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RECURSION RELATIONS OF PRODUCTS OF LINEAR RECURSION SEQUENCES

BROTHER ALFRED BROUSSEAU
St. Mary's College, Moraga, California 94575

Given two sequences S_i and T_i governed respectively by linear recursion relations

(1) $$S_n = \sum_{i=1}^{p} a_i S_{n-i}$$

of order p and

(2) $$T_n = \sum_{i=1}^{q} b_i T_{n-1}$$

of order q. Required to find the recursion relation of the term-by-term product of the two sequences $Z_i = S_i T_i$.

Initially we shall assume that the roots of the auxiliary equations corresponding to the above recursion relations are distinct so that:

(3) $$S_n = \sum_{i=1}^{p} A_i s_i^n \;,$$

where s_i $(i = 1,p)$ are the roots of $x^p - a_1 x^{p-1} - a_2 x^{p-2} \cdots - a_p = 0$. Similarly,

(4) $$T_n = \sum_{i=1}^{q} B_i t_i^n \;,$$

where t_i $(i = 1,q)$ are the roots of $x^q - b_1 x^{q-1} - b_2 x^{q-2} \cdots - b^q = 0.$

No universal formulation applying to all orders has been arrived at so that the results will be given as a series of algorithms applying to particular cases. The method employed is to find the products of the general terms (3) and (4) and then note the new set of roots for the recursion relation of the product. By finding the symmetric functions of these roots one can arrive at the recursion relation of the term-by-term product.

1. GEOMETRIC PROGRESSION BY ANOTHER SEQUENCE

A geometric progression is a linear recursion relation of the first order:

$$S_n = rS_{n-1}$$

whose general term can be taken as $S_n = Ar^n$. If such a term be multiplied by (4) one has:

(5) $$S_n T_n = \sum_{i=1}^{q} B_i (rt_i)^n$$

Thus these terms behave as belonging to an auxiliary equation whose roots are rt_i $(i = 1,q)$. Consequently by finding the symmetric functions of these quantities one arrives at the linear recursion relation governing the terms $Z_n = S_n T_n$. It is not difficult to verify that this leads to:

(6) $$Z_n = \sum_{i=1}^{q} B_i r^i Z_{n-i}$$

2. TWO RELATIONS OF THE SECOND ORDER

Let the auxiliary equations corresponding to two linear recursion relations of the second order be:

$$x^2 + a_1x + b_1 = 0$$
$$x^2 + a_2x + b_2 = 0$$

Let the terms of the sequence governed by the first relation be:

$$S_n = Ar^n + Bs^n$$

and the terms governed by the second sequence be:

$$T_n = Cu^n + Dv^n .$$

Then

$$Z_n = S_nT_n = AC(ru)^n + AD(rv)^n + BC(su)^n + BD(sv)^n .$$

The roots of the auxiliary equation for Z_n are ru, rv, su, sv. To obtain the coefficients of this equation we calculate the symmetric functions of these roots.

$$S_{4,1} = (r+s)(u+v) = (-a_1)(-a_2) = a_1a_2$$

$$S_{4,2} = r^2uv + rsu^2 + rsuv + rsuv + rsv^2 + s^2uv = uv(r^2+s^2) + rs(u^2+v^2) + 2rsuv$$
$$= b_1(a_2^2 - 2b_2^2) + b_2(a_1^2 - 2b_1) + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$

$$S_{4,3} = r^2su^2v + r^2suv^2 + rs^2u^2v + rs^2uv^2 = rsuv(r+s)(u+v) = b_1b_2a_1a_2$$

$$S_{4,4} = r^2s^2u^2v^2 = b_1^2b_2^2 .$$

The recursion relation for the product of two sequences of the second order is thus

$$x^4 - a_1a_2x^3 + (b_1a_2^2 + b_2a_1^2 - 2b_1b_2)x^2 - a_1a_2b_1b_2x + b_1^2b_2^2 = 0 .$$

