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SYMMETRIC SEQUENCES

BROTHER ALFRED BROUSSEAU
St. Mary's College, California 94575

This paper deals with integer sequences governed by linear recursion relations. To avoid useless duplication, sequences with terms having a common factor greater than one will be considered equivalent to the sequence with the greatest common factor of the terms eliminated. The recursion relation governing a sequence will be taken as the recursion relation of lowest order which it obeys.

Symmetric sequences are of two types:

A. Sequences with an Unmatched Zero Term

(1) $$\cdots T_{-3}, T_{-2}, T_{-1}, T_0, T_1, T_2, T_3, \cdots$$

with

$$T_n = T_{-n}$$

B. Sequences with All Matched Terms

(2) $$\cdots T_{-3}, T_{-2}, T_{-1}, T_1, T_2, T_3, \cdots$$

FIRST-ORDER SEQUENCES

The recursion relation of the first order is:

(3) $$T_{n+1} = aT_n$$

which will have all terms integers only if $a = \pm 1$. The only sequences governed by such relations subject to the initial restrictions given above are:

$$\cdots 1, 1, 1, 1, 1, 1, \cdots$$

$$\cdots -1, 1, -1, 1, -1, 1, \cdots$$

These sequences and the sequence $\cdots 0, 0, 0, 0, \cdots$ will be eliminated from consideration in the work that follows.

SECOND-ORDER SEQUENCES

For a recursion relation

$$T_{n+1} = aT_n + bT_{n-1}$$

to have all integer terms, the quantity b must be $+1$ or -1. The same applies to sequences of higher order. These will be denoted Case I $(+1)$ and Case II (-1).

Case I. $$T_{n+1} = aT_n + T_{n-1}$$

A. Zero Term

$$T_0 = T_2 - aT_1, \qquad T_{-1} = T_1 - aT_0 = T_1 - aT_2 + a^2T_1 = T_1, \qquad a(aT_1 - T_2) = 0.$$

Thus either $a = 0$ or $T_0 = 0$. $a = 0$ leads to sequences such as:

$$\cdots 2, 3, 2, 3, 2, 3, 2, 3, \cdots$$

If $T_0 = 0$,

$$T_{-2} = T_2 = T_0 - aT_{-1} = -aT_1 .$$

Hence $T_2 = aT_1$ and $T_2 = -aT_1$ with the result that $a = 0$.

B. No Zero Term

$$T_{-1} = T_2 - aT_1 = T_1, \qquad (a+1)T_1 = T_2, \qquad T_{-2} = T_2 = T_1 - aT_{-1} = (1-a)T_1 .$$

Therefore $aT_1 = 0$. If $T_1 = 0$, all the terms are zero. If $a = 0$, we have the type of sequence given above for this value.

Case II. $$T_{n+1} = aT_n - T_{n-1} .$$

A. Zero Term

(4) $$T_0 = aT_1 - T_2, \qquad T_{-1} = T_1 = aT_0 - T_1 = a^2T_1 - aT_2 - T_1$$
$$(a^2-2)T_1 - aT_2 = 0, \qquad T_{-2} = T_2 = aT_{-1} - T_0 = aT_{-1} - aT_1 + T_2 = T_2 .$$

If symmetry holds up to T_n, then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and hence the entire sequence will be symmetrical.

EXAMPLES

For any value of a, select T_1 and T_2 to satisfy (4) in order to generate a symmetric sequence. Thus for $a = 3$, $7T_1 = 3T_2$, giving the sequence:

$$\cdots 47, 18, 7, 3, 2, 3, 7, 18, 47, \cdots$$

governed by

$$T_{n+1} = 3T_n - T_{n-1} .$$

For $a = 8$, $62T_1 = 8T_2$, giving the sequence:

$$\cdots 1921, 244, 31, 4, 1, 4, 31, 244, 1921, \cdots$$

governed by $T_{n+1} = 8T_n - T_{n-1}$.

