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A Note About Invariant Polynomial Transformations of Integer Sequences

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Abstract

We present an algorithm to find invariant poynomial transformations of integer sequences, using an approach based on classical invariant theory.

1 Introduction

Let $\mathcal{A} = (a_n)_{n\geq 0}$ be an integer sequence. A sequence $\mathbf{F}(\mathcal{A}) = \big(b_n = f_n(a_0, a_1, \ldots, a_m)\big)_{n\geq 0}$, where $f_n \in \mathbb{Z}[x_0, x_1, \ldots, x_m]$ and $m \geq n$, is called a polynomial transformation of the sequence $\mathcal{A}$. In the sequel, only the polynomial transformations are considered. The composition $\mathbf{F} \circ \mathbf{G} := \mathbf{F}(\mathbf{G}(\mathcal{A}))$ of the two transformations $\mathbf{F}$ and $\mathbf{G}$ can be defined in a natural way. A transformation $\mathbf{G}$ is called the *inverse* transformation of $\mathbf{F}$, and it is denoted by $\mathbf{F}^{-1}$, if for every sequence $\mathcal{A}$ we have $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathcal{A}$. A transformation $\mathbf{F}$ is called $\mathbf{G}$-*invariant* if for every sequence $\mathcal{A}$ we have $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathbf{F}(\mathcal{A})$.

For instance, it is well known (see Layman [1] or Spivey and Steil [2]) that the Hankel transformation $\mathbf{H}$ is $\mathbf{B}_\mu$-invariant. Here, for $\mu \in \mathbb{Q}$,

$$\mathbf{B}_\mu(\mathcal{A}) = \left(b_n = \sum_{i=0}^{n} \binom{n}{i} a_i \mu^{n-i} \right)_{n\geq 0},$$

denotes the μ-binomial transformation and $\mathbf{H}(\mathcal{A}) = (h_n)_{n\geq 0}$, where h_n is the determinant of the Hankel matrix for the elements $a_0, a_1, \ldots, a_{2n}$:

$$h_n = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & a_{n-1} & \cdots & a_{2n-1} \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{vmatrix}.$$

This determinant arises in classical invariant theory as the *catalecticant* of a binary form, see Grace and Young [4, p.232]. It was first introduced by Sylvester [5]. Also, the transformation $\mathbf{B}_\mu$ appears in Hilbert's book [7, p. 25]. We can prove that the Hankel transformation is $\mathbf{B}_\mu$-invariant via classical invariant theory, as follows. Write ∂_i for the partial derivative $\partial/\partial a_i$ and let $\mathcal{D}$ be the differential operator:

$$\mathcal{D} = a_0\partial_1 + 2a_1\partial_2 + 3a_2\partial_3 + \cdots + 2na_{2n-1}\partial_{2n}.$$

Define

$$h'_n = \begin{vmatrix} b_0 & b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & b_3 & \cdots & b_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & b_n & b_{n-1} & \cdots & b_{2n-1} \\ b_n & b_{n+1} & b_{n+2} & \cdots & b_{2n} \end{vmatrix}.$$

Then, as in Lie [6], we have

$$h'_n = h_n + \mathcal{D}(h_n)\mu + \mathcal{D}^2(h_n)\frac{\mu^2}{2!} + \cdots + \mathcal{D}^i(h_n)\frac{\mu^i}{i!} + \cdots .$$

By applying the determinant derivative rule we get, after some calculation, that $\mathcal{D}(h_n) = 0$ for all n. Therefore $h'_n = h_n$. This condition is exactly equivalent to the $\mathbf{B}_\mu$-invariance of the Hankel transformation.

This example motivates us to consider the following two general problems:

Problem 1. For a fixed transformation $\mathbf{F}$, find all $\mathbf{F}$-invariant transformations.

Problem 2. For a fixed transformation $\mathbf{F}$, find a transformation $\mathbf{G}$ such that $\mathbf{F}$ is a $\mathbf{G}$-invariant transformation.

The aim of this paper is to develop an effective method to solve these two problems for some special kinds of transformations. The method is inspired by results in classical invariant theory and the theory of locally nilpotent derivations.

