

# q-Special functions, a tutorial

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**Abstract** A tutorial introduction is given to  $q$ -special functions and to  $q$ -analogues of the classical orthogonal polynomials, up to the level of Askey-Wilson polynomials.

## 0. Introduction

It is the purpose of this paper to give a tutorial introduction to  $q$ -hypergeometric functions and to orthogonal polynomials expressible in terms of such functions. An earlier version of this paper was written for an intensive course on special functions aimed at Dutch graduate students, it was elaborated during seminar lectures at Delft University of Technology, and later it was part of the lecture notes of my course on “Quantum groups and  $q$ -special functions” at the European School of Group Theory 1993, Trento, Italy.

I now describe the various sections in some more detail. Section 1 gives an introduction to  $q$ -hypergeometric functions. The more elementary  $q$ -special functions like  $q$ -exponential and  $q$ -binomial series are treated in a rather self-contained way, but for the higher  $q$ -hypergeometric functions some identities are given without proof. The reader is referred, for instance, to the encyclopedic treatise by Gasper & Rahman [15]. Hopefully, this section succeeds to give the reader some feeling for the subject and some impression of general techniques and ideas.

Section 2 gives an overview of the classical orthogonal polynomials, where “classical” now means “up to the level of Askey-Wilson polynomials” [8]. The section starts with the “very classical” situation of Jacobi, Laguerre and Hermite polynomials and next discusses the Askey tableau of classical orthogonal polynomials (still for  $q = 1$ ). Then the example of big  $q$ -Jacobi polynomials is worked out in detail, as a demonstration how the main formulas in this area can be neatly derived. The section continues with the  $q$ -Hahn tableau and then gives a self-contained introduction to the Askey-Wilson polynomials. Both sections conclude with some exercises.

*Notation* The notations  $\mathbb{N} := \{1, 2, \dots\}$  and  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$  will be used.

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## 1. Basic hypergeometric functions

A standard reference for this section is the book by Gasper and Rahman [15]. In particular, the present section is quite parallel to their introductory Chapter 1. See also the useful compendia of formulas in the Appendices to that book. The foreword to [15] by R. Askey gives a succinct historical introduction to the subject. For first reading on the subject I can also recommend Andrews [2].

**1.1. Preliminaries.** We start with briefly recalling the definition of the general hypergeometric series (see Erdélyi e.a. [13, Ch. 4] or Bailey [11, Ch. 2]). For  $a \in \mathbb{C}$  the *shifted factorial* or *Pochhammer symbol* is defined by  $(a)_0 := 1$  and

$$(a)_k := a(a+1)\dots(a+k-1), \quad k = 1, 2, \dots$$

The general *hypergeometric series* is defined by

$${}_rF_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k k!} z^k. \quad (1.1)$$

Here  $r, s \in \mathbb{Z}_+$  and the upper parameters  $a_1, \dots, a_r$ , the lower parameters  $b_1, \dots, b_s$  and the argument  $z$  are in  $\mathbb{C}$ . However, in order to avoid zeros in the denominator, we require that  $b_1, \dots, b_s \notin \{0, -1, -2, \dots\}$ . If, for some  $i = 1, \dots, r$ ,  $a_i$  is a non-positive integer then the series (1.1) is terminating. Otherwise, we have an infinite power series with radius of convergence equal to 0, 1 or  $\infty$  according to whether  $r - s - 1 > 0, = 0$  or  $< 0$ , respectively. On the right hand side of (1.1) we have that

$$\frac{(k+1)\text{th term}}{k\text{th term}} = \frac{(k+a_1)\dots(k+a_r)z}{(k+b_1)\dots(k+b_s)(k+1)} \quad (1.2)$$

is rational in  $k$ . Conversely, any rational function in  $k$  can be written in the form of the right hand side of (1.2). Hence, any series  $\sum_{k=0}^{\infty} c_k$  with  $c_0 = 1$  and  $c_{k+1}/c_k$  rational in  $k$  is of the form of a hypergeometric series (1.1).

The cases  ${}_0F_0$  and  ${}_1F_0$  are elementary: exponential resp. binomial series. The case  ${}_2F_1$  is the familiar Gaussian hypergeometric series, cf. [13, Ch. 2].

We next give the definition of basic hypergeometric series. For  $a, q \in \mathbb{C}$  define the *q-shifted factorial* by  $(a; q)_0 := 1$  and

$$(a; q)_k := (1-a)(1-aq)\dots(1-aq^{k-1}), \quad k = 1, 2, \dots$$

For  $|q| < 1$  put

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1-aq^k).$$

We also write

$$(a_1, a_2, \dots, a_r; q)_k := (a_1; q)_k (a_2; q)_k \dots (a_r; q)_k, \quad k = 0, 1, 2, \dots \text{ or } \infty.$$

Then a *basic hypergeometric series* or *q-hypergeometric series* is defined by

$$\begin{aligned} {}_r\phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right] &= {}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &:= \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s, q; q)_k} \left( (-1)^k q^{k(k-1)/2} \right)^{1+s-r} z^k, \quad r, s \in \mathbb{Z}_+. \end{aligned} \quad (1.3)$$

On the right hand side of (1.3) we have that

$$\frac{(k+1)\text{th term}}{k\text{th term}} = \frac{(1 - a_1 q^k) \dots (1 - a_r q^k) (-q^k)^{1+s-r} z}{(1 - b_1 q^k) \dots (1 - b_s q^k) (1 - q^{k+1})} \quad (1.4)$$

is rational in  $q^k$ . Conversely, any rational function in  $q^k$  can be written in the form of the right hand side of (1.4). Hence, any series  $\sum_{k=0}^{\infty} c_k$  with  $c_0 = 1$  and  $c_{k+1}/c_k$  rational in  $q^k$  is of the form of a *q-hypergeometric series* (1.3). This characterization is one explanation why we allow  $q$  raised to a power quadratic in  $k$  in (1.3).

Because of the easily verified relation

$$(a; q^{-1})_k = (-1)^k a^k q^{-k(k-1)/2} (a^{-1}; q)_k,$$

any series (1.3) can be transformed into a series with base  $q^{-1}$ . Hence, it is sufficient to study series (1.3) with  $|q| \leq 1$ . The tricky case  $|q| = 1$  has not yet been studied much and will be ignored by us. Therefore we may assume that  $|q| < 1$ . In fact, for convenience we will always assume that  $0 < q < 1$ , unless it is otherwise stated.

In order to have a well-defined series (1.3), we require that

$$b_1, \dots, b_s \neq 1, q^{-1}, q^{-2}, \dots.$$

The series (1.3) will terminate iff, for some  $i = 1, \dots, r$ , we have  $a_i \in \{1, q^{-1}, q^{-2}, \dots\}$ . If  $a_i = q^{-n}$  ( $n = 0, 1, 2, \dots$ ) then all terms in the series with  $k > n$  will vanish. In the non-vanishing case, the convergence behaviour of (1.3) can be read off from (1.4) by use of the ratio test. We conclude:

$$\text{convergence radius of (1.3)} = \begin{cases} \infty & \text{if } r < s + 1, \\ 1 & \text{if } r = s + 1, \\ 0 & \text{if } r > s + 1. \end{cases}$$

We can view the  $q$ -shifted factorial as a  $q$ -analogue of the shifted factorial by the limit formula

$$\lim_{q \uparrow 1} \frac{(q^a; q)_k}{(1 - q)^k} = (a)_k := a(a+1) \dots (a+k-1).$$

Hence  ${}_r\phi_s$  is a  $q$ -analogue of  ${}_rF_s$  by the formal (termwise) limit

$$\lim_{q \uparrow 1} {}_r\phi_s \left[ \begin{matrix} q^{a_1}, \dots, q^{a_r} \\ q^{b_1}, \dots, q^{b_s} \end{matrix}; q, (q-1)^{1+s-r} z \right] = {}_rF_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right]. \quad (1.5)$$

However, we get many other  $q$ -analogues of  ${}_rF_s$  by adding upper or lower parameters to the left hand side of (1.5) which are equal to 0 or which depend on  $q$  in such a way that they tend to a limit  $\neq 1$  as  $q \uparrow 1$ . Note that the notation (1.3) has the drawback that some rescaling is necessary before we can take limits for  $q \rightarrow 1$ . On the other hand, parameters can be put equal to zero without problem in (1.3), which would not be the case if we worked with  $a_1, \dots, a_r, b_1, \dots, b_s$  as in (1.5). In general,  $q$ -hypergeometric series can be studied in their own right, without much consideration for the  $q = 1$  limit case. This philosophy is often (but not always) reflected in the notation generally in use.

It is well known that the confluent  ${}_1F_1$  hypergeometric function can be obtained from the Gaussian  ${}_2F_1$  hypergeometric function by a limit process called *confluence*. A similar phenomenon occurs for  $q$ -series. Formally, by taking termwise limits we have

$$\lim_{a_r \rightarrow \infty} {}_r\phi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, \frac{z}{a_r} \right] = {}_{r-1}\phi_s \left[ \begin{matrix} a_1, \dots, a_{r-1} \\ b_1, \dots, b_s \end{matrix}; q, z \right]. \quad (1.6)$$

In particular, this explains the particular choice of the factor  $((-1)^k q^{k(k-1)/2})^{1+s-r}$  in (1.3). If  $r = s + 1$  this factor is lacking and for  $r < s + 1$  it is naturally obtained by the confluence process. The proof of (1.6) is by the following lemma.

**Lemma 1.1** Let the complex numbers  $a_k, k \in \mathbb{Z}_+$ , satisfy the estimate  $|a_k| \leq R^{-k}$  for some  $R > 0$ . Let

$$F(b; q, z) := \sum_{k=0}^{\infty} a_k (b; q)_k z^k, \quad |z| < R, \quad b \in \mathbb{C}, \quad 0 < q < 1.$$

Then

$$\lim_{b \rightarrow \infty} F(b; q, z/b) = \sum_{k=0}^{\infty} a_k (-1)^k q^{k(k-1)/2} z^k, \quad (1.7)$$

uniformly for  $z$  in compact subsets of  $\mathbb{C}$ .

**Proof** For the  $k$ th term of the series on the left hand side of (1.7) we have

$$\begin{aligned} c_k &:= a_k (b; q)_k (z/b)^k \\ &= a_k (b^{-1} - 1)(b^{-1} - q) \dots (b^{-1} - q^{k-1}) z^k. \end{aligned}$$

Now let  $|b| > 1$  and let  $N \in \mathbb{N}$  be such that  $q^{N-1} > |b|^{-1} \geq q^N$ . If  $k \leq N$  then

$$|c_k| \leq |a_k| q^{k(k-1)/2} (2|z|)^k.$$

If  $k > N$  then

$$\begin{aligned} |(b^{-1} - 1)(b^{-1} - q) \dots (b^{-1} - q^{k-1})| &\leq 2^k q^{N(N-1)/2} |b|^{N-k} \\ &\leq 2^k |b|^{(1-N)/2} |b|^{N-k} \\ &\leq 2^k |b|^{-k/2}, \end{aligned}$$

so

$$|c_k| \leq |a_k| (2|z| |b|^{-1/2})^k.$$

Now fix  $M > 0$ . Then, for each  $\varepsilon > 0$  we can find  $K > 0$  and  $B > 1$  such that

$$\sum_{k=K}^{\infty} |c_k| < \varepsilon \quad \text{if } |z| < M \text{ and } |b| > B.$$

Combination with the termwise limit result completes the proof. □

It can be observed quite often in literature on  $q$ -special functions that no rigorous limit proofs are given, but only formal limit statements like (1.6) and (1.5). Sometimes, in heuristic reasonings, this is acceptable and productive for quickly finding new results. But in general, I would say that rigorous limit proofs have to be added.

Any terminating power series

$$\sum_{k=0}^n c_k z^k$$

can also be written as

$$z^n \sum_{k=0}^n c_{n-k} (1/z)^k.$$

When we want to do this for a terminating  $q$ -hypergeometric series, we have to use that

$$(a; q)_{n-k} = \frac{(a; q)_n}{(q^{n-1}a; q^{-1})_k} = (a; q)_n \frac{(-1)^k q^{k(k-1)/2} (a^{-1}q^{1-n})^k}{(q^{1-n}a^{-1}; q)_k}$$

and

$$\frac{(q^{-n}; q)_{n-k}}{(q; q)_{n-k}} = \frac{(q^{-n}; q)_n}{(q; q)_n} \frac{(q^n; q^{-1})_k}{(q^{-1}; q^{-1})_k} = (-1)^n q^{-n(n+1)/2} \frac{(q^{-n}; q)_k}{(q; q)_k} q^{(n+1)k}.$$

Thus we obtain, for  $n \in \mathbb{Z}_+$ ,

$$\begin{aligned} {}_{s+1}\phi_s \left[ \begin{matrix} q^{-n}, a_1, \dots, a_s \\ b_1, \dots, b_s \end{matrix}; q, z \right] &= (-1)^n q^{-n(n+1)/2} \frac{(a_1, \dots, a_s; q)_n}{(b_1, \dots, b_s; q)_n} z^n \\ &\quad \times {}_{s+1}\phi_s \left[ \begin{matrix} q^{-n}, q^{-n+1}b_1^{-1}, \dots, q^{-n+1}b_s^{-1} \\ q^{-n+1}a_1^{-1}, \dots, q^{-n+1}a_s^{-1} \end{matrix}; q, \frac{q^{n+1}b_1 \dots b_s}{a_1 \dots a_s z} \right]. \end{aligned} \quad (1.8)$$

Similar identities can be derived for other  ${}_r\phi_s$  and for cases where some of the parameters are 0.

Thus, any explicit evaluation of a terminating  $q$ -hypergeometric series immediately implies a second one by inversion of the direction of summation in the series, while any identity between two terminating  $q$ -hypergeometric series implies three other identities.

**1.2. The  $q$ -integral.** Standard operations of classical analysis like differentiation and integration do not fit very well with  $q$ -hypergeometric series and can be better replaced by  $q$ -derivative and  $q$ -integral.

The  $q$ -derivative  $D_q f$  of a function  $f$  on an open real interval is given by

$$(D_q f)(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad (1.9)$$

and  $(D_q f)(0) := f'(0)$  by continuity, provided  $f'(0)$  exists. Note that  $\lim_{q \uparrow 1} (D_q f)(x) = f'(x)$  if  $f$  is differentiable. Note also that, analogous to  $d/dx (1-x)^n = -n(1-x)^{n-1}$ , we have

$$f(x) = (x; q)_n \implies (D_q f)(x) = -\frac{1-q^n}{1-q} (qx; q)_{n-1}. \quad (1.10)$$

Now recall that  $0 < q < 1$ . If  $D_q F = f$  and  $f$  is continuous then, for real  $a$ ,

$$F(a) - F(0) = a(1 - q) \sum_{k=0}^{\infty} f(aq^k) q^k. \quad (1.11)$$

This suggests the definition of the  $q$ -integral

$$\int_0^a f(x) d_q x := a(1 - q) \sum_{k=0}^{\infty} f(aq^k) q^k. \quad (1.12)$$

Note that it can be viewed as an infinite Riemann sum with nonequidistant mesh widths. In the limit, as  $q \uparrow 1$ , the right hand side of (1.12) will tend to the classical integral  $\int_0^a f(x) dx$ .

From (1.11) we can also obtain  $F(a) - F(b)$  expressed in terms of  $f$ . This suggests the definition

$$\int_a^b f(x) d_q x := \int_0^a f(x) d_q x - \int_0^b f(x) d_q x. \quad (1.13)$$

Note that (1.12) and (1.13) remain valid if  $a$  or  $b$  is negative.

There is no unique canonical choice for the  $q$ -integral from 0 to  $\infty$ . We will put

$$\int_0^{\infty} f(x) d_q x := (1 - q) \sum_{k=-\infty}^{\infty} f(q^k) q^k$$

(provided the sum converges absolutely). The other natural choices are then expressed by

$$a \int_0^{\infty} f(ax) d_q x = a(1 - q) \sum_{k=-\infty}^{\infty} f(aq^k) q^k, \quad a > 0.$$

Note that the above expression remains invariant when we replace  $a$  by  $aq^n$  ( $n \in \mathbb{Z}$ ).