B. No Zero Term

The relations

$$T_{-1} = T_1 = aT_1 - T_2 \qquad \text{and} \qquad T_{-2} = aT_{-1} - T_1$$

both lead to

$$(a-1)T_1 = T_2 .$$

If $T_{-n} = T_n$ holds up to n, then

$$T_{-n-1} = aT_{-n} - T_{-n+1} = aT_n - T_{n-1} = T_{n+1}$$

and the symmetry will be maintained throughout the sequence.

For $a = 5$, $T_2 = 4T_1$ giving a sequence

$$\cdots 19, 4, 1, 1, 4, 19, 91, 436, \cdots$$

governed by

$$T_{n+1} = 5T_n - T_{n-1} .$$

THIRD-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + T_{n-2} .$$

A. Zero Term

$$T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \qquad T_0 = T_3 - aT_2 - bT_1,$$
$$T_{-1} = T_1 = T_2 - aT_1 - bT_0 = T_2 - aT_1 - bT_3 + abT_2 + b^2T_1$$

(5) $$(b^2 - a - 1)T_1 + (ab+1)T_2 = bT_3 .$$

Also

$$T_{-2} = T_2 = T_1 - aT_0 - bT_{-1} = T_1 - aT_3 + a^2T_2 + abT_1 - bT_1$$

from which

(6) $$(ab - b + 1)T_1 + (a^2-1)T_2 = aT_3$$

$$T_{-3} = T_3 = T_0 - aT_{-1} - bT_{-2} = T_3 - aT_2 - bT_1 - aT_1 - bT_2$$

so that

(7) $$(a+b)(T_1+T_2) = 0 .$$

Equation (7) will hold if $b=-a$ which makes (5) and (6):

(5′) $$(a^2-a-1)T_1+(1-a^2)T_2 = -aT_3$$

(6′) $$(-a^2+a+1)T_1+(a^2-1)T_2 = aT_3$$

which are the same relation. Since

$$T_4 = aT_3-bT_2+T_1 \quad \text{and} \quad T_{-4} = T_{-1}-aT_{-2}-bT_{-3} = T_1-aT_2+aT_3 = T_4$$

the symmetry persists up to this point. An entirely similar argument shows that it holds in general.

EXAMPLE. For a given value of a, many symmetric sequences can be determined. For $a=5$,

$$19T_1-24T_2 = -5T_3$$

from which one may derive any number of symmetric sequences obeying the relation

$$T_{n+1} = 5T_n-5T_{n-1}+T_{n-2}.$$

Examples are:

$$\cdots 1350, 361, 96, 25, 6, 1, 0, 1, 6, 25, 96, 361, 1350, \cdots$$

$$\cdots 363, 98, 27, 8, 3, 2, 3, 8, 27, 98, 363, \cdots, \qquad \cdots 362, 97, 26, 7, 2, 1, 2, 7, 26, 97, 362, \cdots$$

B. No Zero Term

$$T_{n+1} = aT_n+bT_{n-1}+T_{n-2}, \quad T_{n-2} = T_{n+1}-aT_n-bT_{n-1}, \quad T_{-1} = T_1 = T_3-aT_2-bT_1$$

(8) $$(b+1)T_1+aT_2 = T_3$$

$$T_{-2} = T_2 = T_2-aT_1-bT_{-1}$$

(9) $$(a+b)T_1 = 0$$

which is satisfied if $b=-a$

$$T_{-3} = T_3 = T_1-aT_{-1}-bT_{-2}$$

(10) $$T_3 = (1-a)T_1+aT_2$$

which agrees with (8) when $b=-a$.

If the symmetry holds to $T_n = T_{-n}$, then

$$T_{-n-1} = T_{-n+2}-aT_{-n+1}+aT_{-n} = T_{n-2}-aT_{n-1}+aT_n = T_{n+1}$$

so that all corresponding pairs are equal.