In the next section we introduce exponential transformations and then prove that for such transformations Problem 1 is always solvable.

In Section 2 we also give a short introduction to the theory of locally nilpotent derivations and offer algorithms to solve Problems 1 and 2 for special classes of transformations.

In Section 3 we give another proof that the Hankel transformation is $\mathbf{B}_\mu$-invariant and introduce several new $\mathbf{B}_\mu$-invariant transformations. All of these transformations come from classical invariant theory. Further, we describe all $\mathbf{B}_\mu$-invariant polynomial transformations in terms of derivations.

In Section 4 we give some examples to illustrate the theory.

2 Derivations and automorphisms

Let $R = \mathbb{Z}[x_0, x_1, \ldots]$ be the polynomial ring in countably many variables and let $\varphi : R \to R$ be a ring homomorphism. Such a map φ is uniquely determined by the sequence of polynomials $(\varphi(x_n) \mid n = 0, 1, \ldots)$. To any polynomial map φ and an integer sequence $(a_n)_{n\geq 0}$ we assign the transformation $(\varphi(a_n))_{n\geq 0}$. A polynomial map φ is said to be a *polynomial automorphism* if there is a polynomial map ψ such that $\varphi(\psi(x_n)) = x_n$ for all n.

Denote by $\mathbb{Z}[x_0, x_1, \ldots, x_m]^\varphi$ the *algebra of φ-invariants*:

$$\mathbb{Z}[x_0, x_1, \ldots, x_m]^\varphi := \{f \in \mathbb{Z}[x_0, x_1, \ldots, x_m] \mid f(\varphi(x_0), \varphi(x_1), \ldots, \varphi(x_m)) = f(x_0, x_1, \ldots, x_m)\}\,.$$

The following theorem will be our main computing tool in finding invariant polynomial transformations.

Theorem 1. *Let φ be a polynomial map and let $\boldsymbol{F}(\mathcal{A}) = (\varphi(a_n))_{n\geq 0}$ be the corresponding integer transformation. Then the transformation*

$$\boldsymbol{G}(\mathcal{A}) = (g_n(a_0, a_1, \ldots, a_m))_{n\geq 0}\,,$$

is $\boldsymbol{F}$-invariant if and only if $g_n(x_0, a_1, \ldots, x_m) \in \mathbb{Z}[x_0, x_1, \ldots, x_m]^\varphi$.

The proof follows immediately from the definitions above.

In general, the problem of finding the algebras of φ-invariants is difficult. But we will show that when φ is an exponential automorphism this problem can be reduced to the calculation of the kernel of a derivation.

A *derivation* of the algebra $\mathbb{Z}[x_0, x_1, \ldots, x_n]$ is a linear map D satisfying the Leibniz rule:

$$D(f_1\, f_2) = D(f_1) f_2 + f_1 D(f_2), \text{ for all } f_1, f_2 \in \mathbb{Z}[x_0, x_1, \ldots, x_n].$$

A derivation D is called *locally nilpotent* if for every $f \in \mathbb{Z}[x_0, x_1, \ldots, x_n]$ there is an $r \in \mathbb{N}$ such that $D^r(f) = 0$. The subalgebra

$$\ker D := \{f \in \mathbb{Z}[x_0, x_1, \ldots, x_n] | D(f) = 0\}\,,$$

is called the *kernel* of the derivation D.

Any derivation D is completely determined by the elements $D(x_i)$. A derivation D is called *linear* if $D(x_i)$ is a linear form. A linear locally nilpotent derivation is called a *Weitzenböck derivation*. The Weitzenböck derivation defined by $\mathcal{D}(x_0) = 0, \mathcal{D}(x_i) = ix_{i-1}$ is called the *basic Weitzenböck derivation*. There exists an isomorphism between the kernel $\ker \mathcal{D}$ and the algebra of covariants of a binary form, a major object of research in classical invariant theory during the 19th century. Here are a few examples of covariants: the discriminant, the resultant, the Jacobian, the Hessian, the catalectiant and the transvectant. The following theorem gives a description of the algebra $\ker \mathcal{D}$.