As an example consider

$$\int_0^1 x^\alpha d_q x = (1 - q) \sum_{k=0}^{\infty} q^{k(\alpha+1)} = \frac{1 - q}{1 - q^{\alpha+1}}, \quad \operatorname{Re} \alpha > -1, \quad (1.14)$$

which tends, for  $q \uparrow 1$ , to

$$\frac{1}{\alpha + 1} = \int_0^1 x^\alpha dx. \quad (1.15)$$

From the point of view of explicitly computing definite integrals by Riemann sum approximation this is much more efficient than the approximation with equidistant mesh widths

$$\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^\alpha.$$

The  $q$ -approximation (1.14) of the definite integral (1.15) (viewed as an area) goes essentially back to Fermat.

**1.3. Elementary examples.** The  $q$ -binomial series is defined by

$${}_1\phi_0(a; -; q, z) := \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k, \quad |z| < 1. \quad (1.16)$$

Here the ‘–’ in a  ${}_r\phi_s$  expression denotes an empty parameter list. The name “ $q$ -binomial” is justified since (1.16), with  $a$  replaced by  $q^a$ , formally tends, as  $q \uparrow 1$ , to the binomial series

$${}_1F_0(a; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k = (1 - z)^{-a}, \quad |z| < 1. \quad (1.17)$$

The limit transition is made rigorous in [22, Appendix A]. It becomes elementary in the case of a terminating series ( $a := q^{-n}$  in (1.16) and  $a := -n$  in (1.17), where  $n \in \mathbb{Z}_+$ ).

The  $q$ -analogue of the series evaluation in (1.17) is as follows.

**Proposition 1.2** We have

$${}_1\phi_0(a; -; q, z) = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad |z| < 1. \quad (1.18)$$

In particular,

$${}_1\phi_0(q^{-n}; -; q, z) = (q^{-n}z; q)_n. \quad (1.19)$$

**Proof** Put  $h_a(z) := {}_1\phi_0(a; -; q, z)$ . Then

$$(1 - z)h_a(z) = (1 - az)h_a(qz), \quad \text{hence} \quad h_a(z) = \frac{(1 - az)}{(1 - z)}h_a(qz).$$

Iteration gives

$$h_a(z) = \frac{(az; q)_n}{(z; q)_n} h_a(q^n z), \quad n \in \mathbb{Z}_+.$$

Now use that  $h_a$  is analytic and therefore continuous at 0 and use that  $h_a(0) = 1$ . Thus, for  $n \rightarrow \infty$  we obtain (1.18).  $\square$

The two most elementary  $q$ -hypergeometric series are the two  $q$ -exponential series

$$e_q(z) := {}_1\phi_0(0; -; q, z) = \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1, \quad (1.20)$$

and

$$E_q(z) := {}_0\phi_0(-; -; q, -z) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{(q; q)_k} = (-z; q)_{\infty}, \quad z \in \mathbb{C}. \quad (1.21)$$

The evaluation in (1.20) is a specialization of (1.18). On the other hand, (1.21) is a confluent limit of (1.18):

$$\lim_{a \rightarrow \infty} {}_1\phi_0(a; -; q, -z/a) = {}_0\phi_0(-; -; q, -z)$$

(cf. (1.5)). The limit is uniform for  $z$  in compact subsets of  $\mathbb{C}$ , see Lemma 1.1. Thus the evaluation in (1.21) follows also from (1.18).

It follows from (1.20) and (1.21) that

$$e_q(z) E_q(-z) = 1. \quad (1.22)$$

This identity is a  $q$ -analogue of  $e^z e^{-z} = 1$ . Indeed, the two  $q$ -exponential series are  $q$ -analogues of the exponential series by the limit formulas

$$\lim_{q \uparrow 1} E_q((1-q)z) = e^z = \lim_{q \uparrow 1} e_q((1-q)z).$$

The first limit is uniform on compacta of  $\mathbb{C}$  by the majorization

$$\left| \frac{q^{k(k-1)/2} (1-q)^k z^k}{(q; q)_k} \right| \leq \frac{|z|^k}{k!}.$$

The second limit then follows by use of (1.22).

Although we assumed the convention  $0 < q < 1$ , it is often useful to find out for a given  $q$ -hypergeometric series what will be obtained by changing  $q$  into  $q^{-1}$  and then rewriting things again in base  $q$ . This will establish a kind of duality for  $q$ -hypergeometric series. For instance, we have

$$e_{q^{-1}}(z) = E_q(-qz),$$

which can be seen from the power series definitions.

The following four identities, including (1.20), are obtained from each other by trivial rewriting.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(1-q)^k z^k}{(q; q)_k} &= \frac{1}{((1-q)z; q)_{\infty}}, \\ \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} &= \frac{1}{(z; q)_{\infty}}, \\ (1-q)^{1-b} \sum_{k=0}^{\infty} q^{kb} (q^{k+1}; q)_{\infty} &= \frac{(q; q)_{\infty}}{(q^b; q)_{\infty} (1-q)^{b-1}}, \\ \int_0^{(1-q)^{-1}} t^{b-1} ((1-q)qt; q)_{\infty} d_q t &= \frac{(q; q)_{\infty}}{(q^b; q)_{\infty} (1-q)^{b-1}}, \quad \text{Re } b > 0. \end{aligned} \quad (1.23)$$

As  $q \uparrow 1$ , the first identity tends to

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

while the left hand side of (1.23) tends formally to

$$\int_0^{\infty} t^{b-1} e^{-t} dt.$$



Since this last integral can be evaluated as  $\Gamma(b)$ , it is tempting to consider the right hand side of (1.23) as a the  $q$ -gamma function. Thus we put

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty (1-q)^{z-1}} = \int_0^{(1-q)^{-1}} t^{z-1} E_q(-(1-q)qt) d_q t, \quad \operatorname{Re} z > 0.$$

where the last identity follows from (1.23). It was proved in [22, Appendix B] that

$$\lim_{q \uparrow 1} \Gamma_q(z) = \Gamma(z), \quad z \neq 0, -1, -2, \dots \quad (1.24)$$

We have just seen an example how an identity for  $q$ -hypergeometric series can have two completely different limit cases as  $q \uparrow 1$ . Of course, this is achieved by different rescaling. In particular, reconsideration of a power series as a  $q$ -integral is often helpful and suggestive for obtaining distinct limits.

Regarding  $\Gamma_q$  it can yet be remarked that it satisfies the functional equation

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z)$$

and that

$$\Gamma_q(n+1) = \frac{(q; q)_n}{(1-q)^n}, \quad n \in \mathbb{Z}_+.$$

Similarly to the chain of equivalent identities including (1.20) we have a chain including (1.18):

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(q^a; q)_k z^k}{(q; q)_k} &= \frac{(q^a z; q)_\infty}{(z; q)_\infty}, \\ (1-q) \sum_{k=0}^{\infty} q^{kb} \frac{(q^{k+1}; q)_\infty}{(q^{k+a}; q)_\infty} &= \frac{(1-q)(q, q^{a+b}; q)_\infty}{(q^a, q^b; q)_\infty}, \\ \int_0^1 t^{b-1} \frac{(qt; q)_\infty}{(q^a t; q)_\infty} d_q t &= \frac{\Gamma_q(a) \Gamma_q(b)}{\Gamma_q(a+b)}, \quad \operatorname{Re} b > 0. \end{aligned} \quad (1.25)$$

In the limit, as  $q \uparrow 1$ , the first identity tends to

$$\sum_{k=0}^{\infty} \frac{(a)_k z^k}{k!} = (1-z)^{-a},$$

while (1.25) formally tends to

$$\int_0^1 t^{b-1} (1-t)^{a-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}. \quad (1.26)$$

Thus (1.25) can be considered as a  $q$ -beta integral and we define the  $q$ -beta function by

$$B_q(a, b) := \frac{\Gamma_q(a) \Gamma_q(b)}{\Gamma_q(a+b)} = \frac{(1-q)(q, q^{a+b}; q)_\infty}{(q^a, q^b; q)_\infty} = \int_0^1 t^{b-1} \frac{(qt; q)_\infty}{(q^a t; q)_\infty} d_q t. \quad (1.27)$$

**1.4. Heine's  ${}_2\phi_1$  series.** Euler's integral representation for the  ${}_2F_1$  hypergeometric function

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \quad (1.28)$$

$\operatorname{Re} c > \operatorname{Re} b > 0, |\arg(1-z)| < \pi,$

(cf. [13, 2.1(10)]) has the following  $q$ -analogue due to Heine.

$${}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{(tq; q)_\infty}{(tq^{c-b}; q)_\infty} \frac{(tzq^a; q)_\infty}{(tz; q)_\infty} d_q t, \quad (1.29)$$

$\operatorname{Re} b > 0, |z| < 1.$

Note that the left hand side and right hand side of (1.29) tend formally to the corresponding sides of (1.28). The proof of (1.29) is also analogous to the proof of (1.28). Expand  $(tzq^a; q)_\infty / (tz; q)_\infty$  as a power series in  $tz$  by (1.16), interchange summation and  $q$ -integration, and evaluate the resulting  $q$ -integrals by (1.25).

If we rewrite the  $q$ -integral in (1.29) as a series according to the definition (1.12), and if we replace  $q^a, q^b, q^c$  by  $a, b, c$  then we obtain the following transformation formula:

$${}_2\phi_1(a, b; c; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty} \frac{(b; q)_\infty}{(c; q)_\infty} {}_2\phi_1(c/b, z; az; q, b). \quad (1.30)$$

Although (1.29) and (1.30) are equivalent, they look quite different. In fact, in its form (1.30) the identity has no classical analogue. We see a new phenomenon, not occurring for  ${}_2F_1$ , namely that the independent variable  $z$  of the left hand side mixes on the right hand side with the parameters. So, rather than having a function of  $z$  with parameters  $a, b, c$ , we deal with a function of  $a, b, c, z$  satisfying certain symmetries.

Just as in the classical case (cf. [13, 2.1(14)]), substitution of some special value of  $z$  in the  $q$ -integral representation (1.29) reduces it to a  $q$ -beta integral which can be explicitly evaluated. We obtain

$${}_2\phi_1(a, b; c; q, c/(ab)) = \frac{(c/a, c/b; q)_\infty}{(c, c/(ab); q)_\infty}, \quad |c/(ab)| < 1, \quad (1.31)$$

where the more relaxed bounds on the parameters are obtained by analytic continuation. The terminating case of (1.31) is

$${}_2\phi_1(q^{-n}, b; c; q, cq^n/b) = \frac{(c/b; q)_n}{(c; q)_n}, \quad n \in \mathbb{Z}_+. \quad (1.32)$$

The two fundamental transformation formulas

$$\begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{-a} {}_2F_1(a, c-b; c; z/(z-1)) \\ &= (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) \end{aligned} \quad (1.33)$$

(cf. [13, 2.1(22) and (23)]) have the following  $q$ -analogues.

$${}_2\phi_1(a, b; c; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty} {}_2\phi_2(a, c/b; c, az; q, bz) \quad (1.34)$$

$$= \frac{(abz/c; q)_\infty}{(z; q)_\infty} {}_2\phi_1(c/a, c/b; c; q, abz/c). \quad (1.35)$$

Formula (1.35) can be proved either by threefold iteration of (1.30) or by twofold iteration of (1.34). The proof of (1.34) is more involved. Write both sides as power series in  $z$ . Then make both sides into double series by substituting for  $(b; q)_k/(c; q)_k$  on the left hand side a terminating  ${}_2\phi_1$  (cf. (1.32)) and by substituting for  $(aq^k z; q)_\infty/(z; q)_\infty$  on the right hand side a  $q$ -binomial series (cf. (1.16)). The result follows by some rearrangement of series. See [15, §1.5] for the details.

Observe the difference between (1.33) and its  $q$ -analogue (1.34). The argument  $z/(z-1)$  in (1.33) no longer occurs in (1.34) as a rational function of  $z$ , but the  $z$ -variable is distributed over the argument and one of the lower parameters of the  ${}_2\phi_2$ . Also we do not stay within the realm of  ${}_2\phi_1$  functions.

Equation (1.8) for  $s := 1$  becomes

$${}_2\phi_1(q^{-n}, b; c; q, z) = q^{-n(n+1)/2} \frac{(b; q)_n}{(c; q)_n} (-z)^n {}_2\phi_1\left(q^{-n}, \frac{q^{-n+1}}{c}; \frac{q^{-n+1}}{b}; q, \frac{q^{n+1}c}{bz}\right), \quad n \in \mathbb{Z}_+. \quad (1.36)$$

We may apply (1.36) to the preceding evaluation and transformation formulas for  ${}_2\phi_1$  in order to obtain new ones in the terminating case. From (1.32) we obtain

$${}_2\phi_1(q^{-n}, b; c; q, q) = \frac{(c/b; q)_n b^n}{(c; q)_n}, \quad n \in \mathbb{Z}_+. \quad (1.37)$$

By inversion of direction of summation on both sides of (1.34) (with  $a := q^{-n}$ ) we obtain

$${}_2\phi_1(q^{-n}, b; c; q, z) = \frac{(c/b; q)_n}{(c; q)_n} {}_3\phi_2\left[\begin{matrix} q^{-n}, b, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{matrix}; q, q\right], \quad n \in \mathbb{Z}_+. \quad (1.38)$$

A terminating  ${}_2\phi_1$  can also be transformed into a terminating  ${}_3\phi_2$  with one of the upper parameters zero (result of Jackson):

$${}_2\phi_1(q^{-n}, b; c; q, z) = (q^{-n}bz/c; q)_n {}_3\phi_2\left[\begin{matrix} q^{-n}, c/b, 0 \\ c, cq b^{-1} z^{-1} \end{matrix}; q, q\right] \quad (1.39)$$

This formula can be proved by applying (1.19) to the factor  $(cq^{k+1}b^{-1}z^{-1}; q)_{n-k}$  occurring in the  $k$ th term of the right hand side. Then interchange summation in the resulting double sum and substitute (1.37) for the inner sum. See [21, p.101] for the details.

**1.5. A three-term transformation formula.** Formula (1.36) has the following generalization for non-terminating  ${}_2\phi_1$ :

$$\begin{aligned} & {}_2\phi_1(a, b; c; q, z) + \frac{(a, q/c, c/b, bz/q, q^2/bz; q)_\infty}{(c/q, aq/c, q/b, bz/c, cq/(bz); q)_\infty} {}_2\phi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, z\right) \\ &= \frac{(abz/c, q/c, aq/b, cq/(abz); q)_\infty}{(bz/c, q/b, aq/c, cq/(bz); q)_\infty} {}_2\phi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abz}\right), \quad \left|\frac{cq}{ab}\right| < |z| < 1. \end{aligned} \quad (1.40)$$

This identity is a  $q$ -analogue of [13, 2.1(17)]. In the following proposition we rewrite (1.40) in an equivalent form and next sketch the elegant proof due to [28].

**Proposition 1.3** Suppose  $\{q^n \mid n \in \mathbb{Z}_+\}$  is disjoint from  $\{a^{-1}q^{-n}, b^{-1}q^{-n} \mid n \in \mathbb{Z}_+\}$ . Then

$$\begin{aligned} (z, qz^{-1}; q)_\infty {}_2\phi_1(a, b; c; q, z) &= (az, qa^{-1}z^{-1}; q)_\infty \frac{(c/a, b; q)_\infty}{(c, b/a; q)_\infty} \\ &\quad \times {}_2\phi_1(a, qa/c; qa/b; q, qc/(abz)) + (a \longleftrightarrow b). \end{aligned} \quad (1.41)$$

**Proof** Consider the function

$$F(w) := \frac{(a, b, cw, q, qwz^{-1}, w^{-1}z; q)_\infty}{(aw, bw, c, w^{-1}; q)_\infty} \frac{1}{w}.$$

Its residue at  $q^n$  ( $n \in \mathbb{Z}_+$ ) equals the  $n^{\text{th}}$  term of the series on the left hand side of (1.41). The negative of its residue at  $a^{-1}q^{-n}$  ( $n \in \mathbb{Z}_+$ ) equals the  $n^{\text{th}}$  term of the first series on the right hand side of (1.41), and the residue at  $b^{-1}q^{-n}$  is similarly related to the second series on the right hand side. Let  $\mathcal{C}$  be a positively oriented closed curve around 0 in  $\mathbb{C}$  which separates the two sets mentioned in the Proposition. Then  $(2\pi i)^{-1} \int_{\mathcal{C}} F(w) dw$  can be expressed in two ways as an infinite sum of residues: either by letting the contour shrink to  $\{0\}$  or by blowing it up to  $\{\infty\}$ .  $\square$

If we put  $z := q$  in (1.40) and substitute (1.31) then we obtain a generalization of (1.37) for non-terminating  ${}_2\phi_1$ :

$${}_2\phi_1(a, b; c; q, q) + \frac{(a, b, q/c; q)_\infty}{(aq/c, bq/c, c/q; q)_\infty} {}_2\phi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; q, q\right) = \frac{(abq/c, q/c; q)_\infty}{(aq/c, bq/c; q)_\infty}. \quad (1.42)$$

By (1.13) this identity (1.42) can be equivalently written in  $q$ -integral form:

$$\int_{q^{-1}}^1 \frac{(ct, qt; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{(1-q)(abq/c, q/c, c, q; q)_\infty}{(aq/c, bq/c, a, b; q)_\infty}. \quad (1.43)$$

Replace in (1.43)  $c, a, b$  by  $-q^c, -q^a, q^b$ , respectively, and let  $q \uparrow 1$ . Then we obtain formally

$$\int_{-1}^1 (1+t)^{a-c} (1-t)^{b-1} dt = \frac{2^{b+a-c} \Gamma(a-c+1) \Gamma(b)}{\Gamma(a+b-c+1)}. \quad (1.44)$$

Note that, although it is trivial to obtain (1.44) from (1.26) by an affine transformation of the integration variable, their  $q$ -analogues (1.25) and (1.43) are by no means trivially equivalent.