EXAMPLES. For $a=4$, $T_3 = 4T_2-3T_1$ yields many sequences governed by

$$T_{n+1} = 4T_n-4T_{n-1}+T_{n-2}$$

$$\cdots 233, 89, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, 89, 233, \cdots$$

$$\cdots 177, 67, 25, 9, 3, 1, 1, 3, 9, 25, 67, 177, \cdots$$

$$\cdots 265, 100, 37, 13, 4, 1, 1, 4, 13, 37, 100, 265, \cdots$$

Case II. $$T_{n+1} = aT_n+bT_{n-1}-T_{n-2}, \qquad T_{n-2} = aT_n+bT_{n-1}-T_{n+1}$$

A. Zero Term

$$T_0 = aT_2+bT_1-T_3, \; T_{-1}=T_1 = aT_1+bT_0-T_2 = aT_1+baT_2+b^2T_1-bT_3-T_2$$

(11) $$(a+b^2-1)T_1+(ba-1)T_2-bT_3 = 0$$

$$T_{-2} = T_2 = aT_0-bT_{-1}-T_1 = a^2T_2+abT_1-aT_3+bT_{-1}-T_1$$

(12) $$(ab+b-1)T_1+(a^2-1)T_2-aT_3 = 0$$

$$T_{-3} = T_3 = aT_1+bT_2-aT_2-bT_1+T_3$$

(13) $$(a-b)(T_1-T_2) = 0$$

so that $b=a$ satisfies this relation.

Equations (11) and (12) both become for $b = a$:

(14) $$(a^2 + a - 1)T_1 + (a^2 - 1)T_2 - aT_3 = 0.$$

For $a = 2$, $2T_3 = 5T_1 + 3T_2$ yields an infinity of sequences satisfying

$$T_{n+1} = 2T_n + 2T_{n-1} - T_{n-2}$$

$$\cdots 64, 25, 9, 4, 1, 1, 0, 1, 1, 4, 9, 25, 64, \cdots$$

$$\cdots 129, 49, 19, 7, 3, 1, 1, 1, 3, 7, 19, 49, 129, \cdots$$

$$\cdots 194, 73, 29, 10, 5, 1, 2, 1, 5, 10, 29, 73, 194, \cdots$$

$$\cdots 259, 97, 39, 13, 7, 1, 3, 1, 7, 13, 39, 97, 259, \cdots$$

B. No Zero Term

$$T_{n-2} = T_{n+1} - aT_n - bT_{n-1}, \qquad T_{-1} = T_3 - aT_2 - bT_1$$

(15) $$(b + 1)T_1 + aT_2 = T_3$$

$$T_{-2} = T_2 = T_2 - aT_1 - bT_{-1}$$

(16) $$(a + b)T_1 = 0 .$$

Equation (15) becomes $T_3 = (1 - a)T_1 + aT_2$ for $b = -a$. Now, $T_{-3} = T_3 = T_1 - aT_{-1} - bT_{-2}$

(17) $$T_3 = (1 - a)T_1 + aT_2$$

in agreement with (15) if $b = -a$.

$$T_{-4} = T_{-1} - aT_{-2} + aT_{-3} = aT_3 - aT_2 + T_1$$

whereas

$$T_4 = aT_3 - aT_2 - T_1$$

so that $T_1 = 0$ if $T_{-4} = T_4$.

Similarly setting $T_{-5} = T_5$ makes $T_2 = 0$, etc. Hence this case yields nothing more than the trivial result $\cdots 0, 0, 0, 0, 0, \cdots$.

FOURTH-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + T_{n-3}$$

A. Zero Term

$$T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \qquad T_0 = T_4 - aT_3 - bT_2 - cT_1$$

$$T_{-1} = T_1 = T_3 - aT_2 - bT_1 - cT_0 = T_3 - aT_2 - bT_1 - cT_4 + acT_3 + bcT_2 + c^2T_1$$

(18) $$(c^2 - b - 1)T_1 + (bc - a)T_2 + (ac + 1)T_3 - cT_4 = 0$$

$$T_{-2} = T_2 = T_2 - aT_1 - bT_0 - cT_{-1} = T_2 - aT_1 - bT_4 + abT_3 + b^2T_2 + bcT_1 - cT_1$$