Theorem 2. *The kernel of the basic Weitzenböck derivation $\mathcal{D}$ of $\mathbb{Q}[x_0, x_1, \ldots, x_n]$ is a finitely generated algebra and*

$$\ker \mathcal{D} = \mathbb{Q}[z_2, z_3, \ldots, z_n][x_0, x_0^{-1}] \cap \mathbb{Q}[x_0, x_1, \ldots, x_n],$$

where

$$z_k = \sum_{i=0}^{k-2} (-1)^i \binom{k}{i} x_{k-i} x_1^i x_0^{k-i-1} + (k-1)(-1)^{k+1} x_1^k.$$

Theorem 2 is a classical result due to Cayley, see Glenn [3, p. 164]. One can also find more modern proofs in Nowicki [8] and van den Essen [9].

How are we to find the kernel of an arbitrary linear locally nilpotent derivation D? Let us consider the vector space (over $\mathbb{Q}$) $X_n = \langle x_0, x_1, \ldots, x_n \rangle$. Suppose that there exists an isomorphism $\Psi : X_n \to X_n$ such that $\Psi \mathcal{D} = D \Psi$. This implies that $\ker D = \Psi(\ker \mathcal{D})$, i.e.,

$$\ker D = \mathbb{Q}[\Psi(z_2), \Psi(z_3), \ldots, \Psi(z_n)][\Psi(x_0), \Psi(x_0)^{-1}] \cap \mathbb{Q}[x_0, x_1, \ldots, x_n].$$

Such an isomorphism Ψ is called a $(\mathcal{D}, D)$-*intertwining* isomorphism. Therefore, to describe the kernel of an arbitrary Weitzenbök derivation D it is enough to know the explicit form of any $(\mathcal{D}, D)$-*intertwining* isomorphism.

An automorphism φ is called *exponential* if there exists a locally nilpotent derivation D such that

$$\varphi = \exp(D) = D^0 + D + \frac{1}{2!} D^2 + \cdots .$$

For instance, any automorphism of the form

$$\varphi(x_n) = x_n + f(x_0, x_1, \ldots, x_{n-1}), f \in \mathbb{Z}[x_0, x_1, \ldots, x_{n-1}],$$

is exponential, see Drensky and Yu [10]. Nowicki [8, Proposition 6.1.4] shows that for any exponential automorphism $\varphi = \exp(D)$,

$$\mathbb{Q}[x_0, x_1, \ldots, x_n]^\varphi = \ker D. \tag{1}$$

We introduce an analogue of these notions for the integer polynomial transformations.

Definition 3. The transformation $D(\mathbf{F}(\mathcal{A})) := (D(f_n(x_0, x_1, \ldots, x_m))|_{(a_0, a_1, \ldots, a_m)})_{n \geq 0}$, is called the D-derivative of the polynomial transformation $\mathbf{F} = (f_n(a_0, a_1, \ldots, a_m))_{n \geq 0}$, $f \in \mathbb{Z}[x_0, \ldots, x_m]$.

Definition 4. A transformation $\mathbf{F}$ is called exponential if there exists a locally nilpotent derivation D such that

$$\mathbf{F}(\mathcal{A}) = \exp D(\mathcal{A}).$$

We can now rewrite Theorem 1 for an exponential transformation.

Theorem 5. *Suppose a transformation $\boldsymbol{F}$ is exponential and $\mathbf{F}(\mathcal{A}) = \exp D(\mathcal{A})$ for some locally nilpotent derivation D. Then a polynomial transformation $\boldsymbol{G}$ is $\mathbf{F}$-invariant if and only if $D(\boldsymbol{G}(\mathcal{A})) = \mathbf{0}$, where $\mathbf{0}$ stands for the zero sequence* $(0, 0, 0, \ldots)$.

Proof. Suppose that the transformation $\mathbf{G}$ is $\mathbf{F}$-invariant. Then by Theorem 1 we have

$$\mathbf{G}(\mathcal{A}) = (g_n(a_0, a_1, \ldots, a_m))_{n\geq 0}\,,$$

where $g_n(x_0, x_1, \ldots, x_m) \in \mathbb{Z}[x_0, x_1, \ldots, x_m]^{\varphi}$, for $\varphi = \exp D$. Since the automorphism φ is exponential, we have that $D(g_n(x_0, x_1, \ldots, x_m)) = 0$ by (1). Thus $D(\mathbf{G}(\mathcal{A})) = 0$.

