

# CONVERGENCE OF THE COEFFICIENTS IN A RECURRING POWER SERIES

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## 1. INTRODUCTION

In this paper we use the following notation

$$\left( \sum_{w=0}^{\infty} c_w x^w \right)^k = \sum_{w=0}^{\infty} c_w^{(k)} x^w ,$$

(For convenience, we shall write  $c_w$  instead of  $c_w^{(1)}$ .)

We define

$$\sum_{w=0}^f b_w x^w = F(x) \neq 0$$

for a finite  $f$ ,

$$\sum_{w=0}^t a_w x^w = \prod_{w=1}^m (1 - r_w x)^{d_w} = Q(x)$$

for finite  $t$  and  $m$ , where the  $d_w \neq 0$  and are positive integers. The  $r_w \neq 0$  and are distinct and we say  $|r_1|$  is the greatest  $|r|$  in the  $|r_w|$ .

## 2. THEOREM 1

If

$$F(x)/Q(x) = \sum_{w=0}^{\infty} u_w x^w ,$$

then

$$(2.1) \quad \lim_{n \rightarrow \infty} \left| \frac{u_n}{u_{n-j}} \right| \quad (\text{for a finite } j = 0, 1, 2, \dots)$$

converges to  $|r_1^j|$ , where the  $r_w \neq 0$  in  $Q(x)$  are distinct with distinct moduli and  $|r_1|$  is the greatest  $|r|$  in the  $|r_w|$ .

Proof. It has been shown by Poincaré [1] that

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{u_n}{u_{n-1}}$$

converges to some root  $(r)$  in  $Q(x)$ . (We must then prove that this root  $(r)$  in  $Q(x)$  is  $|r_1|$ .)

Let

$$(2.3) \quad M(x) = \prod_{w=1}^m (1 - r_w x)^{p_w},$$

where the  $p_w$  are positive integers or  $=0$  and

$$d_1 + p_1 = d_2 + p_2 = \dots = p_w + d_w = k \quad (k = 1, 2, 3, \dots)$$

for a finite  $w = 1, 2, 3, \dots, m$ .

Then,

$$M(x)Q(x) = \prod_{w=1}^m (1 - r_w x)^k = \phi_k(x),$$

so that

$$(2.4) \quad F(x)M(x)/Q(x)M(x) = F(x)M(x)/\phi_k(x)$$

$$= \sum_{w=0}^{\infty} u_w x^w = \sum_{w=0}^{\infty} c(k, w) x^w,$$

where it is evident

$$u_n = c(k, n) .$$

Now let

$$\phi_k(x) = \sum_{w=0}^v c_w^{(k)} x^w \quad (\text{where } v \text{ is finite}) ,$$

where combining this with (2.4), we write

$$(2.5) \quad F(x)M(x)/\phi_{k-1}(x) = \sum_{w=0}^{\infty} c(k-1, w)x^w \\ = \left( \sum_{w=0}^v c_w^{(k-1)} x^w \right) \left( \sum_{w=0}^{\infty} c(k, w)x^w \right) ,$$

and combining coefficients leads to

$$(2.5.1) \quad c(k-1, n) = \sum_{w=0}^v c(k, n-w)c_w^{(k-1)} = \sum_{w=0}^v u_{n-w} c_w^{(k-1)} , \\ k = 2, 3, 4, \dots .$$

In (2.5.1), we replace  $k$  with  $k+1$  (where  $k = 1, 2, 3, \dots$ ) where combining this result with (2.2) leads to

$$\lim_{n \rightarrow \infty} |c(k+1, n)/c(k+1, n-1)| \text{ converges to some root } (r) \text{ in } Q(x).$$

For convenience, we write the convergence as

$$(2.5.2) \quad c(k+1, n) = g_{k+1} c(k+1, n-1) .$$

Combining (2.5.1) with  $k$  replaced by  $k + 1$  with (2.5.2), it is easily shown, that for a finite  $v$ , we have

$$\begin{aligned}
 (2.5.3) \quad c(k, n)/c(k, n - 1) &= g_k \\
 &= \sum_{w=0}^v c(k + 1, n - w)c_w / \sum_{w=0}^v c(k + 1, n - w - 1)c_w \\
 &= g_{k+1} \quad ,
 \end{aligned}$$

so that

$$(2.5.4) \quad g_{k+1} = g_k = \cdots = g_1 .$$

Thus to complete the proof of Theorem 1, it remains to show that

$$|g_1| = |r_1| .$$

Then we consider the following (we refer to (2.3) )

$$(2.6) \quad (\phi(x))^{-1} = \prod_{w=1}^m (1 - r_w x)^{-1} = \sum_{w=0}^{\infty} e(m, w)x^w \quad (\text{for a finite } m)$$

for the convergence properties of  $e(m, n)/e(m, n - 1)$ , where the  $|r_w|$  are distinct and  $|r_1|$  is the greatest root.

