equivalent to the stated result.  $\square$

The probabilistic approach presented here, provides a direct proof of a symmetry property established in [10]. It extends the classical relation  $B_n(1-x) = (-1)^n B_n(x)$  of the Bernoulli polynomials.

**Theorem 7.9.** The polynomials  $B_n^{(a,b)}(x)$  satisfy the symmetry

$$(7.15) \quad B_n^{(a,b)}(1-x) = (-1)^n B_n^{(b,a)}(x).$$

*Proof.* The moment representation

$$(7.16) \quad B_n^{(a,b)}(x) = \mathbb{E}(x + \mathfrak{J}_{a,b})^n$$

and using (6.11) yields

$$\begin{aligned} B_n^{(a,b)}(1-x) &= \mathbb{E}(1-x + \mathfrak{J}_{a,b})^n \\ &= \mathbb{E}(-x - \mathfrak{J}_{b,a})^n \\ &= (-1)^n \mathbb{E}(x + \mathfrak{J}_{b,a})^n. \end{aligned}$$

$\square$

The next result presents a linear recurrence for the polynomials  $B_n^{(a,b)}(x)$ .



**Theorem 7.10.** Let  $X$  and  $Y$  be conjugate random variables. Define the polynomials

$$(7.17) \quad P_n(z) = \mathbb{E}(z + X)^n \text{ and } Q_n(z) = \mathbb{E}(z + Y)^n.$$

Then  $P_n$  and  $Q_n$  satisfy the recurrences

$$(7.18) \quad P_{n+1}(z) - zP_n(z) = \sum_{j=0}^n \binom{n}{j} \kappa_X(j+1) P_{n-j}(z)$$

and

$$(7.19) \quad Q_{n+1}(z) - zQ_n(z) = - \sum_{j=0}^n \binom{n}{j} \kappa_X(j+1) Q_{n-j}(z).$$

*Proof.* Let  $X_1$  and  $X_2$  be two independent random variables distributed as  $X$  and let

$$\begin{aligned} f(z) &= \mathbb{E}[X_1(X_1 + Y + z + X_2)^n - X_1(z + X_2)^n] \\ &= \sum_{j=0}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + Y)^j (z + X_2)^{n-j}] - \mathbb{E}X_1 (z + X_2)^n \\ &= \sum_{j=1}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + Y)^j] \mathbb{E}(z + X_2)^{n-j}. \end{aligned}$$

Theorem 3.3 in [8] shows that the cumulants satisfy

$$(7.20) \quad \kappa_X(p) = \mathbb{E}[X(X + Y)^{p-1}], \text{ for } p \geq 1.$$

Therefore

$$(7.21) \quad f(z) = \sum_{j=1}^n \binom{n}{j} \kappa_X(j+1) P_{n-j}(z).$$

The function  $f(z)$  may also be expressed as

$$(7.22) \quad f(z) = \sum_{j=0}^n \binom{n}{j} \mathbb{E}[X_1(X_1 + z)^{n-j} (Y + X_2)^j] - \mathbb{E}X_1 \mathbb{E}(z + X_2)^n.$$

The relation  $\mathbb{E}(Y + X_2)^j = \delta_j$  holds since  $X_2$  and  $Y$  are conjugate random variables. This reduces the previous expression for  $f$  to

$$(7.23) \quad f(z) = \mathbb{E}X_1(X_1 + z)^n - \mathbb{E}X_1 P_n(z).$$

This can be simplified using

$$\begin{aligned} \mathbb{E}X_1(X_1 + z)^n &= \mathbb{E}(X_1 + z)^{n+1} - z\mathbb{E}(X_1 + z)^n \\ &= P_{n+1}(z) - zP_n(z). \end{aligned}$$

The function  $f$  has been expressed as

$$(7.24) \quad f(z) = P_{n+1}(z) - (z + \kappa_X(1))P_n(z)$$

using  $\mathbb{E}(X) = \kappa_X(1)$ . The recurrence for  $P_n$  comes by comparing (7.21) and (7.24).

The second identity is obtained by replacing  $X$  and  $Y$  and remarking that  $\kappa_X(p) = -\kappa_Y(p)$  and  $\mathbb{E}(X + Y) = 0$ , since  $X$  and  $Y$  are conjugate random variables.  $\square$

**Theorem 7.11.** The hypergeometric Bernoulli polynomials  $B_n^{(a,b)}(z)$  and the companion family  $C_n^{(a,b)}(z)$  defined by

$$(7.25) \quad B_n^{(a,b)}(z) = \mathbb{E}(z + \mathfrak{Z}_{a,b})^n \text{ and } C_n^{(a,b)}(z) = \mathbb{E}(z + \mathfrak{B}_{a,b})^n$$

satisfy the recurrences

$$(7.26) \quad B_{n+1}^{(a,b)}(z) - zB_n^{(a,b)}(z) = \sum_{j=0}^n \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) B_{n-j}^{(a,b)}(z)$$

and

$$(7.27) \quad C_{n+1}^{(a,b)}(z) - zC_n^{(a,b)}(z) = - \sum_{j=0}^n \frac{n!}{(n-j)!} \zeta_{a,b}^H(j+1) C_{n-j}^{(a,b)}(z).$$

*Proof.* The result now follows from Theorem 7.10 and the cumulants for the beta distribution given in Example 6.10.  $\square$

Our last result provides a probabilistic approach to the linear recurrences for the hypergeometric zeta function.

For a random variable  $X$ , the moments  $\mathbb{E}X^n$  and its cumulants  $\kappa_X(n)$  satisfy the relation (6.22). This is now used to produce a linear recurrence for the hypergeometric zeta function.

**Theorem 7.12.** The hypergeometric zeta function  $\zeta_{a,b}^H$  satisfies

$$(7.28) \quad (n-1)! \sum_{j=2}^n \frac{B_{n-j}^{(a,b)}}{(n-j)!} \zeta_{a,b}^H(j) = \frac{a}{a+b} B_{n-1}^{(a,b)} + B_n^{(a,b)}.$$

*Proof.* Use the identity (6.22) to the random variable  $\mathfrak{Z}_{a,b}$ . Its moments are the hypergeometric Bernoulli numbers

$$(7.29) \quad \mathbb{E}\mathfrak{Z}_{a,b}^p = B_p^{(a,b)}$$

and its cumulants are

$$(7.30) \quad \kappa_{\mathfrak{Z}_{a,b}}(n) = \begin{cases} (n-1)! \zeta_{a,b}^H(n), & \text{for } n \geq 2 \\ -\frac{a}{a+b} & \text{for } n = 1, \end{cases}$$

since  $\mathfrak{Z}_{a,b}$  and  $\mathfrak{B}_{a,b}$  are conjugate random variables. A second proof is obtained by letting  $z = 0$  in (7.26).  $\square$

**Note 7.13.** Surprisingly, the two linear recurrences for  $\zeta_{a,b}^H$ , given in Theorem 3.2 and in Theorem 7.12 are different. For example, choosing  $a = 5$ ,  $b = 3$  these produce for  $n = 3$  the relations

$$\begin{aligned} 2\zeta_{5,3}^H(3) + \frac{5}{4}\zeta_{5,3}^H(2) + \frac{1}{32} &= 0 \\ 2\zeta_{5,3}^H(3) - \frac{5}{4}\zeta_{5,3}^H(2) - \frac{13}{384} &= 0. \end{aligned}$$

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