EXAMPLE. The sequence 1, 4, 17, 72, 305, ⋯ is governed by $T_{n+1} = 4T_n + T_{n-1}$ while 1, −5, 26, −135, 701, ⋯ is governed by $T_{n+1} = -5T_n + T_{n-1}$. The product sequence is 1, −20, 442, −9720, 213805, ⋯. In terms of the above formulation, $a_1 = -4$, $b_1 = -1$, $a_2 = 5$, $b_2 = -1$. The auxiliary equation for the product sequence is given by:

$$x^4 + 20x^3 - 43x^2 + 20x + 1 = 0.$$

$$(-9720)(-20) + 442*43 + (-20)(-20) - 1 = 213805 .$$

SECOND- AND THIRD-ORDER RECURSION RELATIONS

Given two sequences S_n, T_n governed respectively by the relations

(7) $$x^2 + a_1x + b_1 = 0$$

(8) $$x^3 + a_2x^2 + b_2x + c_2 = 0$$

with roots r_1, s_1 and r_2, s_2, t_2, respectively. The recursion relation of the product S_nT_n will have for roots r_1r_2, r_1s_2, r_1t_2, s_1r_2, s_1s_2, s_1t_2. The symmetric functions of these roots are as follows.

(9) $$S_{6,1} = (r_1+s_1)(r_2+s_2+t_2) = a_1a_2$$

(10) $$S_{6,2} = r_1s_1(s_2^2+t_2^2+r_2^2) + (r_1^2+s_1^2)(r_2s_2+r_2t_2+s_2t_2) + 2r_1s_1(r_2s_2+r_2t_2+s_2t_2)$$
$$= b_1(a_2^2 - 2b_2) + (a_1^2 - 2b_1)b_2 + 2b_1b_2 = b_1a_2^2 + b_2a_1^2 - 2b_1b_2$$

(11) $$S_{6,3} = (r_1^3+s_1^3)(r_2s_2t_2) + r_1s_1(r_1+s_1)(r_2+s_2+t_2)(r_2s_2+r_2t_2+s_2t_2)$$
$$= (a_1^3 - 3a_1b_1)c_2 + a_1b_1a_2b_2$$

(12) $$S_{6,4} = r_1s_1(r_1^2+s_1^2)r_2s_2t_2 \quad +s_2+t_2) + r_1^2s_1^2(r_2s_2+r_2t_2+s_2t_2)^2$$
$$= b_1a_1^2a_2c_2 + b_1^2b_2^2 - 2b_1^2a_2c_2$$

(13) $$S_{6,5} = r_1^2s_1^2(r_1+s_1)r_2s_2t_2(r_2s_2+r_2t_2+s_2t_2) = b_1^2a_1b_2c_2$$

(14) $$S_{6,6} = r_1^3s_1^3r_2^2s_2^2t_2^2 = b_1^3c_2^2$$

EXAMPLE

$$x^2 - 4x + 3 = 0 \qquad x^3 - 3x^2 + 6x - 3 = 0$$

2	3	6
7	−21	-147
22	−60	-1320
67	−45	-3015
202	162	32724
607	576	349632
1822	621	1131462
5467	−1107	−6051969
16402	−5319	−87242238

$$a_1 = -4, \quad b_1 = 3, \quad a_2 = -3, \quad b_2 = 6, \quad c_2 = -3$$
$$S_{6,1} = 12$$
$$S_{6,2} = 3*9 + 6*16 - 2*18 = 87$$
$$S_{6,3} = (-64+36)(-3) + 216 = 300$$
$$S_{6,4} = 3*16*9 + 9*36 - 2*9*9 = 594$$
$$S_{6,5} = 9(-4)*6(-3) = 648$$
$$S_{6,6} = 27*9 = 243\,.$$

The recursion relation corresponds to:

$$x^6 - x^5 - 12x^5 + 87x^4 - 300x^3 + 594x^2 - 648x + 243 = 0\,.$$

CHECK

12(−6051969) − 87(1131462) + 300(349632) − 594(32724) + 648(−3015) − 243(−1320) = −87242238 .