**1.6. Bilateral series.** Put  $a := q$  in (1.40) and substitute (1.16). Then we can combine the two series from 0 to  $\infty$  into a series from  $-\infty$  to  $\infty$  without “discontinuity” in the summand:

$$\sum_{k=-\infty}^{\infty} \frac{(b; q)_k}{(c; q)_k} z^k = \frac{(q, c/b, bz, q/(bz); q)_{\infty}}{(c, q/b, z, c/(bz); q)_{\infty}}, \quad |c/b| < |z| < 1. \quad (1.45)$$

Here we have extended the definition of  $(b; q)_k$  by

$$(b; q)_k := \frac{(b; q)_{\infty}}{(bq^k; q)_{\infty}}, \quad k \in \mathbb{Z}.$$

Formula (1.45) was first obtained by Ramanujan (*Ramanujan’s  ${}_1\psi_1$ -summation formula*). It reduces to the  $q$ -binomial formula (1.18) for  $c := q^n$  ( $n \in \mathbb{Z}_+$ ). In fact, this observation can be used for a proof of (1.45), cf. [17], [2, Appendix C]. Formula (1.45) is a  $q$ -analogue of the explicit Fourier series evaluation

$$\sum_{k=-\infty}^{\infty} \frac{(b)_k}{(c)_k} e^{ik\theta} = \frac{\Gamma(c)\Gamma(1-b)}{\Gamma(c-b)} e^{i(1-c)(\theta-\pi)} (1 - e^{i\theta})^{c-b-1},$$

$$0 < \theta < 2\pi, \quad \operatorname{Re}(c-b-1) > 0,$$

where  $(b)_k := \Gamma(b+k)/\Gamma(b)$ , also for  $k = -1, -2, \dots$

Define *bilateral  $q$ -hypergeometric series* by

$${}_r\psi_s \left[ \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right]$$

$$:= \sum_{k=-\infty}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} ((-1)^k q^{k(k-1)/2})^{s-r} z^k \quad (1.46)$$

$$= {}_{r+1}\phi_s \left[ \begin{matrix} a_1, \dots, a_r, q \\ b_1, \dots, b_s \end{matrix}; q, z \right]$$

$$+ \frac{(b_1 - q) \dots (b_s - q)}{(a_1 - q) \dots (a_r - q)} z^{s+1} \phi_s \left[ \begin{matrix} q^2/b_1, \dots, q^2/b_s, q \\ q^2/a_1, \dots, q^2/a_r, 0, \dots, 0 \end{matrix}; q, \frac{b_1 \dots b_s}{a_1 \dots a_r z} \right], \quad (1.47)$$

where  $a_1, \dots, a_r, b_1, \dots, b_s \neq 0$  and  $s \geq r$ . The Laurent series in (1.46) is convergent for

$$\left| \frac{b_1 \dots b_s}{a_1 \dots a_r} \right| < |z| \quad \text{if } s > r$$

and for

$$\left| \frac{b_1 \dots b_s}{a_1 \dots a_r} \right| < |z| < 1 \quad \text{if } s = r.$$

If some of the lower parameters in the  ${}_r\psi_s$  are 0 then a suitable confluent limit has to be taken in the  ${}_{s+1}\phi_s$  of (1.47).

Thus we can write (1.45) as

$${}_1\psi_1(b; c; q, z) = \frac{(q, c/b, bz, q/(bz); q)_{\infty}}{(c, q/b, z, c/(bz); q)_{\infty}}, \quad |c/b| < |z| < 1. \quad (1.48)$$

Replace in (1.48)  $z$  by  $z/b$ , substitute (1.47), let  $b \rightarrow \infty$ , and apply (1.6) Then we obtain

$${}_0\psi_1(-; c; q, z) := \sum_{k=-\infty}^{\infty} \frac{(-1)^k q^{k(k-1)/2} z^k}{(c; q)_k} = \frac{(q, z, q/z; q)_{\infty}}{(c, c/z; q)_{\infty}}, \quad |z| > |c|. \quad (1.49)$$

In particular, for  $c = 0$ , we get the *Jacobi triple product identity*

$${}_0\psi_1(-; 0; q, z) := \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} z^k = (q, z, q/z; q)_{\infty}, \quad z \neq 0. \quad (1.50)$$

The series in (1.50) is essentially a *theta function*. With the notation [14, 13.19(9)] we get

$$\begin{aligned} \theta_4(x; q) &:= \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2\pi i k x} \\ &= {}_0\psi_1(-; 0; q^2, q e^{2\pi i x}) \\ &= (q^2, q e^{2\pi i x}, q e^{-2\pi i x}; q^2)_{\infty} \\ &= \prod_{k=1}^{\infty} (1 - q^{2k}) (1 - 2q^{2k-1} \cos(2\pi x) + q^{4k-2}), \end{aligned}$$

and similarly for the other theta functions  $\theta_i(x; q)$  ( $i = 1, 2, 3$ ).

**1.7. The  $q$ -hypergeometric  $q$ -difference equation.** Just as the *hypergeometric differential equation*

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - ab u(z) = 0$$

(cf. [13, 2.1(1)]) has particular solutions

$$u_1(z) := {}_2F_1(a, b; c; z), \quad u_2(z) := z^{1-c} {}_2F_1(a - c + 1, b - c + 1; 2 - c; z),$$

the  $q$ -hypergeometric  $q$ -difference equation

$$\begin{aligned} z(q^c - q^{a+b+1}z)(D_q^2 u)(z) + \left[ \frac{1-q^c}{1-q} - \left( q^b \frac{1-q^a}{1-q} + q^a \frac{1-q^{b+1}}{1-q} \right) z \right] (D_q u)(z) \\ - \frac{1-q^a}{1-q} \frac{1-q^b}{1-q} u(z) = 0 \end{aligned} \quad (1.51)$$

has particular solutions

$$u_1(z) := {}_2\phi_1(q^a, q^b; q^c; q, z), \quad (1.52)$$

$$u_2(z) := z^{1-c} {}_2\phi_1(q^{1+a-c}, q^{1+b-c}; q^{2-c}; q, z). \quad (1.53)$$

There is an underlying theory of  $q$ -difference equations with regular singularities, similarly to the theory of differential equations with regular singularities discussed for instance in Olver [30, Ch. 5]. It is not difficult to prove the following proposition.

**Proposition 1.4** Let  $A(z) := \sum_{k=0}^{\infty} a_k z^k$  and  $B(z) := \sum_{k=0}^{\infty} b_k z^k$  be convergent power series. Let  $\lambda \in \mathbb{C}$  be such that

$$\frac{(1 - q^{\lambda+k})(1 - q^{\lambda+k-1})}{(1 - q)^2} + a_0 \frac{1 - q^{\lambda+k}}{1 - q} + b_0 \begin{cases} = 0, & k = 0, \\ \neq 0, & k = 1, 2, \dots \end{cases} \quad (1.54)$$

Then the  $q$ -difference equation

$$z^2 (D_q^2 u)(z) + z A(z) (D_q u)(z) + B(z) u(z) = 0 \quad (1.55)$$

has an (up to a constant factor) unique solution of the form

$$u(z) = \sum_{k=0}^{\infty} c_k z^{\lambda+k}. \quad (1.56)$$

Note that (1.51) can be rewritten in the form (1.55) with  $a_0 = (q^{-c} - 1)/(1 - q)$ ,  $b_0 = 0$ , so (1.54) has solutions  $\lambda = 0$  and  $-c + 1 \pmod{(2\pi i \log q^{-1})\mathbb{Z}}$  provided  $c \notin \mathbb{Z} \pmod{(2\pi i \log q^{-1})\mathbb{Z}}$ . For the coefficients  $c_k$  in (1.56) we find the recursion

$$\frac{c_{k+1}}{c_k} = \frac{(1 - q^{a+\lambda+k})(1 - q^{b+\lambda+k})}{(1 - q^{c+\lambda+k})(1 - q^{\lambda+k+1})}.$$

Thus we obtain solutions  $u_1, u_2$  as given in (1.52), (1.53).

Proposition 1.4 can also be applied to the case  $z = \infty$  of (1.51). Just make the transformation  $z \mapsto z^{-1}$  in (1.51). One solution then obtained is

$$u_3(z) := z^{-a} {}_2\phi_1 \left[ \begin{matrix} q^a, q^{a-c+1} \\ q^{a-b+1} \end{matrix}; q, q^{-a-b+c+1} z^{-1} \right].$$

Now (1.40) can be rewritten as

$$\begin{aligned} u_1(z) + \frac{(q^a, q^{1-c}, q^{c-b}; q)_{\infty}}{(q^{c-1}, q^{a-c+1}, q^{1-b}; q)_{\infty}} \frac{(q^{b-1}z, q^{2-b}z^{-1}; q)_{\infty} z^{c-1}}{(q^{b-c}z, q^{c-b+1}z^{-1}; q)_{\infty}} u_2(z) \\ = \frac{(q^{1-c}, q^{a-b+1}; q)_{\infty}}{(q^{1-b}, q^{a-c+1}; q)_{\infty}} \frac{(q^{a+b-c}z, q^{c-a-b+1}z^{-1}; q)_{\infty} z^a}{(q^{b-c}z, q^{c-b+1}z^{-1}; q)_{\infty}} u_3(z). \end{aligned} \quad (1.57)$$

Note that  $u_3$  is not a linear combination of  $u_1$  and  $u_2$ , as the coefficients of  $u_2$  and  $u_3$  in (1.57) depend on  $z$ . However, since an expression of the form

$$z \mapsto \frac{(q^{\alpha}z, q^{1-\alpha}z^{-1}; q)_{\infty} z^{\alpha-\beta}}{(q^{\beta}z, q^{1-\beta}z^{-1}; q)_{\infty}}$$

is invariant under transformations  $z \mapsto qz$ , each term in (1.57) is a solution of (1.51). Thus everything works fine when we restrict ourselves to a subset of the form  $\{z_0 q^k \mid k \in \mathbb{Z}\}$ .

The  $q$ -analogue of the regular singularity at  $z = 1$  for the ordinary hypergeometric differential equation has to be treated in a different way. We can rewrite (1.51) as

$$(q^c - q^{a+b}z) u(qz) + (-q^c - q + (q^a + q^b)z) u(z) + (q - z) u(q^{-1}z) = 0.$$

It can be expected that the points  $z = q^{c-a-b}$  and  $z = q$ , where the coefficient of  $u(qz)$  respectively  $u(q^{-1}z)$  vanishes, will replace the classical regular singularity  $z = 1$ , see also (1.31), (1.37) and (1.42). A systematic theory for such singularities has not yet been developed.

**1.8.  $q$ -Bessel functions.** The classical *Bessel function* is defined by

$$\begin{aligned} J_\nu(z) &:= (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+k+1)k!} (z/2)^{2k} \\ &= \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1(-; \nu+1; -z^2/4), \end{aligned}$$

cf. [14, Ch. 7]). Jackson (1905) introduced two  $q$ -analogues of the Bessel function:

$$\begin{aligned} J_\nu^{(1)}(z; q) &:= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^\nu {}_2\phi_1(0, 0; q^{\nu+1}; q, -z^2/4) \\ &= (z/2)^\nu \sum_{k=0}^{\infty} \frac{(q^{\nu+k+1}; q)_\infty}{(q; q)_\infty} \frac{(-1)^k (z/2)^{2k}}{(q; q)_k}, \end{aligned} \quad (1.58)$$

$$\begin{aligned} J_\nu^{(2)}(z; q) &:= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^\nu {}_0\phi_1(-; q^{\nu+1}; q, -q^{\nu+1}z^2/4) \\ &= (z/2)^\nu \sum_{k=0}^{\infty} \frac{(q^{\nu+k+1}; q)_\infty}{(q; q)_\infty} \frac{q^{k(k+\nu)} (-1)^k (z/2)^{2k}}{(q; q)_k}. \end{aligned} \quad (1.59)$$

Formally we have

$$\lim_{q \uparrow 1} J_\nu^{(i)}((1-q)z; q) = J_\nu(z), \quad i = 1, 2$$

(cf. (1.24)). The two  $q$ -Bessel functions  $J_\nu^{(i)}(z; q)$  can be simply expressed in terms of each other. From (1.35) and (1.6) we obtain

$${}_2\phi_1(0, b; c; q, z) = \frac{1}{(z; q)_\infty} {}_1\phi_1(c/b; c; q, bz).$$

A further confluence with  $b \rightarrow 0$  yields

$${}_2\phi_1(0, 0; c; q, z) = \frac{1}{(z; q)_\infty} {}_0\phi_1(-; c; q, cz).$$

This can be rewritten as

$$J_\nu^{(2)}(z; q) = (-z^2/4; q)_\infty J_\nu^{(1)}(z; q).$$

Yet another  $q$ -analogue of the Bessel function is as follows.

$$\begin{aligned} J_\nu(z; q) &:= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu {}_1\phi_1(0; q^{\nu+1}; q, qz^2) \\ &= z^\nu \sum_{k=0}^{\infty} \frac{(q^{\nu+k+1}; q)_\infty}{(q; q)_\infty} \frac{(-1)^k q^{k(k+1)/2} z^{2k}}{(q; q)_k}. \end{aligned} \quad (1.60)$$

Formally we have

$$\lim_{q \uparrow 1} J_\nu((1-q^{1/2})z; q) = J_\nu(z).$$

This so-called *Jackson's third  $q$ -Bessel function* is not simply related to the  $q$ -Bessel functions (1.58), (1.59), but they were all introduced by Jackson. Koornwinder & Swarttouw [24] gave a satisfactory  $q$ -analogue of the Hankel transform in terms of the  $q$ -Bessel function (1.60).



**1.9. Various results.** Goursat’s list of *quadratic transformations* for Gaussian hypergeometric functions can be found in [13, §2.11]. In a recent paper Rahman & Verma [31] have given a full list of  $q$ -analogues of Goursat’s table. However, all their formulas involve on at least one of both sides an  ${}_8\phi_7$  series. Moreover, a good foundation from the theory of  $q$ -difference equations is not yet available. For terminating series many of their  ${}_8\phi_7$ ’s will simplify to  ${}_4\phi_3$ ’s. In section 2 we will meet some natural examples of these transformations coming from orthogonal polynomials.

An important part of the book by Gasper & Rahman [15] deals with the derivation of summation and transformation formulas of  ${}_{s+1}\phi_s$  functions with  $s > 1$ . A simple example is the  $q$ -Saalschütz formula

$${}_3\phi_2(a, b, q^{-n}; c, abc^{-1}q^{1-n}; q, q) = \frac{(c/a, c/b; q)_n}{(c, c/(ab); q)_n}, \quad n \in \mathbb{Z}_+, \quad (1.61)$$

which follows easily from (1.35) by expanding the quotient on the right hand side of (1.35) with the aid of (1.18), and next comparing equal powers of  $z$  at both sides of (1.35). Formula (1.61) is the  $q$ -analogue of the *Pfaff-Saalschütz formula*

$${}_3F_2(a, b, -n; c, 1 + a + b - c - n; 1) = \frac{(c - a)_n (c - b)_n}{(c)_n (c - a - b)_n}, \quad n \in \mathbb{Z}_+,$$

which can be proved in an analogous way as (1.61).