NOTE. For convenience, we write

$$e(m, n)/e(m, n - j) = r_1^j \quad (\text{for a finite } j = 0, 1, 2, \cdots) ,$$

in place of

$$\lim_{n \rightarrow \infty} |e(m, n)/e(m, n - j)| \text{ converges to } |r_1^j| .$$

For  $m = 1$ , we have

$$(2.7) \quad (1 - r_1x)^{-1} = \sum_{w=0}^{\infty} e(1, w) x^w,$$

where

$$e(1, n) = r_1^n,$$

so that

$$e(1, n)/e(1, n - j) = r_1^j.$$

For  $m = 2$ , we have

$$(2.8) \quad [(1 - r_1x)(1 - r_2x)]^{-1} = \sum_{w=0}^{\infty} e(2, w) x^w.$$

where

$$e(2, n) = (r_1^{n+1} - r_2^{n+1})/(r_1 - r_2),$$

so that

$$e(2, n)/e(2, n - j) = r_1^j.$$

It now remains to consider for finite  $m = 3, 4, 5, \dots$ , let

$$(2.9) \quad \left(1 - \sum_{s=0}^{t-1} a_s x^{t-s}\right)^{-1} = \prod_{s=1}^t (1 - r_s x)^{-1} = 1 + \sum_{s=1}^{\infty} U_s x^s,$$

for a finite  $t = 3, 4, 5, \dots$ , where  $U_0 = 1$ .

Equating the coefficients in this leads to

$$(2.10) \quad U_n = \sum_{s=1}^t a_{t-s} U_{n-s} \quad (U_0 = 1) ,$$

and

$$U_1 = U_0 a_{t-1} , \quad U_2 = U_1 a_{t-1} + U_0 a_{t-2}, \dots, \quad U_t = \sum_{s=0}^{t-1} U_s a_s .$$

Also, since in (2.9), we have

$$\prod_{s=1}^t (1 - r_s x) = 1 - \sum_{s=0}^{t-1} a_s x^{t-s} ,$$

we may write

$$(2.11) \quad \prod_{s=1}^t (x - r_s) = x^t - \sum_{s=0}^{t-1} a_s x^s = 0 .$$

We now combine (2.10) with (2.11) and write

$$(2.12) \quad x^t = U_1 x^{t-1} + \sum_{s=2}^t \left( U_s - \sum_{r=1}^{s-1} U_r a_{t+r-s} \right) x^{t-s} .$$

Multiplying (2.12) by  $x$  and combining the result with

$$U_1 x^t = U_1 \sum_{s=0}^{t-1} a_s x^s$$



Next, we consider the  $t$  equations obtained from (2.16). These  $t$  equations in the  $t$  unknown can be solved by Cramer's rule to obtain

$$(2.17) \quad U_n D_2 = D_1(n) ,$$

where  $D_1(n)$  and  $D_2$  are the determinants given below:

$$(2.18) \quad D_1(n) = \begin{vmatrix} r_1^{t+n-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_t^{t+n-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix}$$

$$(2.19) \quad D_2 = \begin{vmatrix} r_1^{t-1} & r_1^{t-2} & \cdots & r_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_t^{t-1} & r_t^{t-2} & \cdots & r_t & 1 \end{vmatrix}$$

We now replace  $n$  with  $n - 1$  in (2.17) to get

$$(2.20) \quad U_{n-1} D_2 = D_1(n-1) ,$$

and dividing (2.17) by (2.20), we get

$$(2.21) \quad U_n / U_{n-1} = D_1(n) / D_1(n-1) .$$

Since the  $r_t \neq 0$  and are distinct, then one root (say  $|r_1|$  is greater than the other roots, and we write

$$(2.22) \quad U_n / U_{n-1} = (D_1(n) / r_1^{t+n-2}) / (D_1(n-1) / r_1^{t+n-2})$$