(19) $$(bc - c - a)T_1 + b^2T_2 + abT_3 - bT_4 = 0$$

$$T_{-3} = T_3 = T_1 - aT_0 - bT_{-1} - cT_{-2} = T_1 - aT_4 + a^2T_3 + abT_2 + acT_1 - bT_1 - cT_2$$

(20) $$(ac - b + 1)T_1 + (ab - c)T_2 + (a^2 - 1)T_3 - aT_4 = 0$$

$$T_{-4} = T_4 = T_0 - aT_{-1} - bT_{-2} - cT_{-3} = T_4 - aT_3 - bT_2 - cT_1 - aT_1 - bT_2 - cT_3$$

(21) $$(a + c)T_1 + 2bT_2 + (a + c)T_3 = 0 .$$

If this set of four equations in T_1, T_2, T_3, T_4 is to have a non-zero solution, the determinant of the coefficients must be zero.

$$\begin{vmatrix} c^2 - b - 1 & bc - a & ac + 1 & -c \\ bc - c - a & b^2 & ab & -b \\ ac - b + 1 & ab - c & a^2 - 1 & -a \\ a + c & 2b & a + c & 0 \end{vmatrix} = 0$$

from which

(22) $$(a + b + c)(-a + b - c)(a^2 - c^2 + 4b) = 0 .$$

Before proceeding to further analysis some relations will be derived from equations (18) to (20). From (18) and (19)

(23) $$(c^2+ac-b^2-b)T_1 - abT_2 + bT_3 = 0 .$$

From (19) and (20)

(24) $$(b^2-b-ac-a^2)T_1 + bcT_2 + bT_3 = 0$$

and from (23) and (24)

(25) $$(c^2+a^2+2ac-2b^2)T_1 = b(a+c)T_2 .$$

THE CONDITION $a+b+c=0$

$b=-a-c$ substituted into (25) gives

$$(c^2+a^2+2ac-2c^2-2a^2-4ac)T_1 = -(a+c)^2T_2$$

so that $T_1 = T_2$. Then by (21)

$$(a+c)T_1 + 2(-a-c)T_1 + (a+c)T_3 = 0$$

so that $T_3 = T_1$. By (18),

$$(c^2+a+c-1-c^2-ac-a+ac+1)T_1 = cT_4$$

so that $T_4 = T_1$. If the terms up to T_n are all equal to T_1, then

$$T_{n+1} = aT_1 + (-a-c)T_1 + cT_1 + T_1 = T_1$$

so that all terms of the sequence are the same.

THE CONDITION $-a+b-c=0$

$b=a+c$ leads to

$$T_2 = -T_1, \quad T_3 = T_1, \quad T_4 = -T_1 .$$

If this alternation holds up to T_n, then

$$T_{n+1} = [a(-1)^{n-1} + (a+c)(-1)^n + c(-1)^{n-1} + (-1)^n]T_1 = (-1)^nT_1$$

so that the alternation continues.

THE CONDITION $a^2-c^2+4b=0$

a and c must be of the same parity.

EXAMPLE: $a = 1, \quad b = 12, \quad c = 7 .$

Using Eqs. (18), (19) and (20) we obtain:

$$36T_1 + 83T_2 + 8T_3 - 7T_4 = 0, \qquad 76T_1 + 144T_2 + 12T_3 - 12T_4 = 0, \qquad -4T_1 + 5T_2 + 0T_3 - T_4 = 0 .$$

from which $T_1 : T_2 : T_3 : T_4 = 3 : -7 : 18 : -47$.