An example of a much more involved transformation formula, which has many important special cases, is *Watson’s transformation formula*

$$\begin{aligned} {}_8\phi_7 \left[ \begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, \frac{a^2 q^{2+n}}{bcde} \right] \\ = \frac{(aq, aq/(de); q)_n}{(aq/d, aq/e; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, d, e, aq/(bc) \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q \right], \quad n \in \mathbb{Z}_+, \end{aligned} \quad (1.62)$$

cf. Gasper & Rahman [15, §2.5]. Then, for  $a^2 q^{n+1} = bcde$ , the right hand side can be evaluated by use of (1.61). The resulting evaluation

$$\frac{(aq, aq/(bc), aq/(bd), aq/(cd); q)_n}{(aq/b, aq/c, aq/d, aq/(bcd); q)_n}$$

of the left hand side of (1.62) subject to the given relation between  $a, b, c, d, e, q$  is called *Jackson’s summation formula*, cf. [15, §2.6].

The famous *Rogers-Ramanujan identities*

$$\begin{aligned} {}_0\phi_1(-; 0; q, q) &:= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{(q^2, q^3, q^5; q^5)_{\infty}}{(q; q)_{\infty}}, \\ {}_0\phi_1(-; 0; q, q^2) &:= \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q; q)_k} = \frac{(q, q^4, q^5; q^5)_{\infty}}{(q; q)_{\infty}} \end{aligned}$$

have been proved in many different ways (cf. Andrews [2]), not only analytically but also by an interpretation in combinatorics or in the framework of Kac-Moody algebras. A quick analytic proof starting from (1.62) is described in [15, §2.7].

**Exercises to §1**

1.1 Prove that

$$(a; q)_n = (-a)^n q^{n(n-1)/2} (a^{-1}q^{1-n}; q)_n.$$

1.2 Prove that

$$\begin{aligned} (a; q)_{2n} &= (a; q^2)_n (aq; q^2)_n, \\ (a^2; q^2)_n &= (a; q)_n (-a; q)_n, \\ (a; q)_\infty &= (a^{1/2}, -a^{1/2}, (aq)^{1/2}, -(aq)^{1/2}; q)_\infty. \end{aligned}$$

1.3 Prove the following identity of Euler:

$$(-q; q)_\infty (q; q^2)_\infty = 1.$$

1.4 Prove that

$$\sum_{l=-\infty}^{\infty} \frac{(-1)^{l-m} q^{(l-m)(l-m-1)/2}}{\Gamma_q(n-l+1) \Gamma_q(l-m+1)} = \delta_{n,m}, \quad 0 < q \leq 1.$$

Do it first for  $q = 1$ . Start for instance with  $e^z e^{-z} = 1$  and, in the  $q$ -case, with  $e_q(z) E_q(-z) = 1$ .

1.5 Let

$$\frac{1}{(q; q)_\infty} = \sum_{n=0}^{\infty} a_n q^n$$

be the power series expansion of the left hand side in terms of  $q$ . Show that  $a_n$  equals the number of partitions of  $n$ .

Show also that the coefficient  $b_{k,n}$  in

$$\frac{q^k}{(q; q)_k} = \sum_{n=k}^{\infty} b_{k,n} q^n$$

is the number of partitions of  $n$  with highest part  $k$ . Give now a partition theoretic proof of the identity

$$\frac{1}{(q; q)_\infty} = \sum_{k=0}^{\infty} \frac{q^k}{(q; q)_k}.$$

1.6 In the same way, give a partition theoretic proof of the identity

$$\frac{1}{(q^m; q)_\infty} = \sum_{k=0}^{\infty} \frac{q^{mk}}{(q; q)_k}, \quad m \in \mathbb{N}.$$

1.7 Give also a partition theoretic proof of

$$\sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} q^{mk}}{(q; q)_k} = (-q^m; q)_\infty, \quad m \in \mathbb{N}.$$

(Consider the problem first for  $m = 1$ .)

1.8 Let  $GF(p)$  be the finite field with  $p$  elements. Let  $A$  be a  $n \times n$  matrix with entries chosen independently and at random from  $GF(p)$ , with equal probability for the field elements to be chosen. Let  $q := p^{-1}$ . Prove that the probability that  $A$  is invertible is  $(q; q)_n$ . (See SIAM News 23 (1990) no.6, p.8.)

1.9 Let  $GF(p)$  be as in the previous exercise. Let  $V$  be an  $n$ -dimensional vector space over  $GF(p)$ . Prove that the number of  $k$ -dimensional linear subspaces of  $V$  equals the  $q$ -binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_p := \frac{(p; p)_n}{(p; p)_k (p; p)_{n-k}}.$$

1.10 Show that

$$(ab; q)_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a; q)_k (b; q)_{n-k}.$$

Show that both Newton's binomial formula and the formula

$$(a + b)_n = \sum_{k=0}^n \binom{n}{k} (a)_k (b)_{n-k}$$

are limit cases of the above formula.

## 2. $q$ -Analogues of the classical orthogonal polynomials

Originally by classical orthogonal polynomials were only meant the three families of Jacobi, Laguerre and Hermite polynomials, but recent insights consider a much bigger class of polynomials as “classical”. On the one hand, there is an extension to hypergeometric orthogonal polynomials up to the  ${}_4F_3$  level and including certain discrete orthogonal polynomials. These are brought together in the Askey tableau, cf. Table 1. On the other hand there are  $q$ -analogues of all the families in the Askey tableau, often several  $q$ -analogues for one classical family (cf. Table 2 for some of them). The master class of all these  $q$ -analogues is formed by the celebrated Askey-Wilson polynomials. They contain all other families described in this chapter as special cases or limit cases. Good references for this chapter are Andrews & Askey [4] and Askey & Wilson [8]. See Koekoek & Swarttouw [19] for a quite comprehensive list of formulas. See also Atakishiyev, Rahman & Suslov [10] for a somewhat different approach.

Some parts of this section contain surveys without many proofs. However, the subsections 2.3 and 2.4 on big and little  $q$ -Jacobi polynomials and 2.5 and 2.6 on the Askey-Wilson integral and related polynomials are rather self-contained.

**2.1. Very classical orthogonal polynomials.** An introduction to the traditional classical orthogonal polynomials can be found, for instance, in [14, Ch. 10]. One possible characterization is as systems of orthogonal polynomials  $\{p_n\}_{n=0,1,2,\dots}$  which are eigenfunctions of a second order differential operator not involving  $n$  with eigenvalues  $\lambda_n$  depending on  $n$ :

$$a(x) p_n''(x) + b(x) p_n'(x) + c(x) p_n(x) = \lambda_n p_n(x). \quad (2.1)$$

Because we will extend the definition of classical orthogonal polynomials in this section, we will call the orthogonal polynomials satisfying (2.1) *very classical orthogonal polynomials*. The classification shows that, up to an affine transformation of the independent variable, the only cases are as follows.

*Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2), \quad \alpha, \beta > -1, \quad (2.2)$$

orthogonal on  $[-1, 1]$  with respect to the measure  $(1-x)^\alpha (1+x)^\beta dx$ ;

*Laguerre polynomials*

$$L_n^\alpha(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1(-n; \alpha+1; x), \quad \alpha > -1, \quad (2.3)$$

orthogonal on  $[0, \infty)$  with respect to the measure  $x^\alpha e^{-x} dx$ ;

*Hermite polynomials*

$$H_n(x) := (2x)^n {}_2F_0(-n/2, (1-n)/2; -; -x^{-2}), \quad (2.4)$$

orthogonal on  $(-\infty, \infty)$  with respect to the measure  $e^{-x^2} dx$ .

In fact, Jacobi polynomials are the generic case here, while the other two classes are limit cases of Jacobi polynomials:

$$L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha,\beta)}(1-2x/\beta), \quad (2.5)$$

$$H_n(x) = 2^n n! \lim_{\alpha \rightarrow \infty} \alpha^{-n/2} P_n^{(\alpha,\alpha)}(\alpha^{-1/2}x). \quad (2.6)$$

Hermite polynomials are also limit cases of Laguerre polynomials:

$$H_n(x) = (-1)^n 2^{n/2} n! \lim_{\alpha \rightarrow \infty} \alpha^{-n/2} L_n^\alpha((2\alpha)^{1/2}x + \alpha). \quad (2.7)$$

The limit (2.5) is immediate from (2.2) and (2.3). The limit (2.6) follows from (2.4) and a similar series representation for *Gegenbauer* or *ultraspherical polynomials* (special Jacobi polynomials with  $\alpha = \beta$ ):

$$P_n^{(\alpha,\alpha)}(x) = \frac{(\alpha+1)_n (\alpha+1/2)_n}{(2\alpha+1)_n n!} (2x)^n {}_2F_1(-n/2, (1-n)/2; -\alpha-n+1/2; x^{-2}),$$

cf. [14, 10.9(4) and (18)]. The limit (2.7) cannot be easily derived by comparison of series representations. One method of proof is to rewrite the three term recurrence relation for Laguerre polynomials

$$(n+1)L_{n+1}^\alpha(x) - (2n+\alpha+1-x)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0$$

(cf. [14, 10.12(8)]) in terms of the polynomials in  $x$  given by the right hand side of (2.7) and, next, to compare it with the three term recurrence relation for Hermite polynomials (cf. [14, 10.13(10)])

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0.$$

Very classical orthogonal polynomials have other characterizations, for instance by the existence of a Rodrigues type formula or by the fact that the first derivatives again form a system of orthogonal polynomials (cf. [14, §10.6]). Here we want to point out that, associated with the last two characterizations, there is a pair of differential recurrence relations from which many of the basic properties of the polynomials can be easily derived. For instance, for Jacobi polynomials we have the pair

$$\frac{d}{dx} P_n^{(\alpha, \beta)}(x) = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1, \beta+1)}(x), \quad (2.8)$$

$$(1-x)^{-\alpha} (1+x)^{-\beta} \frac{d}{dx} ((1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x)) = -2n P_n^{(\alpha, \beta)}(x). \quad (2.9)$$

The differential operators in (2.8), (2.9) are called *shift operators* because of their parameter shifting property. The differential operator in (2.8) followed by the one in (2.9) yields the second order differential operator of which the Jacobi polynomials are eigenfunctions. If we would have defined Jacobi polynomials only by their orthogonality property, not by their explicit expression, then we would have already been able to derive (2.8), (2.9) up to constant factors just by the remarks that the operators  $D_-$  in (2.8) and  $D_+^{(\alpha, \beta)}$  in (2.9) satisfy

$$\begin{aligned} \int_{-1}^1 (D_- f)(x) g(x) (1-x)^{\alpha+1} (1+x)^{\beta+1} dx \\ = - \int_{-1}^1 f(x) (D_+^{(\alpha, \beta)} g)(x) (1-x)^\alpha (1+x)^\beta dx \end{aligned} \quad (2.10)$$

and that  $D_-$  sends polynomials of degree  $n$  to polynomials of degree  $n - 1$ , while  $D_+^{(\alpha, \beta)}$  sends polynomials of degree  $n - 1$  to polynomials of degree  $n$ .

The same idea can be applied again and again for the more general classical orthogonal polynomials we will discuss in this section.

It follows from (2.8), (2.9) and (2.10) that

$$\int_{-1}^1 (P_n^{(\alpha, \beta)}(x))^2 (1-x)^\alpha (1+x)^\beta dx = \text{const.} \int_{-1}^1 (1-x)^{\alpha+n} (1+x)^{\beta+n} dx,$$

where the constant on the right hand side can be easily computed. What is left for computation is a beta integral, which of course is elementary. However, we emphasize this reduction of computation of quadratic norms of classical orthogonal polynomials to computation of the integral of the weight function with shifted parameter, because this phenomenon will also return in the generalizations. Usually, the computation of the integral of the weight function is the only nontrivial part. On the other hand, if we have some deep evaluation of a definite integral with positive integrand, then it is worth to explore the polynomials being orthogonal with respect to the weight function given by the integrand.

**2.2. The Askey tableau.** Similar to the classification discussed in §2.1, one can classify all systems of orthogonal polynomials  $\{p_n\}$  which are eigenfunctions of a second order difference operator:

$$a(x)p_n(x+1) + b(x)p_n(x) + c(x)p_n(x-1) = \lambda_n p_n(x). \quad (2.11)$$

Here the definition of orthogonal polynomials is relaxed somewhat. We include the possibility that the degree  $n$  of the polynomials  $p_n$  only takes the values  $n = 0, 1, \dots, N$  and that the orthogonality is with respect to a positive measure having support on a set of  $N + 1$  points. Hahn [16] studied the  $q$ -analogue of this classification (cf. §2.5) and he pointed out how the polynomials satisfying (2.11) come out as limit cases for  $q \uparrow 1$  of his classification.

The generic case for this classification is given by the *Hahn polynomials*

$$\begin{aligned} Q_n(x; \alpha, \beta, N) &:= {}_3F_2 \left[ \begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix}; 1 \right] \\ &= \sum_{k=0}^n \frac{(-n)_k (n + \alpha + \beta + 1)_k (-x)_k}{(\alpha + 1)_k (-N)_k k!}. \end{aligned} \quad (2.12)$$

Here we assume  $n = 0, 1, \dots, N$  and we assume the notational convention that

$${}_rF_s \left[ \begin{matrix} -n, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; z \right] := \sum_{k=0}^n \frac{(-n)_k (a_2)_k \dots (a_r)_k}{(b_1)_k \dots (b_s)_k k!} z^k, \quad n \in \mathbb{Z}_+, \quad (2.13)$$

remains well-defined when some of the  $b_i$  are possibly non-positive integer but  $\leq -n$ . For the notation of Hahn polynomials and other families to be discussed in this subsection we keep to the notation of Askey & Wilson [8, Appendix] and Labelle's poster [25]. There one can also find further references.

Hahn polynomials satisfy orthogonality relations

$$\sum_{x=0}^N Q_n(x) Q_m(x) \rho(x) = \delta_{n,m} \frac{1}{\pi_n}, \quad n, m = 0, 1, \dots, N, \quad (2.14)$$

where

$$\rho(x) := \binom{N}{x} \frac{(\alpha + 1)_x (\beta + 1)_{N-x}}{(\alpha + \beta + 2)_N} \quad (2.15)$$

and

$$\pi_n := \binom{N}{n} \frac{2n + \alpha + \beta + 1}{\alpha + \beta + 1} \frac{(\alpha + 1)_n (\alpha + \beta + 1)_n}{(\beta + 1)_n (N + \alpha + \beta + 2)_n}. \quad (2.16)$$

We get positive weights  $\rho(x)$  (or weights of fixed sign) if  $\alpha, \beta \in (-1, \infty) \cup (-\infty, -N)$ .

The other orthogonal polynomials coming out of this classification are limit cases of Hahn polynomials. We get the following families.

*Krawtchouk polynomials*

$$K_n(x; p, N) := {}_2F_1(-n, -x; -N; p^{-1}), \quad n = 0, 1, \dots, N, \quad 0 < p < 1, \quad (2.17)$$

with orthogonality measure having weights  $x \mapsto \binom{N}{x} p^x (1-p)^{N-x}$  on  $\{0, 1, \dots, N\}$ ;

*Meixner polynomials*

$$M_n(x; \beta, c) := {}_2F_1(-n, -x; \beta; 1-c^{-1}), \quad 0 < c < 1, \quad \beta > 0, \quad (2.18)$$

with orthogonality measure having weights  $x \mapsto (\beta)_x c^x / x!$  on  $\mathbb{Z}_+$ ;

*Charlier polynomials*

$$C_n(x; a) := {}_2F_0(-n, -x; -; -a^{-1}), \quad a > 0,$$

with orthogonality measure having weights  $x \mapsto a^x / x!$  on  $\mathbb{Z}_+$ .

Note that Krawtchouk polynomials are Meixner polynomials (2.18) with  $\beta := q^{-N}$ . Krawtchouk and Meixner polynomials are limits of Hahn polynomials, while Charlier polynomials are limits of both Krawtchouk and Meixner polynomials. In a certain sense, the very classical orthogonal polynomials are also contained in the class discussed here, since Jacobi, Laguerre and Hermite polynomials are limits of Hahn, Meixner and Charlier polynomials, respectively. For instance,

$$P_n^{(\alpha, \beta)}(1-2x) = \frac{(\alpha+1)_n}{n!} \lim_{N \rightarrow \infty} Q_n(Nx; \alpha, \beta, N).$$

See Table 1 (or rather a part of it) for a pictorial representation of these families and their limit transitions.