Now in (2.22) we let  $r_1^{t+n-2}$  (in the numerator) divide every term of the first column in (2.18) and  $r_1^{t+n-2}$  (in the denominator) divide every term in the first column of (2.18) (with  $n$  replaced by  $n - 1$ ). Then if we let  $n \rightarrow \infty$  it is evident that



$$(2.23) \quad \lim_{n \rightarrow \infty} \left| U_n / U_{n-1} \right| = |r_1| .$$

Now for a finite  $t$  we write

$$\lim_{n \rightarrow \infty} \left| U_{n-j} / U_{n-j-1} \right| = |r_1| \quad (j = 0, 1, 2, \dots, t-1) ,$$

so that

$$(2.24) \quad \lim_{n \rightarrow \infty} \left| U_n / U_{n-t} \right| = |r_1|^t .$$

Multiplying the  $F(x)$  in (1) with

$$\sum_{s=0}^{\infty} U_s x^s$$

in (2.9), we write

$$(2.25) \quad \left( \sum_{w=0}^f b_w x^w \right) \left( \sum_{s=0}^{\infty} U_s x^s \right) = \sum_{s=0}^{\infty} C_s x^s ,$$

where comparing the coefficients we have

$$(2.26) \quad C_n = \sum_{s=0}^f U_{n-s} b_s .$$

Now, since  $f$  is finite, and by the results in (2.23), we write

$$C_n = r_1 \sum_{s=0}^f U_{n-s-1} b_s = r_1 C_{n-1} ,$$

where combining this with the  $r_t \neq 0$  and are distinct (so that we may add that the  $r_t$  have distinct moduli), leads to the completion of the proof for Theorem 1.

From (2.7), (2.8), and (2.17), the following corollary is immediate:

Corollary. If

$$\prod_{s=1}^t (1 - r_s x)^{-1} = \sum_{s=0}^{\infty} U_s x^s \quad (U_0 = 1),$$

where the  $r_s \neq 0$  and are distinct, then

(2.27) It is always possible to solve for the  $U_n$  ( $n = 0, 1, 2, \dots$ ) as a function of the  $r_s$ .

### SECTION 3

Let

$$\left(1 - \sum_{w=1}^t a_w x^w\right)^{-k} = \prod_{w=1}^t (1 - r_w x)^{-k} = \sum_{w=0}^{\infty} c_w^{(k)} x^w$$

( $c_0^{(k)} = 1$  and  $k = 1, 2, 3, \dots$ ) for a finite  $t = 2, 3, 4, \dots$  and the given roots  $r_w \neq 0$  and are distinct. We also define

$$S(x) = \sum_{w=1}^t \sum_{r=w}^t a_r c_{n+w-r} x^{w-1} = 0$$

and

$$b = \sum_{w=2}^t a_w x_1^{w-2}$$

where  $x_1 \neq 0$  and is a root in  $S(x) = S(x_1) = 0$ .

We then have the following:

Theorem 2. If

$$c_0 = 1, \quad c_1 = a_1 c_0, \quad c_2 = a_1 c_1 + a_2 c_0, \quad \dots$$

$$\dots, \quad c_t = \sum_{w=0}^{t-1} a_{w+1} c_{t-w-1}$$

and

$$p_j = a_1(k+n-j) \quad (j = 1, 2, 3, \dots, n),$$

$$q_{m+1} = b(n-m)(2k+n-m-1) \\ (m = 1, 2, 3, \dots, n-1)$$

then

$$(3.1) \quad n c_n^{(k)} / c_{n-1}^{(k)} = E_n / G_n \quad (k, n = 1, 2, 3, \dots),$$

where  $E_n$  and  $G_n$  are the determinants given below.

$$(3.1.1) \quad E_n = \begin{vmatrix} p_1 & q_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & p_2 & q_3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & p_3 & q_4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_4 & q_5 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & p_n \end{vmatrix}$$

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\*It should be noted that since the  $a$ 's are constant for a fixed  $t$ , that the root  $x_1$  will be determined as a variable, since it is a function of the  $c_n$  and will, of course, change values for different  $n$ .

$$(3.1.2) \quad G_n = \begin{pmatrix} p_2 & q_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & p_3 & q_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p_4 & q_5 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & p_5 & q_6 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & p_{n-1} & q_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & p_n \end{pmatrix}$$

Proof. Let

$$(3.2) \quad 1 = \left( 1 - \sum_{w=1}^t a_w x^w \right) \left( \sum_{w=0}^n c_w x^w \right) \quad (\text{for a finite } n),$$

where the  $a_w$  and the  $c_w$  are identical to those in (3). Then multiplying and combining the terms in (3.2) leads to  $S(x_1) = S(x) = 0$  in (3).