TWO THIRD-ORDER RELATIONS

For two sequences governed by the relations:

$$x^3 + a_1x^2 + b_1x + c_1 = 0 \qquad \text{and} \qquad x^3 + a_2x^2 + b_2x + c_2 = 0$$

The coefficients of the recursion relation of the product are found to be:

x^9	1
x^8	$-a_1a_2$
x^7	$a_1^2b_2 + a_2^2b_1 - 2b_1b_2$
x^6	$-a_1^3c_2 - a_2^3c_1 - 3c_1c_2 + 3a_1b_1c_2 - 3a_2b_2c_1 - a_1a_2b_1b_2$
x^5	$b_1^2b_2^2 - 2a_2b_1^2c_2 - 2a_1b_2^2c_1 + a_1^2a_2b_1c_2 + a_1a_2^2b_2c_1 - a_1a_2c_1c_2$
x^4	$-a_1b_1^2b_2c_2 - a_2b_1b_2^2c_1 + 2a_1^2 \;{}_2c_1c_2 + 2a_2^2b_1c_1c_2 - a_1^2a_2^2c_1c_2 + b_1b_2c_1c_2$
x^3	$b_1^3c_2^2 + b_2^3c_1^2 - 3a_1b_1c_1c_2^2 - 3a_2b_2c_1^2c_2 + 3c_1^2c_2^2 + a_1a_2b_1b_2c_1c_2$
x^2	$-a_2b_1^2c_1c_2^2 - a_1b_2^2c_1^2c_2 + 2a_1a_2c_1^2c_2^2$
x	$b_1b_2c_1^2c_2^2$
1	$-c_1^3c_2^3$

EXAMPLE $x^3 - 3x^2 + 5x - 2 = 0 \quad x^3 + 4x^2 - 7x - 3 = 0$

1	0	0
−2	1	−2
−9	2	−18
−15	−1	15
−4	21	−84
45	−85	−3825
125	484	60500
142	−2468	−350456
−109	13005	−1417545
−787	−67844	53393228
−1532	355007	−543870724
−879	−1855921	1631354559
3449	9705201	33473238249

$a_1 = -3, \quad b_1 = 5, \quad c_1 = -2, \quad a_2 = 4, \quad b_2 = -7, \quad c_2 = -3.$

The recursion relation for the product is:

$$x^9 + 12x^8 + 87x^7 - 88x^6 + 97x^5 + 2665x^4 + 563x^3 - 828x^2 - 1260x - 216 = 0.$$

CHECK

$$-12*1631354559 - 87(-543870724) + 88*53393228 - 97(-1417545) - 2665(-350456)$$
$$- 563*60500 + 828(-3825) + 1260(-84) + 216*15 = 33473238249.$$

SECOND AND FOURTH ORDERS

Given two sequences governed by the following relations, respectively:

$$x^2 + a_1x + b_1 = 0$$

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0.$$

The coefficients of the product recursion relation are:

x^8	1
x^7	$-a_1a_2$
x^6	$a_1^2b_2 + a_2^2b_1 - 2b_1b_2$
x^5	$-a_1^3c_2 - a_1a_2b_1b_2 + 3a_1b_1c_2$
x^4	$a_1^4d_2 - 4a_1^2b_1d_2 + 2b_1^2d_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2$
x^3	$-a_1^3a_2b_1d_2 - a_1b_1^2b_2c_2 + 3a_1a_2b_1^2d_2$
x^2	$a_1^2b_1^2b_2d_2 + b_1^3c_2^2 - 2b_1^3b_2d_2$
x	$-a_1b_1^3c_2d_2$
1	$b_1^4d_2^2$

EXAMPLE $x^2 - 3x + 2 = 0 \quad x^4 + 2x^3 - 3x^2 + x - 3 = 0$

10	−8	−80
22	34	748
46	−97	−4462
94	319	29986
190	−987	−187530
382	3130	1195660
766	−9831	−7530546
1534	30996	47547864
3070	−97576	−299558320
6142	307361	1887811262
12286	−967939	−11892098554

CHECK

The recursion relation of the product corresponds to:

$$x^8 + 6x^7 - 7x^6 - 27x^5 + 5x^4 - 144x^3 + 188x^2 - 72x + 144 = 0$$

$$-6(1887811262) + 7(-299558320) + 27*47547864 + (-5)(-7530546) + 144*1195660$$
$$- 188(-187530) + 72*29986 - 144(-4462) = -11892098554.$$