The Krawtchouk, Meixner and Charlier polynomials are *self-dual*, i.e., they satisfy

$$p_n(x) = p_x(n), \quad x, n \in \mathbb{Z}_+ \text{ or } x, n \in \{0, 1, \dots, N\}.$$

Thus the orthogonality relations and dual orthogonality relations of these polynomials essentially coincide and the second order difference equation (2.11) becomes the three term recurrence relation after interchange of  $x$  and  $n$ . Then it can be arranged that the eigenvalue  $\lambda_n$  in (2.11) becomes  $n$ . By the self-duality, the  $L^2$  completeness of the systems in case  $N = \infty$  is also clear.

For Hahn polynomials we have no self-duality, so the dual orthogonal system will be different. Observe that, in (2.12), we can write

$$(-n)_k (n + \alpha + \beta + 1)_k = \prod_{j=0}^{k-1} (-n(n + \alpha + \beta + 1) + j(j + \alpha + \beta + 1)),$$

which is a polynomial of degree  $k$  in  $n(n + \alpha + \beta + 1)$ . Now define

$$R_n(x(x + \alpha + \beta + 1); \alpha, \beta, N) := Q_x(n; \alpha, \beta, N), \quad n, x = 0, 1, \dots, N.$$

Then  $R_n$  ( $n = 0, 1, \dots, N$ ) extends to a polynomial of degree  $n$ :

$$\begin{aligned} R_n(x(x + \alpha + \beta + 1); \alpha, \beta, N) &= {}_3F_2 \left[ \begin{matrix} -n, -x, x + \alpha + \beta + 1 \\ \alpha + 1, -N \end{matrix}; 1 \right] \\ &= \sum_{k=0}^n \frac{(-n)_k (-x)_k (x + \alpha + \beta + 1)_k}{(\alpha + 1)_k (-N)_k k!}. \end{aligned}$$

These are called the *dual Hahn polynomials*. The dual orthogonality relations implied by (2.14) are the orthogonality relations for the dual Hahn polynomials:

$$\sum_{x=0}^N R_m(x(x + \alpha + \beta + 1)) R_n(x(x + \alpha + \beta + 1)) \pi_x = \delta_{m,n} \frac{1}{\rho(n)}.$$

Here  $\pi_x$  and  $\rho(n)$  are as in (2.15) and (2.16). Thus the dual Hahn polynomials are also orthogonal polynomials. The three term recurrence relation for Hahn polynomials translates as a second order difference equation which is a slight generalization of (2.11). It has the form

$$a(x) p_n(\lambda(x + 1)) + b(x) p_n(\lambda(x)) + c(x) p_n(\lambda(x - 1)) = \lambda_n p_n(\lambda(x)), \quad (2.19)$$

where  $\lambda(x) := x(x + \alpha + \beta + 1)$  is a quadratic function of  $x$ . The Krawtchouk and Meixner polynomials are also limit cases of the dual Hahn polynomials.

It is a natural question to ask for other orthogonal polynomials being eigenfunctions of a second order difference equation of the form (2.19). A four-parameter family with this property are the Racah polynomials, essentially known for a long time to the physicists as Racah coefficients, which occur in connection with threefold tensor products of irreducible representations of the group  $SU(2)$ . However, it was not recognized before the late seventies (cf. Wilson [33]) that orthogonal polynomials are hidden in the Racah coefficients. *Racah polynomials* are defined by

$$R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) := {}_4F_3 \left[ \begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix}; 1 \right], \quad (2.20)$$

where  $\alpha + 1$  or  $\beta + \delta + 1$  or  $\gamma + 1 = -N$  for some  $N \in \mathbb{Z}_+$ , and where  $n = 0, 1, \dots, N$ . Similarly as for the dual Hahn polynomials it can be seen that the Racah polynomial  $R_n$  is indeed a polynomial of degree  $n$  in  $\lambda(x) := x(x + \gamma + \delta + 1)$ . The Racah polynomials are orthogonal with respect to weights  $w(x)$  on the points  $\lambda(x)$ ,  $x = 0, 1, \dots, N$ , given by

$$w(x) := \frac{(\gamma + \delta + 1)_x ((\gamma + \delta + 3)/2)_x (\alpha + 1)_x (\beta + \delta + 1)_x (\gamma + 1)_x}{x! ((\gamma + \delta + 1)/2)_x (\gamma + \delta - \alpha + 1)_x (\gamma - \beta + 1)_x (\delta + 1)_x}.$$

It is evident from (2.20) that dual Racah polynomials are again Racah polynomials with  $\alpha, \beta$  interchanged with  $\gamma, \delta$ . Hahn and dual Hahn polynomials can be obtained as limit cases of Racah polynomials.

Each orthogonality relation for the Racah polynomials is an explicit evaluation of a finite sum. It is possible to interpret this finite sum as a sum of residues coming from a



contour integral in the complex plane. This contour integral can also be considered and evaluated for values of  $N$  not necessarily in  $\mathbb{Z}_+$ . For suitable values of the parameters the contour integral can then be deformed to an integral over the imaginary axis and it gives rise to the orthogonality relations for the *Wilson polynomials* (cf. Wilson [33]) defined by

$$W_n(x^2; a, b, c, d) := (a+b)_n (a+c)_n (a+d)_n {}_4F_3 \left[ \begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1 \right]. \quad (2.21)$$

Apparently, the right hand side defines a polynomial of degree  $n$  in  $x^2$ . If  $a, b, c, d$  have positive real parts and complex parameters appear in conjugate pairs then the functions  $x \mapsto W_n(x^2)$  ( $n \in \mathbb{Z}_+$ ) of (2.21) are orthogonal with respect to the measure  $w(x)dx$  on  $[0, \infty)$ , where

$$w(x) := \left| \frac{\Gamma(a+ix) \Gamma(b+ix) \Gamma(c+ix) \Gamma(d+ix)}{\Gamma(2ix)} \right|^2.$$

The normalization in (2.21) is such that the Wilson polynomials are symmetric in their four parameters  $a, b, c, d$ .

The Wilson polynomials satisfy an eigenfunction equation of the form

$$a(x) W_n((x+i)^2) + b(x) W_n(x^2) + c(x) W_n((x-i)^2) = \lambda_n W_n(x^2).$$

So we have the new phenomenon that the difference operator at the left hand side shifts into the complex plane, out of the real interval on which the Wilson polynomials are orthogonal. This difference operator can be factorized as a product of two shift operators of similar type. They have properties and applications analogous to the shift operators for Jacobi polynomials discussed at the end of §2.1. They also reduce the evaluation of the quadratic norms to the evaluation of the integral of the weight function, but this last problem is now much less trivial than in the Jacobi case.

Now, we can descend from the Wilson polynomials by limit transitions, just as we did from the Racah polynomials. On the  ${}_3F_2$  level we thus get continuous analogues of the Hahn and dual Hahn polynomials as follows.

*Continuous dual Hahn polynomials:*

$$S_n(x^2; a, b, c) := (a+b)_n (a+c)_n {}_3F_2 \left[ \begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1 \right],$$

where  $a, b, c$  have positive real parts; if one of these parameters is not real then one of the other parameters is its complex conjugate. The functions  $x \mapsto S_n(x^2)$  are orthogonal with respect to the measure  $w(x)dx$  on  $[0, \infty)$ , where

$$w(x) := \left| \frac{\Gamma(a+ix) \Gamma(b+ix) \Gamma(c+ix)}{\Gamma(2ix)} \right|^2.$$

*Continuous Hahn polynomials:*

$$p_n(x; a, b, \bar{a}, \bar{b}) := i^n \frac{(a+\bar{a})_n (a+\bar{b})_n}{n!} {}_3F_2 \left[ \begin{matrix} -n, n+a+\bar{a}+b+\bar{b}-1, a+ix \\ a+\bar{a}, a+\bar{b} \end{matrix}; 1 \right],$$

where  $a, b$  have positive real part. The polynomials  $p_n$  are orthogonal on  $\mathbb{R}$  with respect to the measure  $|\Gamma(a+ix)\Gamma(b+ix)|^2 dx$ . In Askey & Wilson [8, Appendix] only the symmetric case ( $a, b > 0$  or  $a = \bar{b}$ ) of these polynomials occurs. The general case was discovered by Atakishiyev & Suslov [9].

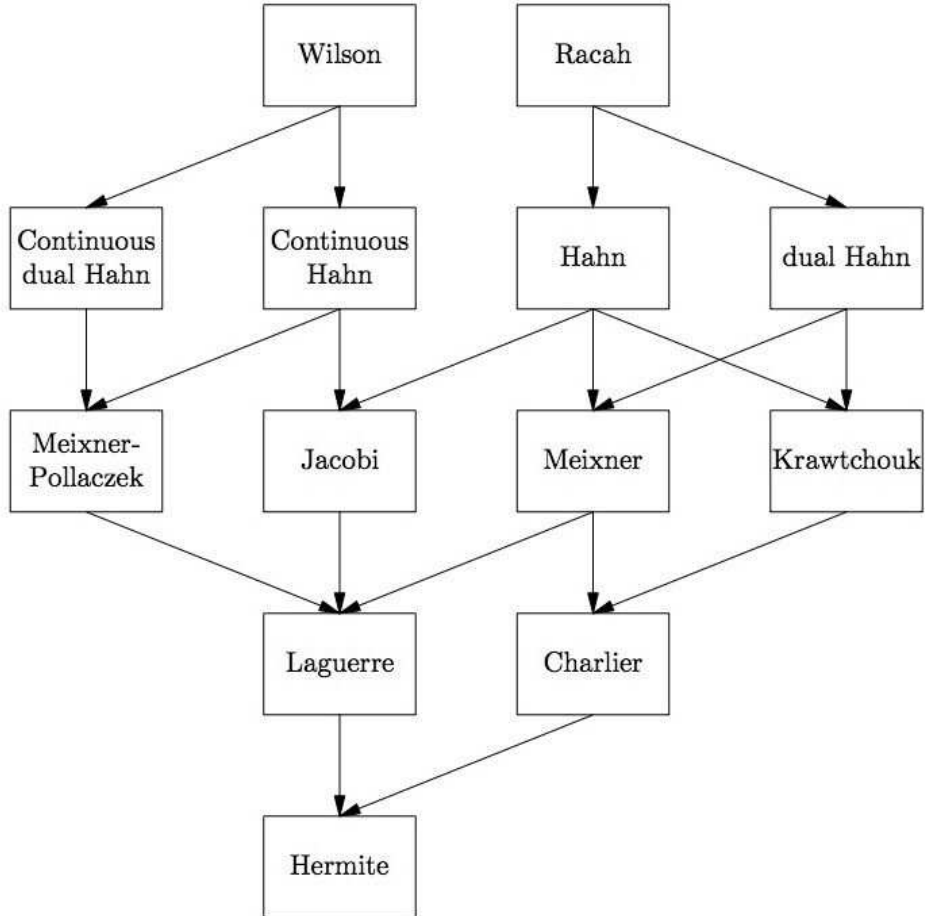


Table 1. The Askey tableau

Jacobi polynomials are limit cases of continuous Hahn polynomials and also directly of Wilson polynomials (with one pair of complex conjugate parameters). There is one further class of orthogonal polynomials on the  ${}_2F_1$  level occurring as limit cases of orthogonal polynomials on the  ${}_3F_2$  level:

*Meixner-Pollaczek polynomials:*

$$P_n^{(a)}(x; \phi) := \frac{(2a)_n}{n!} e^{in\phi} {}_2F_1(-n, a+ix; 2a; 1-e^{-2i\phi}), \quad a > 0, \quad 0 < \phi < \pi.$$

(Here we have chosen the normalization as in Labelle [25], which is the same as in Pollaczek's original 1950 paper.) They are orthogonal on  $\mathbb{R}$  with respect to the weight function  $x \mapsto e^{(2\phi-\pi)x} |\Gamma(a+ix)|^2$ . They can be considered as continuous analogues of the Meixner and Krawtchouk polynomials. They are limits of both continuous Hahn and continuous dual Hahn polynomials. Laguerre polynomials are limit cases of Meixner-Pollaczek polynomials. Note that the last three families are analytic continuations, both in  $x$  and in

the parameters, of dual Hahn polynomials, Hahn polynomials and Meixner polynomials, respectively.

All families of orthogonal polynomials discussed until now, together with the limit transitions between them, form the *Askey tableau* (or *scheme* or *chart*) of *hypergeometric orthogonal polynomials*. See Askey & Wilson [8, Appendix], Labelle [25] or Table 1. See also [20] for group theoretic interpretations.

**2.3. Big  $q$ -Jacobi polynomials.** These polynomials were hinted at by Hahn [16] and explicitly introduced by Andrews & Askey [4]. Here we will show how their basic properties can be derived from a suitable pair of shift operators. We keep the convention of §1 that  $0 < q < 1$ .

First we introduce  $q$ -integration by parts. This will involve *backward* and *forward  $q$ -derivatives*:

$$(D_q^- f)(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q^+ f)(x) := \frac{f(q^{-1}x) - f(x)}{(1-q)x}.$$

Here  $D_q^-$  coincides with  $D_q$  introduced in (1.9).

**Proposition 2.1** If  $f$  and  $g$  are continuous on  $[-d, c]$  ( $c, d \geq 0$ ) then

$$\int_{-d}^c (D_q^- f)(x) g(x) d_q x = f(c) g(q^{-1}c) - f(-d) g(-q^{-1}d) - \int_{-d}^c f(x) (D_q^+ g)(x) d_q x.$$

**Proof**

$$\begin{aligned} \int_0^c (D_q^- f)(x) g(x) d_q(x) &= \sum_{k=0}^{\infty} (f(cq^k) - f(cq^{k+1})) g(cq^k) \\ &= \lim_{N \rightarrow \infty} \left\{ f(c) g(q^{-1}c) - f(cq^{N+1}) g(cq^N) + \sum_{k=0}^N f(cq^k) (g(cq^k) - g(cq^{k-1})) \right\} \\ &= f(c) g(q^{-1}c) - f(0) g(0) + \sum_{k=0}^{\infty} f(cq^k) (g(cq^k) - g(cq^{k-1})) \\ &= f(c) g(q^{-1}c) - f(0) g(0) - \int_0^c f(x) (D_q^+ g)(x) d_q x. \end{aligned}$$

Now apply (1.13). □

Let

$$w(x; a, b, c, d; q) := \frac{(qx/c, -qx/d; q)_{\infty}}{(qax/c, -qbx/d; q)_{\infty}}. \quad (2.22)$$

Note that  $w(x; a, b, c, d; q) > 0$  on  $[-d, c]$  if  $c, d > 0$  and

$$\left[ -\frac{c}{dq} < a < \frac{1}{q} \ \& \ -\frac{d}{cq} < b < \frac{1}{q} \right] \quad \text{or} \quad [a = c\alpha \ \& \ b = -d\bar{\alpha} \ \text{for some } \alpha \in \mathbb{C} \setminus \mathbb{R}.] \quad (2.23)$$

From now on we assume that these inequalities hold. If  $f$  is continuous on  $[-d, c]$  then

$$\lim_{q \uparrow 1} \int_{-d}^c f(x) w(x; q^\alpha, q^\beta, c, d; q) d_q x = \int_{-d}^c f(x) (1 - c^{-1}x)^\alpha (1 + d^{-1}x)^\beta dx, \quad (2.24)$$

Thus the measure  $w(x; a, b, c, d; q) d_q x$  can be considered as a  $q$ -analogue of the Jacobi polynomial orthogonality measure shifted to an arbitrary finite interval containing 0.