Suppose now that the transformation $\mathbf{G}$ has the form

$$\mathbf{G}(\mathcal{A}) = (g_n(a_0, a_1, \ldots, a_m))_{n\geq 0}\,,$$

and $D(\mathbf{G}(\mathcal{A})) = 0$. This implies that $D(g_n(a_0, a_1, \ldots, a_m)) = 0$ for all n. Then we have that the polynomial $g_n(x_0, x_1, \ldots, x_m)$ belongs to $\mathbb{Q}[x_0, x_1, \ldots, x_n]^{\varphi}$ where $\varphi = \exp D$. By Theorem 1 the transformation $\mathbf{G}$ is $\mathbf{F}$-invariant. □

The Weitzenbök derivations are related to some special transformations by the following theorem:

Theorem 6. *Given an integer sequence* $(\alpha_n)_{n\geq 0}$*, the transformation* $F(\mathcal{A})=\left(a_n + \sum\limits_{i=0}^{n-1}\alpha_i a_i\right)_{n\geq 0}$ *is exponential and* $F(\mathcal{A}) = \exp D(\mathcal{A})$*, where the derivation* D *is a Weitzenbök derivation defined by*

$$D(f) = \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}E^i(f),$$

and $E(x_k) = \sum\limits_{i=0}^{k-1}\alpha_k x_k$.

The proof follows from van den Essen [9, Proposition 2.1.3].

Theorem 6 yields an algorithm to solve Problem 1 in the case when the transformation $\mathbf{F}(\mathcal{A}) = (b_n)_{n\geq 0}$ has the special form

$$b_n = a_n + \sum_{i=0}^{n-1}\alpha_i a_i, \alpha_i \in \mathbb{Z}.$$

In this case, for the corresponding polynomial automorphism $\varphi(x_n) = x_n + \sum\limits_{i=0}^{n-1}\alpha_i x_i$, we find the explicit form of the Weitzenbök derivation D such that $\varphi = \exp(D)$ (as predicted in Theorem 5). After that we find any $(\mathcal{D}, D)$-intertwining automorphism Ψ and obtain that $\ker D = \Psi(\ker\mathcal{D})$. Then an arbitrary sequence of kernel elements defines an $\mathbf{F}$-invariant transformation (by Theorem 1).

To solve Problem 2 for a transformation $\mathbf{F}(\mathcal{A}) = (b_n = f_n(a_0, a_1, \ldots, a_m))_{n\geq 0}$ we find a locally nilpotent derivation D of $\mathbb{Z}[x_0, x_1, \ldots, x_m]$ such that $f_n(x_0, x_1, \ldots, x_m) \in \ker D$. This can be done by the method of undetermined coefficients. We define the automorphism $\varphi = \exp D$ and the transformation $\mathbf{G}(\mathcal{A}) = (b_n = \varphi(a_n))_{n\geq 0}$. Then the transformation $\mathbf{F}$ is $\mathbf{G}$-invariant by Theorem 1.

3 The μ-binomial transformations

We use the techniques developed in Section 2 to get another proof of the following well known result.

Theorem 7 ([1, 2])**.** *The Hankel transformation $\boldsymbol{H}$ is $\boldsymbol{B}_\mu$-invariant.*

Proof. We follow the algorithm from Section 2. The automorphism φ_μ corresponding to $\mathbf{B}_\mu$ has the form:

$$\varphi_\mu(x_n) = \sum_{i=0}^{n} \binom{n}{i} x_i \mu^{n-i}.$$

For the basic Weitzenbök derivation $\mathcal{D}$ we have

$$\exp(\mu\mathcal{D})(x_n) = \sum_{i\geq 0} \frac{1}{i!} \left(\mu\mathcal{D}\right)^i (x_n) = \sum_{i=0}^{n} \frac{n(n-1)\cdots(n-(i-1))}{i!} \mu^i x_{n-i} =$$
$$= \sum_{i=0}^{n} \binom{n}{i} \mu^i x_{n-i} = \sum_{i=0}^{n} \binom{n}{i} x_i \mu^{n-i}.$$