THIRD- AND FOURTH-ORDER SEQUENCES

For two sequences governed respectively by the relations corresponding to:

$$x^3 + a_1x^2 + b_1x + c_1 = 0$$

and

$$x^4 + a_2x^3 + b_2x^2 + c_2x + d_2 = 0$$

the coefficients for the auxiliary equation of the product are given by

x^{12}	1
x^{11}	$-a_1a_2$
x^{10}	$a_1^2b_2 + a_2^2b_1 - 2b_1b_2$
x^9	$-a_1^3c_2 - 3c_1c_2 + 3a_1b_1c_2 + 3a_2b_2c_1 - a_1a_2b_1b_2 - a_2^3c_1$
x^8	$a_1^4d_2 - 4a_1^2b_1d_2 - a_1a_2c_1c_2 + a_1^2a_2b_1c_2 - 2a_2b_1^2c_2 + b_1^2b_2^2 - 2a_1b_2^2c_1 + a_1a_2^2b_2c_1 + 2b_1^2d_2 + 4a_1c_1d_2$
x^7	$-a_1^3a_2b_1d_2 - 5a_2b_1c_1d_2 + 3a_1a_2b_1^2d_2 + a_1^2a_2c_1d_2 - a_1b_1^2b_2c_2 + 2a_1^2b_2c_1c_2 + b_1b_2c_1c_2 - a_1^2a_2^2c_1c_2$ $+ 2a_2^2b_1c_1c_2 - a_2b_1b_2^2c_1$
x^6	$a_1^2b_1^2b_2d_2 - 2a_1^3b_2c_1d_2 - 2b_1^3b_2d_2 + 4a_1b_1b_2c_1d_2 - 3b_2c_1^2d_2 + a_1^3a_2^2c_1d_2 + 3a_2^2c_1^2d_2 - 3a_1a_2^2b_1c_1d_2$ $+ b_1^3c_2^2 - 3a_1b_1c_1c_2^2 + 3c_1^2c_2^2 + b_2^3c_1^2 - 3a_2b_2c_1^2c_2 + a_1a_2b_1b_2c_1c_2$
x^5	$-a_1b_1^3c_2d_2 + 3a_1^2b_1c_1c_2d_2 + b_1^2c_1c_2d_2 - 5a_1c_1^2c_2d_2 + a_1a_2b_2c_1^2d_2 - a_1^2a_2b_1b_2c_1d_2 + 2a_2b_1^2b_2c_1d_2$ $- a_2b_1^2c_1c_2^2 + 2a_1a_2c_1^2c_2^2 - a_1b_2^2c_1^2c_2$
x^4	$b_1^4d_2^2 - 4a_1b_1^2c_1d_2^2 + 2a_1^2c_1^2d_2^2 + 4b_1c_1^2d_2^2 + a_1a_2b_1^2c_1c_2d_2 - 2a_1^2a_2c_1^2c_2d_2 - a_2b_1c_1^2c_2d_2 + a_1^2b_2^2c_1^2d_2$ $- 2b_1b_2^2c_1^2d_2 + b_1b_2c_1^2c_2^2$
x^3	$-a_2b_1^3c_1d_2^2 + 3a_1a_2b_1c_1^2d_2^2 - 3a_2c_1^3d_2^2 + 3b_2c_1^3c_2d_2 - a_1b_1b_2c_1^2c_2d_2 - c_1^3c_2^3$
x^2	$b_1^2b_2c_1^2d_2^2 - 2a_1b_2c_1^3d_2^2 + a_1c_1^3c_2^2d_2$
x	$-b_1c_2c_1^3d_2^2$
1	$c_1^4d_2^3$

EXAMPLE

$x^3 - x^2 - x - 1 = 0$	$x^4 - x^3 - x^2 - x - 1 = 0$	PRODUCT
6	21	126
11	39	429
20	76	1520
37	147	5439
68	283	19244
125	545	68125
230	1051	241730
423	2026	856998
778	3905	3038090
1431	7527	10771137
2632	14509	38187688
4841	27967	135388247
8904	53908	479996832
16377	103911	1701750447
30122	200295	6033285990

The recursion relation for the product corresponds to the equation:

$$x^{12} - x^{11} - 4x^{10} - 12x^9 - 17x^8 - 12x^7 - 5x^6 + 10x^5 - 7x^4 - 2x^3 + 0x^2 + x - 1 = 0.$$

CHECK

$$479996832 + 4*135388247 + 12*38187688 + 17*10771137 + 12*3038090 + 5*856998$$
$$- 10*241730 + 7*68125 + 2*19244 - 1520 + 429 = 1701750447 .$$

REPEATED ROOTS

The case of d roots can be handled in the same way but with some modifications in the procedure for finding the symr ıctions. This discussion will be limited to the important case in which one of the sequences has a general te… given by a polynomial function. The recursion relation for such a polynomial function of the n^{th} degree has its coefficients determined by the expansion of

$$(x-1)^{n+1} = 0 .$$

In other words there are $n+1$ roots all equal to unity.