Since

$$w(q^{-1}c; a, b, c, d; q) = 0 = w(-q^{-1}d; a, b, c, d; q),$$

we get from Proposition 2.1 that for any two polynomials  $f, g$  the following holds.

$$\begin{aligned} \int_{-d}^c (D_q^- f)(x) g(x) w(x; qa, qb, c, d; q) d_q x \\ = - \int_{-d}^c f(x) [D_q^+ (g(\cdot) w(\cdot; qa, qb, c, d; q))](x) d_q x. \end{aligned} \quad (2.25)$$

Define

$$\begin{aligned} (D_q^{+,a,b} f)(x) &:= \frac{[D_q^+ (w(\cdot; qa, qb, c, d; q) f(\cdot))](x)}{w(x; a, b, c, d; q)} \\ &= (1 - q)^{-1} x^{-1} (1 - x/c) (1 + x/d) f(q^{-1}x) \\ &\quad - (1 - q)^{-1} x^{-1} (1 - qax/c) (1 + qbx/d) f(x). \end{aligned} \quad (2.26)$$

In the following  $D_q^- f(x)$  or  $D_q^-(f(x))$  will mean  $(D_q^- f)(x)$ . Also  $D_q^{+,a,b} f(x)$  or  $D_q^{+,a,b}(f(x))$  will mean  $(D_q^{+,a,b} f)(x)$ . A simple computation yields the following.

$$D_q^- x^n = \frac{1 - q^n}{1 - q} x^{n-1}, \quad (2.27)$$

$$\begin{aligned} D_q^{+,a,b} ((q^2 ax/c; q)_{n-1}) \\ = \frac{q^{-n} a^{-1} - qb}{(1 - q)d} (qax/c; q)_n + \text{terms of degree } \leq n - 1 \text{ in } x. \end{aligned} \quad (2.28)$$

Define the *big  $q$ -Jacobi polynomial*  $\tilde{P}_n(x; a, b, c, d; q)$  as the monic orthogonal polynomial of degree  $n$  in  $x$  with respect to the measure  $w(x; a, b, c, d; q) d_q x$  on  $[-d, c]$ . Later we will introduce another normalization for these polynomials and we will then write them as  $P_n$  instead of  $\tilde{P}_n$ . It follows from (2.24) that

$$\lim_{q \uparrow 1} \tilde{P}_n(x; q^\alpha, q^\beta, c, d; q) = \text{const. } P_n^{(\alpha, \beta)} \left( \frac{d - c + 2x}{d + c} \right),$$

a Jacobi polynomial shifted to the interval  $[-d, c]$ . Big  $q$ -Jacobi polynomials with  $d = 0$ ,  $c = 1$  are called *little  $q$ -Jacobi polynomials*.

Two simple consequences of the definition are:

$$\tilde{P}_n(-x; a, b, c, d; q) = (-1)^n \tilde{P}_n(x; b, a, d, c; q) \quad (2.29)$$

and

$$\tilde{P}_n(\lambda x; a, b, c, d; q) = \lambda^n \tilde{P}_n(x; a, b, \lambda^{-1}c, \lambda^{-1}d; q), \quad \lambda > 0.$$

It follows from (2.25), (2.27) and (2.28) that  $D_q^-$  and  $D_q^{+,a,b}$  act as shift operators on the big  $q$ -Jacobi polynomials:

$$D_q^- \tilde{P}_n(x; a, b, c, d; q) = \frac{1 - q^n}{1 - q} \tilde{P}_{n-1}(x; qa, qb, c, d; q), \quad (2.30)$$

$$D_q^{+,a,b} \tilde{P}_{n-1}(x; qa, qb, c, d; q) = \frac{q^2 ab - q^{-n+1}}{(1 - q)cd} \tilde{P}_n(x; a, b, c, d; q). \quad (2.31)$$

Composition of (2.30) and (2.31) yields a second order  $q$ -difference equation for the big  $q$ -Jacobi polynomials:

$$D_q^{+,a,b} D_q^- \tilde{P}_n(x; a, b, c, d; q) = \frac{q(1 - q^{-n})(1 - q^{n+1}ab)}{(1 - q)^2 cd} \tilde{P}_n(x; a, b, c, d; q). \quad (2.32)$$

The left hand side of (2.32) can be rewritten as

$$\begin{aligned} A(x) ((D_q^-)^2 \tilde{P}_n)(q^{-1}x) + B(x) (D_q^- \tilde{P}_n)(q^{-1}x) \\ = a(x) \tilde{P}_n(q^{-1}x) + b(x) \tilde{P}_n(x) + c(x) \tilde{P}_n(qx) \end{aligned} \quad (2.33)$$

for certain polynomials  $A, B$  and  $a, b, c$ . Here

$$\begin{aligned} A(x) &= q^{-1} (1 - qax/c) (1 + qbx/d), \\ B(x) &= \frac{(1 - qb)c - (1 - qa)d - (1 - q^2 ab)x}{(1 - q)cd}. \end{aligned} \quad (2.34)$$

Compare (2.32), (2.33) with (2.1), (2.11). Apparently we have here another extension of the concept of classical orthogonal polynomials. Two other properties point into the same direction. First, by (2.30) the polynomials  $D_q^- \tilde{P}_{n+1}$  ( $n = 0, 1, 2, \dots$ ) form again a system of orthogonal polynomials. Second, iteration of (2.31) yields a Rodrigues type formula.

It follows from (2.26) that

$$(D_q^{+,a,b} f) \left( \frac{c}{qa} \right) = - \frac{(1 - qa)(1 + qad/c)}{qad(1 - q)} f \left( \frac{c}{q^2 a} \right).$$

Combination with (2.31) yields the recurrence

$$\begin{aligned} \frac{q^2 ab - q^{-n+1}}{(1 - q)cd} \tilde{P}_n \left( \frac{c}{qa}; a, b, c, d; q \right) \\ = - \frac{(1 - qa)(1 + qad/c)}{qad(1 - q)} \tilde{P}_{n-1} \left( \frac{c}{q^2 a}; qa, qb, c, d; q \right). \end{aligned}$$

By iteration we get an evaluation of the big  $q$ -Jacobi polynomial at a special point:

$$\tilde{P}_n \left( \frac{c}{qa}; a, b, c, d; q \right) = \left( \frac{c}{qa} \right)^n \frac{(qa; q)_n (-qad/c; q)_n}{(q^{n+1}ab; q)_n}. \quad (2.35)$$

Now we will normalize the big  $q$ -Jacobi polynomials such that they take the value 1 at  $c/(qa)$ :

$$P_n(x; a, b, c, d; q) := \frac{\tilde{P}_n(x; a, b, c, d; q)}{\tilde{P}_n(c/(qa); a, b, c, d; q)}. \quad (2.36)$$

Note that the value at  $-d/(qb)$  now follows from (2.29) and (2.35):

$$P_n\left(\frac{-d}{qb}; a, b, c, d; q\right) = \left(-\frac{ad}{bc}\right)^n \frac{(qb; q)_n (-qbc/d; q)_n}{(qa; q)_n (-qad/c; q)_n}. \quad (2.37)$$

It follows from (2.29) and (2.37) that

$$P_n(-x; a, b, c, d; q) = \left(-\frac{ad}{bc}\right)^n \frac{(qb; q)_n (-qbc/d; q)_n}{(qa; q)_n (-qad/c; q)_n} P_n(x; b, a, d, c; q).$$

The following lemma, proved by use of (1.10), gives the  $q$ -analogue of a Taylor series expansion.

**Lemma 2.2** If  $f(x) := \sum_{k=0}^n c_k (qax/c; q)_k$  then

$$((D_q^-)^k f) \left( \frac{c}{q^{k+1}a} \right) = c_k (-1)^k \left( \frac{qa}{c} \right)^k q^{k(k-1)/2} \frac{(q; q)_k}{(1-q)^k}.$$

Now put  $f(x) := P_n(x; a, b, c, d; q)$  in Lemma 2.2 and substitute (2.30) (iterated) and (2.36). Then the  $c_k$  can be found explicitly and we obtain a representation by a  $q$ -hypergeometric series:

$$\begin{aligned} P_n(x; a, b, c, d; q) &= \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{n+1}ab; q)_k (qax/c; q)_k q^k}{(qa; q)_k (-qad/c; q)_k (q; q)_k} \\ &= {}_3\phi_2 \left[ \begin{matrix} q^{-n}, q^{n+1}ab, qax/c \\ qa, -qad/c \end{matrix}; q, q \right]. \end{aligned} \quad (2.38)$$

Andrews & Askey [4, (3,28)] use the notation  $P_n^{(\alpha, \beta)}(x; c, d; q)$ , which coincides, up to a constant factor, with our  $p_n(x; q^\alpha, q^\beta, c, d; q)$ .

It follows by combination of (2.29) and (2.38) that

$$\frac{P_n(x; a, b, c, d; q)}{P_n(-d/(qb); a, b, c, d; q)} = {}_3\phi_2 \left[ \begin{matrix} q^{-n}, q^{n+1}ab, -qbx/d \\ qb, -qbc/d \end{matrix}; q, q \right], \quad (2.39)$$

where the denominator of the left hand side is explicitly given by (2.37).

Next we will compute the quadratic norms. By (2.25), (2.30) and (2.31) we get the recurrence

$$\begin{aligned} \int_{-d}^c (\tilde{P}_n^2 w)(x; a, b, c, d; q) d_q x &= \frac{q^{n-1} (1 - q^n) cd}{1 - q^{n+1} ab} \int_{-d}^c (\tilde{P}_{n-1}^2 w)(x; qa, qb, c, d; q) d_q x \\ &= \frac{q^{n(n-1)/2} (q; q)_n (cd)^n}{(q^{n+1}ab; q)_n} \int_{-d}^c w(x; q^n a, q^n b, c, d; q) d_q x, \end{aligned} \quad (2.40)$$

where the second equality follows by iteration. Now the  $q$ -integral of  $w$  from  $-d$  to  $c$  can be rewritten as a sum of two  ${}_2\phi_1$ 's of argument  $q$  of the form of the left hand side of (1.42). Evaluation by (1.42) yields

$$\int_{-d}^c w(x; a, b, c, d; q) d_q x = (1 - q)c \frac{(q, -d/c, -qc/d, q^2 ab; q)_\infty}{(qa, qb, -qbc/d, -qad/c; q)_\infty}. \quad (2.41)$$

Together with (2.40) this yields:

$$\frac{\int_{-d}^c (\tilde{P}_n^2 w)(x; a, b, c, d; q) d_q x}{\int_{-d}^c w(x; a, b, c, d; q) d_q x} = q^{n(n-1)/2} (cd)^n \frac{(q, qa, qb, -qbc/d, -qad/c; q)_n}{(q^2 ab; q)_{2n} (q^{n+1} ab; q)_n}. \quad (2.42)$$

Now we can compute the coefficients in the three term recurrence relation for the big  $q$ -Jacobi polynomials  $\tilde{P}_n(x) := \tilde{P}_n(x; a, b, c, d; q)$ :

$$x \tilde{P}_n(x) = \tilde{P}_{n+1}(x) + B_n \tilde{P}_n(x) + C_n \tilde{P}_{n-1}(x). \quad (2.43)$$

Then  $C_n$  is the quotient of the right hand sides of (2.42) for degree  $n$  and for degree  $n - 1$ , respectively, so

$$C_n = \frac{q^{n-1} (1 - q^n) (1 - q^n a) (1 - q^n b) (1 - q^n ab) (d + q^n bc) (c + q^n ad)}{(1 - q^{2n-1} ab) (1 - q^{2n} ab)^2 (1 - q^{2n+1} ab)}. \quad (2.44)$$

Then we obtain  $B_n$  by substitution of  $x := c/(qa)$  in (2.43), in view of (2.35).

From (2.38) one sees that  $P_n(x; a, b, c, d; q)$  is homogeneous of degree 0 in the three variables  $x, c, d$ . Therefore, only three of the four parameters  $a, b, c, d$  are essential. Probably for this reason, Gasper & Rahman [15, (7.3.10)] use the following notation for big  $q$ -Jacobi polynomials:

$$P_n^{\text{GR}}(x; a, b, c; q) = P_n(x; a, b, c; q) := {}_3\phi_2 \left[ \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}; q, q \right].$$

Their notation is related to the notation (2.38) in the present paper by

$$\begin{aligned} P_n(x; a, b, c, d; q) &= P_n^{\text{GR}}(ac^{-1}qx; a, b, -ac^{-1}d; q), \\ P_n^{\text{GR}}(x; q, b, c; q) &= P_n(x; a, b, aq, -cq; q). \end{aligned}$$

**2.4. Little  $q$ -Jacobi polynomials.** These polynomials are the most straightforward  $q$ -analogues of the Jacobi polynomials. They were first observed by Hahn [16] and studied in more detail by Andrews & Askey [3]. Here we will study them as a special case of the big  $q$ -Jacobi polynomials.

When we specialize the big  $q$ -Jacobi polynomials (2.36) to  $c = 1, d = 0$  and normalize them such that they take the value 1 at 0 then we obtain the *little  $q$ -Jacobi polynomials*

$$\begin{aligned} p_n(x; a, b; q) &:= \frac{P_n(x; b, a, 1, 0; q)}{P_n(0; b, a, 1, 0; q)} \\ &= {}_2\phi_1(q^{-n}, q^{n+1} ab; qa; q, qx) \end{aligned} \quad (2.45)$$

$$= (-qb)^{-n} q^{-n(n-1)/2} \frac{(qb; q)_n}{(qa; q)_n} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, q^{n+1} ab, qbx \\ qb, 0 \end{matrix}; q, q \right]. \quad (2.46)$$

Here (2.45) follows by letting  $d \rightarrow 0$  in (2.39), while (2.46) follows from (2.38) and (2.37). With the equality of (2.45) and (2.46) we have reobtained the transformation formula (1.38) for terminating  ${}_2\phi_1$  series. From (2.46) we obtain

$$p_n(q^{-1}b^{-1}; a, b; q) = (-qb)^{-n} q^{-n(n-1)/2} \frac{(qb; q)_n}{(qa; q)_n}.$$

From (2.45) and (1.37) we obtain

$$p_n(1; a, b; q) := (-a)^n q^{n(n+1)/2} \frac{(qb; q)_n}{(qa; q)_n}.$$

Little  $q$ -Jacobi polynomials satisfy the orthogonality relations

$$\begin{aligned} \frac{1}{B_q(\alpha+1, \beta+1)} \int_0^1 p_n(t; q^\alpha, q^\beta; q) p_m(t; q^\alpha, q^\beta; q) t^\alpha \frac{(qt; q)_\infty}{(q^{\beta+1}t; q)_\infty} d_q t \\ = \delta_{n,m} \frac{q^{n(\alpha+1)} (1 - q^{\alpha+\beta+1}) (q^{\beta+1}; q)_n (q; q)_n}{(1 - q^{2n+\alpha+\beta+1}) (q^{\alpha+1}; q)_n (q^{\alpha+\beta+1}; q)_n}. \end{aligned} \quad (2.47)$$

The orthogonality measure is the measure of the  $q$ -beta integral (1.27). For positivity and convergence we require that  $0 < a < 1/q$ ,  $b < 1/q$  (after we have replaced  $q^\alpha$  by  $a$  and  $q^\beta$  by  $b$  in (2.47)). It is maybe not immediately seen that (2.47) is a limit case of (2.42). However, we can establish this by observing the weak convergence as  $d \downarrow 0$  of the normalized measure  $\text{const.} \times w(x; q^\beta, q^\alpha, 1, d; q) d_q x$  on  $[-d, 1]$  to the normalized measure in (2.47) on  $[0, 1]$ :

$$\begin{aligned} \lim_{d \downarrow 0} \frac{\int_{-d}^1 (q^{\beta+1}t; q)_n w(t; q^\beta, q^\alpha, 1, d; q) d_q t}{\int_{-d}^1 w(t; q^\beta, q^\alpha, 1, d; q) d_q t} &= \lim_{d \downarrow 0} \frac{(q^{\beta+1}; q)_n (-q^{\beta+1}d; q)_n}{(q^{\alpha+\beta+2}; q)_n} \\ &= \frac{(q^{\beta+1}; q)_n}{(q^{\alpha+\beta+2}; q)_n} = \frac{1}{B_q(\alpha+1, \beta+1)} \int_0^1 (q^{\beta+1}t; q)_n \frac{(qt; q)_\infty}{(q^{\beta+1}t; q)_\infty} t^\alpha d_q t. \end{aligned}$$

Here the first equality follows from (2.41) and (2.22), while the last equality follows from (1.27).

Little  $q$ -Jacobi polynomials are the  $q$ -analogues of the Jacobi polynomials shifted to the interval  $[0, 1]$ :

$$\lim_{q \uparrow 1} p_n(x; q^\alpha, q^\beta; q) = \frac{P_n^{(\alpha, \beta)}(1 - 2x)}{P_n^{(\alpha, \beta)}(1)}.$$

If we fix  $b < 1$  (possibly 0 or negative) and put  $a = q^\alpha$  then little  $q$ -Jacobi polynomials tend for  $q \uparrow 1$  to Laguerre polynomials:

$$\lim_{q \uparrow 1} p_n \left( \frac{(1-q)x}{1-b}; q^\alpha, b; q \right) = \frac{L_n^\alpha(x)}{L_n^\alpha(0)}.$$

In particular, little  $q$ -Jacobi polynomials  $p_n(x; a, 0; q)$ , called *Wall polynomials*, are  $q$ -analogues of Laguerre polynomials.