Using the recursion relation

$$T_{n+1} = T_n + 12T_{n-1} + 7T_{n-2} + T_{n-3}$$

and a corresponding backward recursion relation, the following terms were obtained:

$$\cdots 843, -322, 123, -47, 18, -7, 3, -2, 3, -7, 18, -47, 123, -322, 843, \cdots .$$

Second-Order Factor

If the symmetry is to continue beyond a term T_{-n}, the condition for this would be:

$$T_{-n-1} = T_{n+1} = T_{-n+3} - aT_{-n+2} - bT_{-n+1} - cT_{-n} = T_{n-3} - aT_{n-2} - bT_{n-1} - cT_n .$$

But

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + T_{n-3} .$$

Hence there is a relation

$$(a+c)T_n + 2bT_{n-1} + (a+c)T_{n-2} = 0 .$$

But since $4b = (c-a)(c+a)$ we have in fact

$$T_n = (a-c)T_{n-1}/2 - T_{n-2} .$$

Thus if the symmetry is to continue the terms must satisfy a second-order recursion relation. That they do so can be seen from factoring

$$x^4 - ax^3 - bx - c - 1 = 0 \quad \text{into factors} \quad (x^2 + Ex + 1)(x^2 + Fx - 1) = 0,$$

where E is $(c-a)/2$. The conditions would be:

$$(c-a)/2 + F = -a \quad \text{or} \quad F = -(a+c)/2$$

from the coefficient of x cubed and the same value of F comes from the coefficient of x. Then the coefficient of x^2 would be:

$$EF = (-c^2 + a^2)/4 = -b$$

as required. Hence the terms obey this second-order relation and this insures the continuation of symmetry beyond T_{-4}. Note that this is not a proper fourth-order symmetric sequence.

B. No Zero Term

$$T_{n-3} = T_{n+1} - aT_n - bT_{n-1} - cT_{n-2}, \qquad T_{-1} = T_1 = T_4 - aT_3 - bT_2 - cT_1$$

(26) $$(c+1)T_1 + bT_2 + aT_3 - T_4 = 0$$

$$T_{-2} = T_2 = T_3 - aT_2 - bT_1 - cT_{-1}$$

(27) $$(b+c)T_1 + (a+1)T_2 - T_3 = 0$$

$$T_{-3} = T_3 = T_2 - aT_1 - bT_{-1} - cT_{-2}$$

(28) $$(a+b)T_1 + (c-1)T_2 + T_3 = 0$$

$$T_{-4} = T_4 = T_1 - aT_{-1} - bT_{-2} - cT_{-3}$$

(29) $$(a-1)T_1 + bT_2 + cT_3 + T_4 = 0.$$

To have a non-zero solution the following determinant must be zero.

$$\begin{vmatrix} c+1 & b & a & -1 \\ b+c & a+1 & -1 & 0 \\ a+b & c-1 & 1 & 0 \\ a-1 & b & c & 1 \end{vmatrix} = 0$$

or

(30) $$(a+b+c)(c^2 - a^2 - 4b) = 0.$$

As in the zero case, the condition $a + b + c = 0$ leads to a sequence where all terms are the same. The other condition requires that the fourth-order recursion relation have a second-order factor which the terms of the symmetric sequence must obey. Hence this is a degenerate case also.

Case II. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

A. Zero Term

$$T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n+1}$$

If the symmetry is to continue indefinitely

$$T_{-n-1} = aT_{-n+2} + bT_{-n+1} + cT_{-n} - T_{-n+3}$$

$$T_{n+1} = aT_{n-2} + bT_{n-1} + cT_n - T_{n-3} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3}$$

$$(a-c)(T_{n-2} - T_n) = 0$$

so that $a = c$ unless there is to be a recursion relation of lower order.

$$T_0 = aT_3 + bT_2 + aT_1 - T_4, \qquad T_{-1} = T_1 = aT_2 + bT_1 + aT_0 - T_3$$

from which

(31) $$(a^2 + b - 1)T_1 + a(1+b)T_2 + (a^2 - 1)T_3 = aT_4$$

$$T_{-2} = T_2 = aT_1 + b(aT_3 + bT_2 + aT_1 - T_4) + aT_{-1} - T_2$$

from which

(32) $$a(2+b)T_1 + (b^2 - 2)T_2 + abT_3 = bT_4.$$

Other relations simply repeat one of the above. Eliminating T_4 from (31) and (32):