Thus $\varphi_\mu = \exp(\mu\mathcal{D})$. It follows that the transformation $\mathbf{B}_\mu$ is exponential, i.e., $\mathbf{B}_\mu = \exp(\mu\mathcal{D})\mathcal{A}$. Since the catalectiant belongs to the kernel of the derivation $\mathcal{D}$ we have that $\mathcal{D}(\mathbf{H}(\mathcal{A})) = \mathbf{0}$. Then by Theorem 5 we obtain that the transformation $\mathbf{H}$ is $\mathbf{B}_\mu$-invariant.

The map $\exp(\mu\mathcal{D}) : \mathbb{Q}[x_0, x_1, \ldots, x_n] \to \mathbb{Q}[x_0, x_1, \ldots, x_n]$ is a ring homomorphism, see van den Essen [9, Proposition 1.2.24]. It follows that $\varphi_{\mu_1+\mu_2} = \varphi_{\mu_1} \circ \varphi_{\mu_2}$. Therefore $\varphi_\mu \circ \varphi_{-\mu}$ is the identity map and $\mathbf{B}_\mu^{-1} = \mathbf{B}_{-\mu}$. It follows immediately that the inverse transformation $\mathbf{B}_\mu^{-1}$ is also an $\mathbf{H}$-invariant transformation. □

See French [11] for a proof that all $\mathbf{B}_\mu$-invariant transformations form a group. The identity $\varphi_{\mu_1+\mu_2} = \varphi_{\mu_1} \circ \varphi_{\mu_2}$ implies that the group $(\mathbb{Z}, +)$ is a subgroup of those groups.

The following theorem gives a solution to Problem 1 for the μ-binomial transformation.

Theorem 8. *A transformation $\boldsymbol{F}$ is $\boldsymbol{B}_\mu$-invariant if and only if $\mathcal{D}\left(\boldsymbol{F}(\mathcal{A})\right) = \mathbf{0}$.*

The proof follows from Theorem 5.

The next result follows from Bedratyuk [12, Theorem 3.2].

Theorem 9. *Let $\boldsymbol{F}$ be an arbitrary $\boldsymbol{B}_\mu$-invariant transformation. Then $\boldsymbol{F}(\boldsymbol{1}) = \mathbf{0}$, where $\boldsymbol{1} = (1, 1, 1, \ldots, 1, \ldots)$.*

Below we describe some Hankel-type transformations which arise in classical invariant theory. Note that all of these transformations are $\mathbf{B}_\mu$-invariant and $\mathbf{B}_\mu^{-1}$-invariant.

3.1 Cayley transformation

Put $\mathbf{CAYLEY}(\mathcal{A}) = (b_{n+2})_{n\geq 0}$, where

$$b_n = \sum_{i=0}^{n-2} (-1)^i \binom{n}{i} a_{n-i} a_1^i a_0^{n-k-1} + (n-1)(-1)^{n+1} a_1^n.$$

The definition of this transformation is inspired by Theorem 2.

3.2 Transvectant transformation

Let $\mathcal{A} = (a_n)_{n\geq 0}$ and $\mathcal{C} = (c_n)_{n\geq 0}$ be two sequences. The transformation $\mathbf{TR}(\mathcal{A}, \mathcal{C}) = (b_n)_{n\geq 0}$, where

$$b_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i c_{n-i},$$

is called the transvectant transformation. We have

$$\mathbf{Tr}(\mathbf{B}_\mu(\mathcal{A}), \mathbf{B}_\mu(\mathcal{C})) = \mathbf{Tr}(\mathcal{A}, \mathcal{C}).$$

In the case $\mathcal{C} = \mathcal{A}$ we get

$$b_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i a_{n-i}.$$

3.3 Resultant transformation

Let $\mathcal{A} = (a_n)_{n\geq 0}$ and $\mathcal{C} = (c_n)_{n\geq 0}$ be two sequences. The transformation $\mathbf{RES}(\mathcal{A}, \mathcal{C}) = (b_n)_{n\geq 0}$, where b_n is the leading coefficient of the resultant of the polynomials

$$P_n(\mathcal{A}) = \sum_{i=0}^{n} \binom{n}{i} a_i X^{n-i}, P_n(\mathcal{C}) = \sum_{i=0}^{n} \binom{n}{i} c_i X^{n-i},$$

is called the *resultant transformation*.