Analogous to the quadratic transformations for Jacobi polynomials (cf. [14, 10.9(4), (21) and (22)]) we can find *quadratic transformations* between little and big  $q$ -Jacobi polynomials:

$$P_{2n}(x; a, a, 1, 1; q) = \frac{p_n(x^2; q^{-1}, a^2; q^2)}{p_n((qa)^{-2}; q^{-1}, a^2; q^2)}, \quad (2.48)$$

$$P_{2n+1}(x; a, a, 1, 1; q) = \frac{x p_n(x^2; q, a^2; q^2)}{(qa)^{-1} p_n((qa)^{-2}; q, a^2; q^2)}. \quad (2.49)$$

The proof is also analogous: by use of the orthogonality properties and normalization of the polynomials.

Just as Jacobi polynomials tend to Bessel functions by

$$\frac{P_{n_N}^{(\alpha, \beta)}(1 - x^2/(2N^2))}{P_{n_N}^{(\alpha, \beta)}(1)} = {}_2F_1\left(-n_N, n_N + \alpha + \beta + 1; \alpha + 1; \frac{x^2}{4N^2}\right)$$

$$\xrightarrow{N \rightarrow \infty} {}_0F_1(-; \alpha + 1; -(\lambda x/2)^2) = (\lambda x/2)^{-\alpha} \Gamma(\alpha + 1) J_\alpha(\lambda x), \quad n_N/N \rightarrow \lambda \text{ as } N \rightarrow \infty,$$

little  $q$ -Jacobi polynomials tend to Jackson's third  $q$ -Bessel function (1.60):

$$\lim_{N \rightarrow \infty} p_{N-n}(q^N x; a, b; q) = \lim_{N \rightarrow \infty} {}_2\phi_1(q^{-N+n}, abq^{N-n+1}; aq; q, q^{N+1}x)$$

$$= {}_1\phi_1(0; aq; q, q^{n+1}x),$$

cf. Koornwinder & Swarttouw [24, Prop. A.1].

**2.5. Hahn's classification.** Hahn [16] classified all families of orthogonal polynomials  $p_n$  ( $n = 0, 1, \dots, N$  or  $n = 0, 1, \dots$ ) which are eigenfunctions of a second order  $q$ -difference equation, i.e.,

$$A(x) (D_q^2 p_n)(q^{-1}x) + B(x) (D_q p_n)(q^{-1}x) = \lambda_n p_n(x) \quad (2.50)$$

or equivalently

$$a(x) p_n(q^{-1}x) + b(x) p_n(x) + c(x) p_n(qx) = \lambda_n p_n(x),$$

where  $A, B$  and  $a, b, c$  are fixed polynomials. Necessarily,  $A$  is of degree  $\leq 2$  and  $B$  is of degree  $\leq 1$ . The eigenvalues  $\lambda_n$  will be completely determined by  $A$  and  $B$ . One distinguishes cases depending on the degrees of  $A$  and  $B$  and the situation of the zeros of  $A$ . For each case one finds a family of  $q$ -hypergeometric polynomials satisfying (2.50) which, moreover, satisfies an explicit three term recurrence relation

$$x p_n(x) = p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x).$$

(Here we assumed the  $p_n$  to be monic.) Then we will have orthogonal polynomials with respect to a positive orthogonality measure iff  $C_n > 0$  for all  $n$ . A next problem is to find the explicit orthogonality measure. For a given family of  $q$ -hypergeometric polynomials depending on parameters the type of this measure may vary with the parameters.

Finally the limit transitions between the various families of orthogonal polynomials can be examined.

In essence this program has been worked out by Hahn [16], but his paper is somewhat sketchy in details. Unfortunately, there is no later publication, where the details have all been filled in. In Table 2 we give a  $q$ -Hahn tableau: a  $q$ -analogue of that part of the Askey tableau (Table 1) which is dominated by the Hahn polynomials. In the  ${}_r\phi_s$  formulas in the Table we have omitted the last but one parameter denoting the base except when this is different from  $q$ . The arrows denote limit transitions. The box for case 3a, which occurred in an earlier version, is now suppressed, because it does not correspond to a class of orthogonal polynomials. Below we will give a brief discussion of each case. We do not claim completeness.

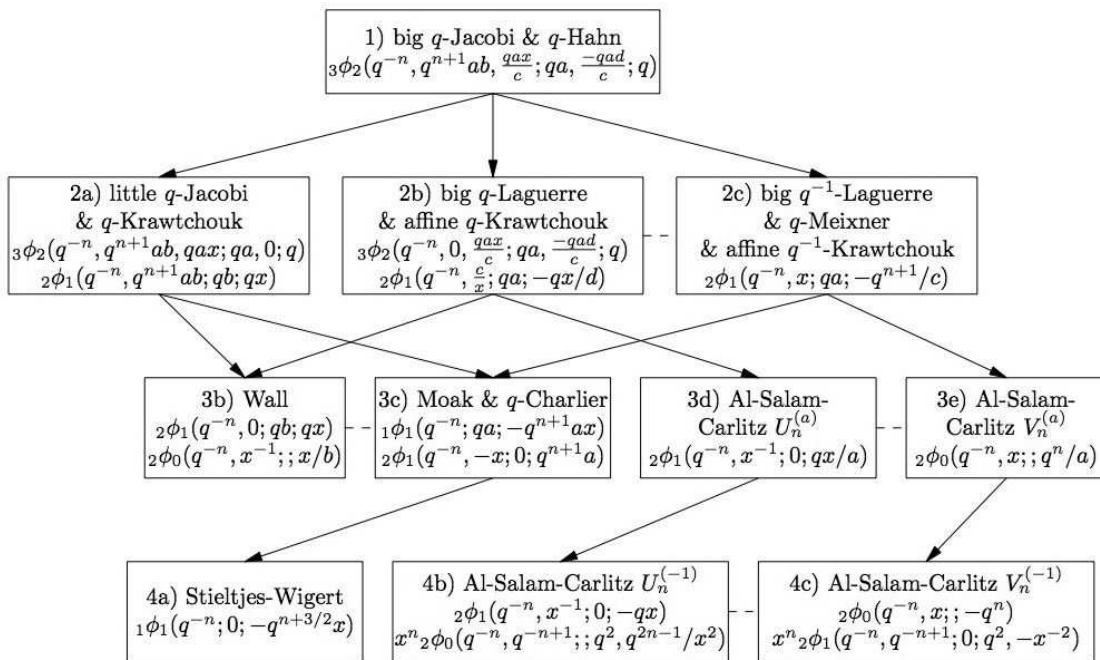


Table 2. The  $q$ -Hahn tableau.

Several new phenomena occur here:

1. Within one class of  $q$ -hypergeometric polynomials we may obtain, depending on the values of the parameters, either a  $q$ -analogue of the Jacobi-Laguerre-Hermite class or of the (discrete) Hahn-Krawtchouk-Meixner-Charlier class.
2.  $q$ -Analogues of Jacobi, Laguerre and Hermite polynomials occur in a “little” version (corresponding to Jacobi polynomials on  $[0, c]$ , Laguerre polynomials on  $[0, \infty)$  and Hermite polynomials which are even or odd functions) and a “big” version ( $q$ -analogues of Jacobi, Laguerre and Hermite polynomials of arbitrarily shifted argument).
3. There is a duality under the transformation  $q \mapsto q^{-1}$ , denoted in Table 2 by dashed lines. We insist on the convention  $0 < q < 1$ , but we can rewrite  $q^{-1}$ -hypergeometric orthogonal polynomials as  $q$ -hypergeometric orthogonal polynomials and thus sometimes obtain another family.
4. Cases may occur where the orthogonality measure is not uniquely determined.

We now briefly discuss each case occurring in Table 2.

1) *Big  $q$ -Jacobi polynomials*  $P_n(x; a, b, c, d; q)$ , defined by (2.36), form the generic case in this classification.  $A(x)$  and  $B(x)$  in (2.50) are given by (2.34) and  $\lambda_n$  is as in the right hand side of (2.32). From the explicit expression for  $C_n$  in (2.44) we get the values of  $a, b, c, d$  for which there is a positive orthogonality measure. For  $c, d > 0$  these are given by (2.23).

The  *$q$ -Hahn polynomials* can be obtained as special big  $q$ -Jacobi polynomials with  $-qad/c = q^{-N}$ ,  $n = 0, 1, \dots, N$  ( $N \in \mathbb{Z}_+$ ). For convenience we may take  $c = qa$ . The  $q$ -Hahn polynomials are usually (cf. [15, (7.2.21)]) notated as

$$Q_n(x; a, b, N; q) := {}_3\phi_2 \left[ \begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, q^{-N} \end{matrix}; q, q \right] = \sum_{k=0}^n \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k (x; q)_k}{(aq; q)_k (q^{-N}; q)_k (q; q)_k} q^k.$$

Here the convention regarding lower parameters  $q^{-N}$  ( $N \in \mathbb{Z}_+$ ) is similar to the convention for (2.13).

$q$ -Hahn polynomials satisfy orthogonality relations

$$\sum_{x=0}^N (Q_n Q_m)(q^{-x}; a, b, N; q) \frac{(aq; q)_x (bq; q)_{N-x}}{(q; q)_x (q; q)_{N-x}} (aq)^{-x} = 0, \quad n \neq m.$$

The weights are positive in one of the three following cases: (i)  $b < q^{-1}$  and  $0 < a < q^{-1}$ ; (ii)  $a, b > q^{-N}$ ; (iii)  $a < 0$  and  $b > q^{-N}$ . For  $q \uparrow 1$  the polynomials tend to ordinary Hahn polynomials (2.12):

$$\lim_{q \uparrow 1} Q_n(q^{-x}; q^\alpha, q^\beta, N; q) = Q_n(x; \alpha, \beta, N).$$

2a) *Little  $q$ -Jacobi polynomials*  $p_n(x; a, b; q)$ , given by (2.45), (2.46), were discussed in §2.4. When we put  $b := q^{-N-1}$  in (2.46), we obtain  $q$ -analogues of Krawtchouk polynomials (2.17): the  *$q$ -Krawtchouk polynomials*

$$K_n(x; b, N; q) := {}_3\phi_2 \left[ \begin{matrix} q^{-n}, -b^{-1}q^n, x \\ 0, q^{-N} \end{matrix}; q, q \right] = \lim_{a \rightarrow 0} Q_n(x; a, -(qba)^{-1}, N; q)$$

with orthogonality relations

$$\sum_{x=0}^N (K_n K_m)(q^{-x}; b, N; q) \frac{(q^{-N}; q)_x}{(q; q)_x} (-b)^x = 0, \quad n \neq m, \quad n, m = 0, 1, \dots, N,$$

and limit transition

$$\lim_{q \uparrow 1} K_n(q^{-x}; b, N; q) = K_n(x; b/(b+1), N).$$

See [15, Exercise 7.8(i)] and the reference given there. Note that the  $q$ -Krawtchouk polynomials  $K_n(q^{-x}; b, N; q)$  are not self-dual under the interchange of  $x$  and  $n$ .

2b) *Big  $q$ -Laguerre polynomials*, these are big  $q$ -Jacobi polynomials with  $b = 0$ :

$$\begin{aligned} P_n(x; a, 0, c, d; q) &= {}_3\phi_2 \left[ \begin{matrix} q^{-n}, 0, qax/c \\ qa, -qad/c \end{matrix}; q, q \right] \\ &= \frac{1}{(-q^{-n}ca^{-1}d^{-1}; q)_n} {}_2\phi_1 \left[ \begin{matrix} q^{-n}, c/x \\ qa \end{matrix}; q, -qx/d \right]. \end{aligned} \quad (2.51)$$

Here the second equality follows by (1.39). Note that

$$\lim_{q \uparrow 1} P_n(x; q^\alpha, 0, c, (1-q)^{-1}; q) = {}_1F_1(-n; \alpha + 1; c - x) = \text{const. } L_n^\alpha(c - x).$$

Another family of  $q$ -analogues of Krawtchouk polynomials, called *affine  $q$ -Krawtchouk polynomials*, can be obtained from (2.51) by putting  $-qad/c = q^{-N}$ :

$$K_n^{Aff}(x; a, N; q) := {}_3\phi_2 \left[ \begin{matrix} q^{-n}, 0, x \\ aq, q^{-N} \end{matrix}; q, q \right] = Q_n(x; a, 0, N; q)$$

with orthogonality relations

$$\sum_{x=0}^N (K_n^{Aff} K_m^{Aff})(q^{-x}; a, N; q) \frac{(aq; q)_x (aq)^{-x}}{(q; q)_x (q; q)_{N-x}} = 0, \quad n \neq m, \quad n, m = 0, 1, \dots, N,$$

and limit transition

$$\lim_{q \uparrow 1} K_n^{Aff}(q^{-x}; a, N; q) = K_n(x; 1 - a, N).$$

See [15, Exercise 7.11] and the references given there.

2c)  *$q$ -Meixner polynomials*

$$M_n(x; a, c; q) := {}_2\phi_1 \left[ \begin{matrix} q^{-n}, x \\ qa \end{matrix}; q, \frac{-q^{n+1}}{c} \right]. \quad (2.52)$$

For certain values of the parameters these can be considered as  $q$ -analogues of the Meixner polynomials (2.18), cf. [15, Exercise 7.12]. However, when we write these polynomials as

$$M_n(qax; a, -\bar{a}^{-1}; q) = {}_2\phi_1 \left[ \begin{matrix} q^{-n}, qax \\ qa \end{matrix}; q, q^{n+1}\bar{a} \right] = \lim_{d \rightarrow \infty} P_n(x; a, -\bar{a}d, 1, d; q)$$

and  $a$  is complex but not real, then these polynomials also become orthogonal (not documented in the literature) and they can be considered as  $q$ -analogues of Laguerre polynomials of shifted argument. Moreover, these polynomials can be obtained from the big  $q$ -Laguerre polynomials (2.51) by the transformation  $q \mapsto q^{-1}$ . Therefore, we call these polynomials also *big  $q^{-1}$ -Laguerre polynomials*.

For  $a := q^{-N-1}$  the polynomials (2.52) become yet another family of  $q$ -analogues of the Krawtchouk polynomials, which we will call *affine  $q^{-1}$ -Krawtchouk polynomials*,

because they are obtained from affine  $q$ -Krawtchouk polynomials by changing  $q$  into  $q^{-1}$ . These polynomials, written as

$$M_n(x; q^{-N-1}, -b^{-1}; q) := {}_2\phi_1(q^{-n}, x; q^{-N}; q, bq^{n+1}) = \lim_{a \rightarrow \infty} Q_n(x; a, b, N; q),$$

where  $n = 0, 1, \dots, N$ , have orthogonality relations

$$\sum_{x=0}^N (M_n M_m)(q^{-x}; q^{-N-1}, -b^{-1}; q) \frac{(bq; q)_{N-x} (-1)^{N-x} q^{x(x-1)/2}}{(q; q)_x (q; q)_{N-x}} = 0, \quad n \neq m,$$

and limit transition

$$\lim_{q \uparrow 1} M_n(q^{-x}; q^{-N-1}, -b^{-1}; q) = K_n(x; b^{-1}, N).$$

See Koornwinder [21].

3b) *Wall polynomials* (cf. Chihara [12, §VI.11]) are special little  $q$ -Jacobi polynomials

$$\begin{aligned} p_n(x; a, 0; q) &= {}_2\phi_1(q^{-n}, 0; qa; q, qx) \\ &= \frac{1}{(q^n a^{-1}; q)_n} {}_2\phi_0(q^{-n}, x^{-1}; -; q, x/a). \end{aligned}$$

The second equality is a limit case of (1.39). Wall polynomials are  $q$ -analogues of Laguerre polynomials on  $[0, \infty)$ , so they might be called little  $q$ -Laguerre polynomials.

3c) Moak's [29]  *$q$ -Laguerre polynomials* (notation as in [15, Exercise 7.43]) are given by

$$\begin{aligned} L_n^\alpha(x; q) &:= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_1\phi_1(q^{-n}; q^{\alpha+1}; q, -xq^{n+\alpha+1}) \\ &= \frac{1}{(q; q)_n} {}_2\phi_1(q^{-n}, -x; 0, q, q^{n+\alpha+1}). \end{aligned}$$

The second equality is a limit case of (1.38). They can be obtained from the Wall polynomials by replacing  $q$  by  $q^{-1}$ . Their orthogonality measure is not unique. For instance, there is a continuous orthogonality measure

$$\int_0^\infty L_m^\alpha(x; q) L_n^\alpha(x; q) \frac{x^\alpha dx}{(-(1-q)x; q)_\infty} = \delta_{m,n}, \quad m \neq n,$$

but also discrete orthogonality measures

$$\int_0^\infty L_m^\alpha(cx; q) L_n^\alpha(cx; q) \frac{x^\alpha d_q x}{(-c(1-q)x; q)_\infty} = \delta_{m,n}, \quad m \neq n, \quad c > 0.$$

Chihara [12]Ch. VI, §2 calls these polynomials generalized Stieltjes-Wigert polynomials and he uses another notation. For certain parameter values these polynomials may be considered as  $q$ -analogues of the Charlier polynomials, see [15, Exercise 7.13].