Now, taking each side of (3.2) to the  $k^{\text{th}}$  power, we write

$$(3.3) \quad 1^k = \left( 1 - \sum_{w=1}^t a_w x^w \right)^k \left( \sum_{w=0}^n c_w^{(k)} x^w + J(x) \right) \quad (k = 2, 3, \dots),$$

(where, of course,  $x_1$  is a root in (3.3)).

Using the corresponding values in (3), we write (3.3) as

$$(3.3.1) \quad 1 = (1 - a_1 x - bx^2)^k \left( \sum_{w=0}^n c_w^{(k)} x^w + J(x) \right)$$

Differentiation of (3.3.1) leads to

$$k(a_1 x + 2bx^2) \left( \sum_{w=0}^n c_w^{(k)} x^w + J(x) \right) = (1 - a_1 x - bx^2) \left( \sum_{w=1}^n n c_w^{(k)} x^{w-1} + W(x) \right)$$

and by comparing coefficients, we conclude that

$$(3.4) \quad nc_n^{(k)} = a_1(k+n-1)c_{n-1}^{(k)} + b(2k+n-2)c_{n-2}^{(k)}$$

for

$$k = 2, 3, \dots, \quad n = 2, 3, \dots, \quad c_0^{(k)} = 1 \quad \text{and} \quad c_1^{(k)} = a_1k.$$

When we divide (3.4) by  $c_{n-1}^{(k)}$ , we get

$$\frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = a_1(k+n-1) + \frac{b(2k+n-2)(n-1)}{\frac{(n-1)c_{n-1}^{(k)}}{c_{n-2}^{(k)}}} \quad (n, k = 2, 3, \dots),$$

which in turn, along with  $c_0^{(k)} = 1$  and  $c_1^{(k)} = a_1k$ , implies (along with the values of  $p$  and  $q$  in (3)),

$$(3.5) \quad \frac{nc_n^{(k)}}{c_{n-1}^{(k)}} = p_1 + \frac{q_2}{p_2} + \frac{q_3}{p_3} + \dots + \frac{q_{n-1}}{p_{n-1}} + \frac{q_n}{p_n} = K(n).$$

We complete the proof of Theorem 2 with Euler's statement [2]

$$K(n) = E_n / G_n;$$

and we resolve for the case when  $k = 1$  with (2.27).

Corollary. In

$$\prod_{w=1}^t (1 - r_w x)^{-k} = \left( 1 - \sum_{w=1}^t a_w x^w \right)^{-k} = 1 + \sum_{w=1}^{\infty} c_w^{(k)} x^w,$$

it is always possible to solve for

$$(3.6) \quad n c_n^{(k)} / c_{n-1}^{(k)} = K(n) = E_n / G_n \quad (k \text{ and } n = 2, 3, \dots)$$

when  $t = 2, 3, 4$ , or  $5$ , if the  $r_w \neq 0$  and are distinct.

Proof. In (2.27), it is seen that the  $c_n$  may be determined. Now, since  $t - 1 = 1, 2, 3$ , or  $4$ , then the roots (each root is a function of the  $c_n$ ) in  $S(x)$  (in 3) may always be found, so that we will obtain values for the  $p$  and  $q$ . We then complete the proof of the corollary by observing that  $E_n$  and  $G_n$  are both functions of the  $p$  and  $q$ .

In conclusion: We solve when  $t = 1$  and we write

$$(1 - r_x)^{-k} = \sum_{w=0}^{\infty} d_w^{(k)} x^w \quad (d_0^{(k)} = 1, r \neq 0) .$$

Now, differentiating, we have

$$xkr \left( \sum_{w=0}^{\infty} d_w^{(k+1)} x^w \right) = \sum_{w=1}^{\infty} w d_w^{(k)} x^w$$

and comparing the coefficients leads to

$$n d_n^{(k)} = d_{n-1}^{(k+1)} r^k$$

so that

$$\prod_{w=1}^n w d_w^{(k+n-w)} = r^n \prod_{w=0}^{n-1} (k+n-w-1) d_w^{(k+n-w)}$$

and we then have

$$d_n^{(k)} = r^n (k+n-1)! / n! (k-1)! .$$

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[Continued from p. 40.]

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