3.4 Discriminant transformation

The transformation $\mathbf{DISCR}(\mathcal{A}) = (b_n)_{n\geq 0}$, where b_n is the discriminant of the polynomial

$$P_{n+2}(\mathcal{A}) = \frac{1}{(n+2)^{n+2}} \sum_{i=0}^{n+2} a_i \binom{n+2}{i} X^{n+2-i},$$

is called the *discriminant transformation*.

Problem 3. What is the explicit form of the $(\mathcal{D}, D)$-intertwining isomorphism $\Psi(\mathbf{F})$ for

$$\mathbf{F} \in \{\mathbf{CAYLEY, H, RES, DISCRIM, TR}\}?$$

4 Examples

4.1 Transformation $\mathrm{PSUM}(\mathcal{A}) = (b_n = a_0 + a_1 + \cdots + a_n)_{n\geq 0}$

The corresponding locally nilpotent derivation (see Theorem 5) has the form

$$D(x_n) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} E^i(x_n).$$

We have

$$E(x_0)=0, E(x_n)=x_0+x_1+x_2+\cdots+x_{n-1},$$
$$E^2(x_n)=\sum_{i=0}^{n-1}E(x_i)=\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}x_j=\sum_{i=0}^{n-2}(n-1-i)x_i.$$

By induction we obtain $E^i(x_n)=\sum_{k=0}^{n-i}\binom{n-i-1}{i-1}x_k$. Then

$$D(x_n)=\sum_{i=1}^{n}\frac{(-1)^{i+1}}{i}\sum_{k=0}^{n-i}\binom{n-i-1}{i-1}x_k=\sum_{k=0}^{n-1}\left(\sum_{i=0}^{n-1-k}\frac{(-1)^i}{i+1}\binom{n-1-k}{i}\right)x_k=\sum_{k=0}^{n-1}\frac{x_k}{n-k}.$$

Let us find a $(\mathcal{D},D)$-intertwining transformation Ψ. We show that

$$\Psi(\mathcal{A})=\left\{\Psi(x_n)=\sum_{k=0}^{n}(-1)^{n+k}k!\left\{{n \atop k}\right\}x_k, \Psi(x_0)=x_0\right\},$$

where $\left\{{n \atop k}\right\}$ is the Stirling number of the second kind, is such a transformation. In fact,

$$D\left(\Psi(x_n)\right)=D\left(\sum_{k=0}^{n}(-1)^{n+k}k!\left\{{n \atop k}\right\}x_k\right)=\sum_{k=0}^{n}(-1)^{n+k}k!\left\{{n \atop k}\right\}\sum_{i=0}^{k-1}\frac{x_i}{k-i}$$
$$=\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}(-1)^{n+j}\left\{{n \atop j}\right\}\frac{j!}{j-i}\,x_i=n\sum_{i=0}^{n-1}(-1)^{n-1+i}\left\{{n-1 \atop i}\right\}i!\,x_i=\Psi(\mathcal{D}(x_n)).$$

Therefore, we may now construct a PSUM-invariant transformation using our known $\mathbf{B}_\mu$-invariant transformations and this $(\mathcal{D},D)$-intertwining transformation Ψ. For instance, the transformation

$$\Psi(\mathbf{H}(\mathcal{A}))=\{a_0,-{a_1}^2-a_1a_0+2\,a_2a_0,-4\,a_1a_2a_0+24\,a_1a_2a_3+24\,a_0a_1a_3+48\,a_0a_2a_4-8\,{a_2}^3-$$
$$-8\,a_0{a_2}^2-12\,a_1{a_2}^2-36\,a_0{a_3}^2-4\,{a_1}^2a_2-24\,{a_1}^2a_4+24\,{a_1}^2a_3-24\,a_0a_1a_4,\ldots\},$$

is PSUM-invariant.