QUADRATIC POLYNOMIAL SEQUENCE

The general procedure can be illustrated by the case of a sequence whose terms are given by a quadratic polynomial function. To keep the resulting formulas reasonably simple, let the other sequence be limited to order five and be in the form:

$$x^5 - a_1x^4 + a_2x^3 - a_3x^2 + a_4x - a_5 = 0$$

so that the quantities a_i are the symmetric functions of the roots. If the roots are given by r_i, the general term of the sequence would be:

$$T_n = \sum A_ir_i^n .$$

If the polynomial function is $f(n) = B_1n^2 + B_2n + B_3$, the product of the terms of the two sequences is:

$$Z_n = T_nf(n) = B_1n^2\sum A_ir_i^n + B_2n\sum A_ir_i^n + B_3\sum A_ir_i^n .$$

This shows that the equation for the product has the roots r_i taken three times. The problem then is to find the symmetric functions for three such sets of roots taken together. Suppose we wish to find $S_{15,9}$, the symmetric function of these fifteen roots taken nine at a time. The various cases can be found by taking the partitions of 9 into three or less parts, the largest being five (since this limitation was set on the order of the recursion relation). These partitions would be: 54, 531, 522, 441, 432, 333. Hence

$$S_{15,9} = 6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3$$

the coefficients being determined by the multinomial coefficient corresponding to the number of ways the various groups of roots can be selected.

For the quadratic polynomial function and linear recursion relations up to the fifth order the coefficients of the product recursion relation are as follows:

$$1$$
$$-3a_1$$
$$3a_2 + 3a_1^2$$
$$-[3a_3 + 6a_1a_2 + a_1^3]$$
$$3a_4 + 6a_1a_3 + 3a_2^2 + 3a_2a_1^2$$
$$-[3a_5 + 6a_4a_1 + 6a_3a_2 + 3a_3a_1^2 + 3a_2^2a_1]$$
$$6a_5a_1 + 6a_4a_2 + 3a_3^2 + 3a_4a_1^2 + 6a_3a_2a_1 + a_2^3$$
$$-[6a_5a_2 + 6a_4a_3 + 3a_5a_1^2 + 6a_4a_2a_1 + 3a_3^2a_1 + 3a_3a_2^2]$$
$$6a_5a_3 + 3a_4^2 + 6a_5a_2a_1 + 6a_4a_3a_1 + 3a_4a_2^2 + 3a_3^2a_2$$
$$-[6a_5a_4 + 6a_5a_3a_1 + 3a_5a_2^2 + 3a_4^2a_1 + 6a_4a_3a_2 + a_3^3]$$
$$3a_5^2 + 6a_5a_4a_1 + 6a_5a_3a_2 + 3a_4^2a_2 + 3a_4a_3^2$$
$$-[3a_5^2a_1 + 6a_5a_4a_2 + 3a_5a_3^2 + 3a_4^2a_3]$$
$$3a_5^2a_2 + 6a_5a_4a_3 + a_4^3$$
$$-[3a_5^2a_3 + 3a_5a_4^2]$$
$$3a_5^2a_4$$
$$-a_5^3$$

EXAMPLE $x^5 - 2x^4 + x^3 + x^2 - 3x - 2 = 0 \quad f(n) = n^2$

		PRODUCT
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
11	36	396
26	49	1274
54	64	3456
94	81	7614
151	100	15100
254	121	30734
477	144	68688
939	169	158691
1788	196	350448
3224	225	725400
5660	256	1448960
10079	289	2912831
18516	324	5999184

The recursion relation of the product corresponds to the equation:

$$x^{15} - 6x^{14} + 15x^{13} - 17x^{12} - 6x^{11} + 42x^{10} - 38x^9 - 21x^8 + 69x^7 - 17x^6 - 54x^5$$
$$+ 33x^4 + 21x^3 - 42x^2 - 36x - 8 = 0.$$

CHECK

6*725400 – 15*350448 + 17*158691 + 6*68688 – 42*30734 + 38*15100 + 21*7614 – 69*3456 + 17*1274 + 54*396 – 33*125 – 21*64 + 42*27 + 36*8 + 8*1 = 1448960.