(33) $$(b^2 - b - 2a^2)T_1 + a(b+2)T_2 - bT_3 = 0$$

For given a and b, a suitable selection of T_1 and T_2 will given an integral value for T_3. Thus for $a = 7, b = -5$,

$$-68T_1 - 21T_2 = -5T_3 .$$

$$T_1 = 1, \quad T_2 = 2, \quad T_3 = 22 .$$

Then from (31), $T_4 = 149$. The symmetric sequence:

$$\cdots 38494, 6029, 946, 149, 22, 2, 1, 2, 1, 2, 22, 149, 946, 6029, 38494, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 7T_n - 5T_{n-1} + 7T_{n-2} - T_{n-3} .$$

B. No Zero Term

As before the continuation of symmetry for all terms requires that $a = c$ in the relation

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} - T_{n-3} .$$

Two relations are obtained from the requirement $T_{-1} = T_1$ and $T_{-2} = T_2$, namely:

(34) $$(a-1)T_1 + bT_2 + aT_3 = T_4$$

(35) $$(b+a)T_1 + (a-1)T_2 = T_3$$

The relations for T_{-3} and T_{-4} repeat these in inverse order.

EXAMPLE: $a = -2, \quad b = 5, \quad -3T_1 - 3T_2 = T_3$

$$T_1 = 4, \quad T_2 = 7, \quad T_3 = -9 .$$

Then from (34), $T_4 = 41$.

The symmetric sequence:

$$\cdots 6399, -1810, 506, -145, 41, -9, 7, 4, 4, 7, -9, 41, -145, 506, -1810, 6399, \cdots$$

obeys the recursion relation:

$$T_{n+1} = -2T_n + 5T_{n-1} - 2T_{n-2} - T_{n-3}$$

FIFTH-ORDER SEQUENCES

Case I. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} + T_{n-4}$$

A. Zero Term

To insure symmetry for all n we set:

$$T_{-n-1} = T_{n+1} = T_{-n+4} - aT_{-n+3} - bT_{-n+2} - cT_{-n+1} - dT_{-n} = T_{n-4} - aT_{n-3} - bT_{n-2} - cT_{n-1} - dT_n .$$

Combining this with the original recursion relation:

$$(a+d)(T_n + T_{n-3}) + (b+c)(T_{n-1} + T_{n-2}) = 0$$

so that $d = -a$ and $b = -c$ are necessary conditions to prevent reduction to a lower order recurrence relation.

Using the same techniques as previously we have the relations:

(36) $$(a^2 + b - 1)T_1 + (ab - b)T_2 + (-ab - a)T_3 + (1 - a^2)T_4 + aT_5 = 0$$

(37) $$(ab - b + a)T_1 + (b^2 - a - 1)T_2 + (1 - b^2)T_3 - abT_4 + bT_5 = 0 .$$

Eliminating T_5 from (36) and (37) gives:

(38) $$(b^2 - b + ab - a^2)T_1 + (a^2 + a - b^2)T_2 + (-ab - a)T_3 + bT_4 = 0 .$$

EXAMPLE: $a = 5, \ b = -3$ from which

$$-28T_1 + 21T_2 + 10T_3 = 3T_4$$

which is satisfied by $T_1 = 1, T_2 = 3, T_3 = 4, T_4 = 25$. Then from (36)

$$21T_1 - 12T_2 + 10T_3 - 24T_4 = -5T_5$$

which gives $T_5 = 115.$

The sequence

$$\cdots 190299, 43060, 9745, 2203, 498, 115, 25, 4, 3, 1, -2, 1, 3, 4, 25, 115, 498, 2203, 9745, 43060, 190299, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

B. No Zero Term

An entirely similar analysis leads to two relations:

(39) $$T_5 = (1-a)T_1 - bT_2 + bT_3 + aT_4$$

(40) $$T_4 = (-b-a)T_1 + (b+1)T_2 + aT_3$$

EXAMPLE. $a = 5,\ b = -3.$ From (40),

$$T_4 = -2T_1 - 2T_2 + 5T_3$$

which is satisfied by $T_1 = 1,\ T_2 = 3,\ T_3 = 4,\ T_4 = 12.$

Then by (39), $T_5 = -4T_1 + 3T_2 - 3T_3 + 5T_4 = 53.$ The sequence

$$\cdots 19428, 4397, 995, 227, 53, 12, 4, 3, 1, 1, 3, 4, 12, 53, 227, 995, 4397, 19428, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 5T_n - 3T_{n-1} + 3T_{n-2} - 5T_{n-3} + T_{n-4} .$$

Case II. $$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} + dT_{n-3} - T_{n-4} .$$

In this case symmetry in the sequence requires that $a = d$ and $b = c.$

A. Zero Case

The final relations obtained from the analysis are:

(41) $$(a^2 + b - 1)T_1 + (ab + b)T_2 + (ab + a)T_3 + (a^2 - 1)T_4 = aT_5$$

(42) $$(ab + a + b)T_1 + (b^2 + a - 1)T_2 + (b^2 - 1)T_3 + abT_4 = bT_5$$

from which

(43) $$(b^2 - b - a^2 - ab)T_1 + (b^2 - a^2 + a)T_2 + (ab + a)T_3 = bT_4 .$$

EXAMPLE. $a = 3,\ b = -7.$ (43) becomes

$$68T_1 + 43T_2 - 18T_3 = -7T_4$$

which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = 9, \quad T_4 = -5 .$$

Then from (41),

$$T_1 - 28T_2 - 18T_3 + 8T_4 = 3T_5 \qquad \text{gives} \qquad T_5 = -95 .$$

The symmetric sequence:

$$\cdots 2203, -191, -305, -95, -5, 9, 3, 1, -1, 1, 3, 9, -5, -95, -305, -191, 2203, \cdots$$

is governed by the recursion relation:

$$T_{n+1} = 3T_n - 7T_{n-1} - 7T_{n-2} + 3T_{n-3} - T_{n-4}$$

B. No Zero Term

The relations obtained are:

(44) $$(a-1)T_1 + bT_2 + bT_3 + aT_4 = T_5$$

(45) $$(a+b)T_1 + (b-1)T_2 + aT_3 = T_4$$

(46) $$bT_1 + aT_2 = T_3$$

EXAMPLE. $a = -5,\ b = 7.$ (46) becomes $7T_1 - 5T_2 = T_3$ which is satisfied by

$$T_1 = 1, \quad T_2 = 3, \quad T_3 = -8 .$$

Then (45)

$$2T_1 + 6T_2 - 5T_3 = T_4 \qquad \text{gives} \qquad T_4 = 60..$$

Finally (44)
$$-6T_1 + 7T_2 + 7T_3 - 5T_4 = T_5$$
gives a value $T_5 = -341$. The symmetric sequence:
$$\cdots 72667, -12195, 2053, -341, 60, -8, 3, 1, 1, 3, -8, 60, -341, 2053, -12195, 72667, \cdots$$
is governed by the recursion relation:
$$T_{n+1} = -5T_n + 7T_{n-1} + 7T_{n-2} - 5T_{n-3} - T_{n-4}\ .$$

CONCLUSION

From this investigation the following general approach to creating symmetric sequences of integers governed by linear recursion relations emerges.

(1) Given a linear recursion relation of order k,
$$T_{n+1} = a_1T_n + a_2T_{n-1} + \cdots + a_{k-1}T_{n-k+2} + T_{n-k+1}$$
the condition of symmetry in the sequence requires that:
$$a_j = -a_{k-j}$$
and for the recursion relation:
$$T_{n+1} = a_1T_n + a_2T_{n-1} + \cdots + a_{k-1}T_{n-k+2} - T_{n-k+1}$$
symmetry requires that $a_j = a_{k-j}$.

(2) For the reduced number of parameters a_i, set up a corresponding number of symmetry conditions using the first few terms of the sequence.

(3) Using these conditions, select values for the parameters a_i and then find starting values in integers that satisfy the given conditions.

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