4.2 The Transformation $\mathrm{SUM}(\mathcal{A})=(b_n=a_n+a_{n-1})_{n\geq 0}$

We have $\varphi(x_n)=x_n+x_{n-1}$, $E(x_n)=\varphi(x_n)-x_n=x_{n-1}$, and

$$D(x_n)=\sum_{i\geq 1}\frac{(-1)^{i+1}}{i}E^i(x_n)=\sum_{i=1}^{n}\frac{(-1)^{i+1}}{i}\,x_{n-i}.$$

Let

$$\Psi(x_0)=x_0, \Psi(x_n)=c_{n,1}x_1+c_{n,2}x_2+\cdots+c_{n,n}x_n.$$

The $(\mathcal{D}, D)$-intertwining map satisfies the conditions $D(\Psi(x_n)) = \Psi(\mathcal{D}(x_n))$, $n = 0, 1, 2, \ldots n$. After a routine calculation we get that $c_{n,i} = i! \left\{ {n \atop i} \right\}$ and a $(\mathcal{D}, D)$-intertwining map is given by

$$\Psi(x_n) = \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} x_i.$$

Thus, the transformation

$$\Psi(\mathbf{H}(\mathcal{A})) = \begin{vmatrix} \Psi(a_0) & \Psi(a_1) & \Psi(a_2) & \cdots & \Psi(a_n) \\ \Psi(a_1) & \Psi(a_2) & \Psi(a_3) & \cdots & \Psi(a_{n+1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Psi(a_{n-1}) & \Psi(a_n) & \Psi(a_{n-1}) & \cdots & \Psi(a_{2n-1}) \\ \Psi(a_n) & \Psi(a_{n+1}) & \Psi(a_{n+2}) & \cdots & \Psi(a_{2n}) \end{vmatrix}.$$

is SUM-invariant.

4.3 Transformation $\mathrm{DIFF}(\mathcal{A}) = (b_n = a_n - a_{n-1})_{n \geq 0}$

The corresponding automorphism has the form $\varphi(x_n) = x_n - x_{n-1}$. This implies that $E(x_n) = -x_{n-1}$ and $E^i(x_n) = (-1)^i x_{n-i}$. Then the derivation D is defined by

$$D(x_n) = \sum_{i \geq 1} \frac{(-1)^{i+1}}{i} E^i(x_n) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} (-1)^i x_{n-i} = -\sum_{i=1}^{n} \frac{x_{n-i}}{i}.$$

A $(\mathcal{D}, D)$-intertwining map has the form

$$\Psi(x_n) = \sum_{i=1}^{n} (-1)^i i! \left\{ {n \atop i} \right\} x_i.$$

4.4 The transformation $\mathbf{F} = (b_n = \sum_{i=0}^{2n} (-1)^i a_i a_{2n-i})_{n \geq 0}$.

Let us try to solve Problem 2 for this transformation. To do so, we need to find a suitable locally nilpotent derivation that satisfies the conditions

$$D\left(\sum_{i=0}^{2n} (-1)^i x_i x_{2n-i} \right) = 0, n = 0, 1, \ldots.$$

Let us consider the locally nilpotent derivation D with $D(x_i) = x_{i-1}$. Now

$$D\left(\sum_{i=0}^{2n} (-1)^i x_i x_{2n-i} \right) = 0,$$

as is shown in Bedratyuk [13]. Let us calculate the exponential automorphism $\varphi = \exp D$. We have

$$\varphi(x_n) = D^0(x_n) + D(x_n) + \frac{1}{2!} D^2(x_n) + \cdots = x_n + x_{n-1} + \frac{1}{2!} x_{n-2} + \frac{1}{n!} x_0.$$

Define a rational transformation by $\mathbf{G}(\mathcal{A}) := (\varphi(x_n))_{n \geq 0}$. Then $\mathbf{F}(\mathbf{G}(\mathcal{A})) = \mathbf{F}(\mathcal{A})$.

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