ARITHMETIC PROGRESSION

An arithmetic progression is given by a polynomial function of the first degree so that its recursion relation corresponds to $(x-1)^2 = 0$ with the root 1 taken twice. For a fifth order linear recursion relation such as given under the quadratic polynomial the coefficients of the equation corresponding to the linear recursion relation for the product are as follows.

$$1$$
$$-2a_1$$
$$2a_2 + a_1^2$$
$$-[2a_3 + 2a_2a_1]$$
$$2a_4 + 2a_3a_1 + a_2^2$$
$$-[2a_5 + 2a_4a_1 + 2a_3a_2]$$
$$2a_5a_1 + 2a_4a_2 + a_3^2$$
$$-[2a_5a_2 + 2a_4a_3]$$
$$2a_5a_3 + a_4^2$$
$$-2a_5a_4$$
$$a_5^2$$

POLYNOMIAL OF THE THIRD DEGREE

For a third-degree polynomial and a recursion relation up to the third order the coefficients of the equation corresponding to the linear recursion relation of the product are given by the following.

$$1$$
$$-4a_1$$
$$4a_2 + 6a_1^2$$
$$-[4a_3 + 12a_2a_1 + 4a_1^3]$$
$$12a_3a_1 + 6a_2^2 + 12a_2a_1^2 + a_1^4$$
$$-[12a_3a_2 + 12a_3a_1^2 + 12a_2^2a_1 + 4a_2a_1^3]$$
$$6a_3^2 + 24a_3a_2a_1 + 4a_3a_1^3 + 4a_2^3 + 6a_2^2a_1^2$$
$$-[12a_3^2a_1 + 12a_3a_2^2 + 12a_3a_2a_1^2 + 4a_2^3a_1]$$
$$12a_3^2a_2 + 6a_3^2a_1^2 + 12a_3a_2^2a_1 + a_2^4$$
$$-[4a_3^3 + 12a_3^2a_2a_1 + 4a_3a_2^3]$$
$$4a_3^3a_1 + 6a_3^2a_2^2$$
$$-4a_3^3a_2$$
$$a_3^4$$

EXAMPLE

$$x^3 - 3x^2 + x - 2 = 0 \quad \text{and} \quad f(n) = n^3$$

1	1	1
2	8	16
3	27	81
9	64	576
28	125	3500
81	216	17496
233	343	79919
674	512	345088
1951	729	1422279
5645	1000	5645000
16332	1331	21737892
47253	1728	81653184
136717	2197	300367249
395562	2744	1085422128
1144475	3375	3862603125

The recursion relation of the product corresponds to the relation:

$$x^{12} - 12x^{11} + 58x^{10} - 152x^9 + 267x^8 - 384x^7 + 442x^6 - 396x^5 + 337x^4 - 184x^3 + 120x^2 - 32x + 16 = 0.$$

CHECK

12*81653184 – 58*21737892 + 152*5645000 – 267*1422279 + 384*345088 – 442*79919 + 396*17496 – 337*3500 + 184*576 – 120*81 + 32*16 – 16 = 300367249

REPEATED ROOTS IN GENERAL

Given a sequence whose recursion relation has p repeated roots and another whose recursion relation has q repeated roots. We would have:

$$S_n = r^n(a_0 + a_1n + \cdots + a_{p-1}n^{p-1})$$
$$T_n = s^n(b_0 + b_1n + \cdots + b_{q-1}n^{q-1})$$
$$Z_n = S_nT_n = (rs)^n(c_0 + c_1n + \cdots + c_{p+q-2}n^{p+q-2})$$

so that the recursion relation of the product is of order $p + q - 1$.

If the first recursion relation has m distinct roots r_i and a repeated root r of multiplicity p, while the second has n distinct roots s_j and a repeated root of multiplicity q, the product recursion relation has the following number of roots: $mn + pn + qm + p + q - 1 = (m + p)(n + q) - (p - 1)(q - 1)$. The symmetric functions of these roots will give the coefficients of a recursion relation of this order. Similar considerations can be applied when there are a number of repeated roots of various multiplicities.

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