3d) *Al-Salam-Carlitz I polynomials*

$$U_n^{(a)}(x) = U_n^{(a)}(x; q) := (-1)^n q^{n(n-1)/2} a^n {}_2\phi_1(q^{-n}, x^{-1}; 0; q, qx/a)$$

(cf. Al-Salam & Carlitz [1]) satisfy the three term recurrence relation

$$x U_n^{(a)}(x) = U_{n+1}^{(a)}(x) + (1+a) q^n U_n^{(a)}(x) - a q^{n-1} (1-q^n) U_{n-1}^{(a)}(x).$$

Thus they are orthogonal polynomials for  $a < 0$ . They can be expressed in terms of big  $q$ -Jacobi polynomials by

$$U_n^{(a)}(x) = \tilde{P}_n(x; 0, 0, 1, -a; q),$$

so they are orthogonal with respect to the measure  $(qx, qx/a; q)_\infty d_q x$  on  $[a, 1]$ , cf. (2.22). By the above recurrence relation these polynomials are  $q$ -analogues of Hermite polynomials of shifted argument, so they may be considered as “big”  $q$ -Hermite polynomials.

3e) *Al-Salam-Carlitz II polynomials*

$$V_n^{(a)}(x) = V_n^{(a)}(x; q) := U_n^{(a)}(x; q^{-1}) = (-1)^n q^{-n(n-1)/2} a^n {}_2\phi_0(q^{-n}, x; -, q^n a^{-1})$$

(cf. [1]). For  $a > 0$  these form another family of  $q$ -analogues of the Charlier polynomials. On the other hand, the polynomials  $x \mapsto V_n^{(\alpha/\bar{\alpha})}(-q\alpha x)$  can be considered as  $q$ -analogues of Hermite polynomials of shifted argument. See Groenevelt [34] for a further discussion of this case.

4a) *Stieltjes-Wigert polynomials*

$$S_n(x; q) := (-1)^n q^{-n(2n+1)/2} {}_1\phi_1(q^{-n}; 0; q, -q^{n+3/2}x)$$

(cf. Chihara [12, §VI.2]) do not have a unique orthogonality measure. It was already noted by Stieltjes that the corresponding moments are an example of a non-determinate (Stieltjes) moment problem. After suitable scaling these polynomials tend to Hermite polynomials as  $q \uparrow 1$ .

4b) *Al-Salam-Carlitz I polynomials*  $U_n^{(a)}$  with  $a := -1$  (these are also known as *discrete  $q$ -Hermite I polynomials*  $h_n(\cdot; q)$ ):

$$\begin{aligned} h_n(x; q) &= U_n^{(-1)}(x; q) = q^{n(n-1)/2} {}_2\phi_1(q^{-n}, x^{-1}; 0; q, -qx) \\ &= \tilde{P}_n(x; 0, 0, 1, 1; q) \\ &= x^n {}_2\phi_0(q^{-n}, q^{-n+1}; -, q^2, q^{2n-1}x^{-2}) \end{aligned}$$

(cf. [1]),  $q$ -analogues of Hermite polynomials. By the last equality there is a quadratic transformation between these polynomials and certain Wall polynomials. This is a limit case of the quadratic transformations (2.48) and (2.49).

4c) *Al-Salam-Carlitz II polynomials*  $V_n^{(a)}$  of imaginary argument and with  $a := -1$  (also known as *discrete  $q$ -Hermite II polynomials*  $\tilde{h}_n(\cdot; q)$ ):

$$\begin{aligned} \tilde{h}_n(x; q) &= i^{-n} V_n^{(-1)}(ix; q) = i^{-n} q^{-n(n-1)/2} {}_2\phi_0(q^{-n}, ix; -, -q^n) \\ &= \lim_{c \rightarrow \infty} \tilde{P}_n(x; ic, ic, qc, qc; q) \\ &= x^n {}_2\phi_1(q^{-n}, q^{-n+1}; 0; q^2, -x^{-2}) \end{aligned}$$

(cf. [1]), also  $q$ -analogues of Hermite polynomials. By the last equality there is a quadratic transformation between these polynomials and certain of Moak’s  $q$ -Laguerre polynomials. This is a limit case of (2.48) and (2.49).

**2.6. The Askey-Wilson integral.** We remarked earlier that, whenever some nontrivial evaluation of an integral can be given, the orthogonal polynomials having the integrand as weight function may be worthwhile to study. In this subsection we will give an evaluation of the integral which corresponds to the Askey-Wilson polynomials. After the original evaluation in Askey & Wilson [8] several easier approaches were given, cf. Gasper & Rahman [15, Ch. 6] and the references given there. Our proof below borrowed ideas from Kalnins & Miller [18] and Miller [27] but is still different from theirs. Fix  $0 < q < 1$ . Let

$$w_{a,b,c,d}(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty}. \quad (2.53)$$

We want to evaluate the integral

$$I_{a,b,c,d} := \frac{1}{2\pi i} \oint_{|z|=1} w_{a,b,c,d}(z) \frac{dz}{z}, \quad |a|, |b|, |c|, |d| < 1. \quad (2.54)$$

Note that  $I_{a,b,c,d}$  is analytic in the four complex variables  $a, b, c, d$  when these are bounded in absolute value by 1. It is also symmetric in  $a, b, c, d$ .

**Lemma 2.3**

$$I_{a,b,c,d} = \frac{1 - abcd}{(1 - ab)(1 - ac)(1 - ad)} I_{qa,b,c,d}. \quad (2.55)$$

**Proof** The integral

$$\oint \frac{w_{q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d}(z) dz}{z - z^{-1}} \frac{dz}{z}$$

equals on the one hand

$$\begin{aligned} & \oint \frac{w_{q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d}(q^{1/2}z) dz}{q^{1/2}z - q^{-1/2}z^{-1}} \frac{dz}{z} \\ &= \oint \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz) w_{a,b,c,d}(z) dz}{q^{1/2}z(1 - z^2)} \frac{dz}{z} \end{aligned}$$

and on the other hand

$$\begin{aligned} & \oint \frac{w_{q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d}(q^{-1/2}z) dz}{q^{-1/2}z - q^{1/2}z^{-1}} \frac{dz}{z} \\ &= - \oint \frac{(1 - a/z)(1 - b/z)(1 - c/z)(1 - d/z) w_{a,b,c,d}(z) dz}{q^{1/2}z^{-1}(1 - z^{-2})} \frac{dz}{z}. \end{aligned}$$

Subtraction yields

$$\begin{aligned} 0 &= q^{-1/2}a^{-1} \oint \{ -(1 - abcd)(1 - az)(1 - az^{-1}) + (1 - ab)(1 - ac)(1 - ad) \} \\ &\times w_{a,b,c,d}(z) \frac{dz}{z} = 2\pi i q^{-1/2}a^{-1} \{ -(1 - abcd)I_{qa,b,c,d} + (1 - ab)(1 - ac)(1 - ad)I_{a,b,c,d} \}. \quad \square \end{aligned}$$

By iteration of (2.55) and use of the analyticity and symmetry of  $I_{a,b,c,d}$  we obtain

$$I_{a,b,c,d} = \frac{(abcd; q)_\infty}{(ab, ac, ad, bc, bd, cd; q)_\infty} I_{0,0,0,0}. \quad (2.56)$$

We might evaluate  $I_{0,0,0,0}$  by the Jacobi triple product identity (1.50), but it is easier to observe that  $I_{1,q^{1/2},-1,-q^{1/2}} = 1$ . (Note that this case can be continuously reached from the domain of definition of  $I_{a,b,c,d}$  in (2.54).) Hence (2.56) yields that  $I_{0,0,0,0} = 2/(q; q)_\infty$ . Thus

$$I_{a,b,c,d} = \frac{2(abcd; q)_\infty}{(ab, ac, ad, bc, bd, cd, q; q)_\infty}. \quad (2.57)$$

By the symmetry of  $w_{a,b,c,d}(z)$  under  $z \mapsto z^{-1}$  we can rewrite (2.54), (2.57) as

$$\frac{1}{2\pi} \int_0^\pi w_{a,b,c,d}(e^{i\theta}) d\theta = \frac{(abcd; q)_\infty}{(ab, ac, ad, bc, bd, cd, q; q)_\infty}, \quad |a|, |b|, |c|, |d| < 1. \quad (2.58)$$

Here  $w$  is still defined by (2.53). The integral (2.58) is known as the *Askey-Wilson integral*.

**2.7. Askey-Wilson polynomials.** We now look for an orthogonal system corresponding to the weight function in the Askey-Wilson integral. First observe that

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{(az, az^{-1}; q)_k}{(ac, ad; q)_k} \frac{(bz, bz^{-1}; q)_l}{(bc, bd; q)_l} w_{a,b,c,d}(z) \frac{dz}{z} &= \frac{I_{q^k a, q^l b, c, d}}{(ac, ad; q)_k (bc, bd; q)_l} \\ &= \frac{(ab; q)_{k+l}}{(abcd; q)_{k+l}} I_{a,b,c,d}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{I_{a,b,c,d}} \frac{1}{2\pi i} \oint \left\{ \sum_{k=0}^n \frac{(q^{-n}, q^{n-1}abcd; q)_k q^k}{(ab, q; q)_k} \frac{(az, az^{-1}; q)_k}{(ac, ad; q)_k} \right\} \frac{(bz, bz^{-1}; q)_l}{(bc, bd; q)_l} w_{a,b,c,d}(z) \frac{dz}{z} \\ &= \sum_{k=0}^n \frac{(q^{-n}, q^{n-1}abcd; q)_k q^k}{(ab, q; q)_k} \frac{(ab; q)_{k+l}}{(abcd; q)_{k+l}} \\ &= \frac{(ab; q)_l}{(abcd; q)_l} {}_3\phi_2 \left[ \begin{matrix} q^{-n}, q^{n-1}abcd, q^l ab \\ ab, q^l abcd \end{matrix}; q, q \right] \\ &= \frac{(ab; q)_l}{(abcd; q)_l} \frac{(q^{-n+1}c^{-1}d^{-1}, q^{-l}; q)_n}{(ab, q^{-n-l+1}/(abcd); q)_n} = 0, \quad l = 0, 1, \dots, n-1. \end{aligned} \quad (2.59)$$

Here we used the  $q$ -Saalschütz formula (1.61).

The above orthogonality suggests to define *Askey-Wilson polynomials* (Askey & Wilson [8])

$$\frac{p_n(\cos \theta; a, b, c, d | q)}{a^{-n} (ab, ac, ad; q)_n} = r_n(\cos \theta; a, b, c, d | q) \quad (2.60)$$

$$:= {}_4\phi_3 \left[ \begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right] \quad (2.61)$$

$$= \sum_{k=0}^n \frac{(q^{-n}, q^{n-1}abcd; q)_k q^k}{(ab, q; q)_k} \frac{(ae^{i\theta}, ae^{-i\theta}; q)_k}{(ac, ad; q)_k}.$$



Since

$$(ae^{i\theta}, ae^{-i\theta}; q)_k = \prod_{j=0}^{k-1} (1 - 2aq^j \cos \theta + a^2 q^{2j}),$$

formula (2.61) defines a polynomial of degree  $n$  in  $\cos \theta$ . It follows from (2.59) that the functions  $\theta \mapsto p_n(\cos \theta; a, b, c, d | q)$  are orthogonal with respect to the measure  $w_{a,b,c,d}(e^{i\theta}) d\theta$  on  $[0, \pi]$ , so we are really dealing with orthogonal polynomials. Now

$$p_n(x; a, b, c, d | q) = k_n x^n + \text{terms of lower degree}, \quad \text{with } k_n := 2^n (q^{n-1}abcd; q)_n, \quad (2.62)$$

so the coefficient of  $x^n$  is symmetric in  $a, b, c, d$ . Since the weight function is also symmetric in  $a, b, c, d$ , the Askey-Wilson polynomial will be itself symmetric in  $a, b, c, d$ .

**Proposition 2.4** Let  $|a|, |b|, |c|, |d| < 1$ . Then

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos \theta) p_m(\cos \theta) w(\cos \theta) d\theta = \delta_{m,n} h_n,$$

where

$$\begin{aligned} p_n(\cos \theta) &= p_n(\cos \theta; a, b, c, d | q), \\ w(\cos \theta) &= \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}, \\ \frac{h_n}{h_0} &= \frac{(1 - q^{n-1}abcd) (q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd) (abcd; q)_n}, \end{aligned}$$

and

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}.$$

The orthogonality measure is positive if  $a, b, c, d$  are real, or if complex, appear in conjugate pairs,

**Proof** Apply (2.59) to

$$\frac{1}{I_{a,b,c,d}} \frac{1}{2\pi i} \oint (p_n p_m) ((z + z^{-1})/2; a, b, c, d | q) w_{a,b,c,d}(z) \frac{dz}{z},$$

where  $p_n$  is expanded according to (2.61) and  $p_m$  similarly, but with  $a$  and  $b$  interchanged.  $\square$

Note that we can evaluate the Askey-Wilson polynomial  $p_n((z + z^{-1})/2)$  for  $z = a$  (and by symmetry also for  $z = b, c, d$ ):

$$p_n((a + a^{-1})/2; a, b, c, d | q) = a^{-n} (ab, ac, ad; q)_n. \quad (2.63)$$

When we write the three term recurrence relation for the Askey-Wilson polynomial as

$$2x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \quad (2.64)$$

then the coefficients  $A_n$  and  $C_n$  can be computed from

$$A_n = \frac{2k_n}{k_{n+1}}, \quad C_n = \frac{2k_{n-1}}{k_n} \frac{h_n}{h_{n-1}},$$

where  $k_n$  and  $h_n$  are given by (2.62) and Proposition 2.4, respectively. Then  $B_n$  can next be computed from (2.63) by substituting  $x := (a + a^{-1})/2$  in (2.64).

**2.8. Various results.** Here we collect without proof some further results about Askey-Wilson polynomials and their special cases and limit cases.

The *q*-ultraspherical polynomials

$$\begin{aligned} C_n(\cos \theta; \beta \mid q) &:= \frac{(\beta^2; q)_n}{\beta^{n/2}(q; q)_n} {}_4\phi_3 \left[ \begin{matrix} q^{-n}, q^n \beta^2, \beta^{1/2} e^{i\theta}, \beta^{1/2} e^{-i\theta} \\ \beta q^{1/2}, -\beta q^{1/2}, -\beta \end{matrix} ; q, q \right] \\ &= \text{const. } p_n(\cos \theta; \beta^{1/2}, \beta^{1/2} q^{1/2}, -\beta^{1/2}, -\beta^{1/2} q^{1/2} \mid q) \\ &= \frac{(\beta; q)_n}{(q; q)_n} \sum_{k=0}^n \frac{(q^{-n}, \beta; q)_k}{(q^{1-n} \beta^{-1}, q; q)_k} (q/\beta)^k e^{i(n-2k)\theta} \end{aligned}$$

are special Askey-Wilson polynomials which were already known to Rogers (1894), however not as orthogonal polynomials. See Askey & Ismail [5].

By easy arguments using analytic continuation, contour deformation and taking of residues it can be seen that Askey-Wilson polynomials for more general values of the parameters than in Proposition 2.4 become orthogonal with respect to a measure which contains both a continuous and discrete part (Askey & Wilson [8])

**Proposition 2.5** Assume  $a, b, c, d$  are real, or if complex, appear in conjugate pairs, and that the pairwise products of  $a, b, c, d$  are not  $\geq 1$ , then

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos \theta) p_m(\cos \theta) w(\cos \theta) d\theta + \sum_k p_n(x_k) p_m(x_k) w_k = \delta_{m,n} h_n,$$

where  $p_n(\cos \theta)$ ,  $w(\cos \theta)$  and  $h_n$  are as in Proposition 2.4, while the  $x_k$  are the points  $(eq^k + e^{-1}q^{-k})/2$  with  $e$  any of the parameters  $a, b, c$  or  $d$  whose absolute value is larger than one, the sum is over the  $k \in \mathbb{Z}_+$  with  $|eq^k| > 1$  and  $w_k$  is  $w_k(a; b, c, d)$  as defined by [8, (2.10)] when  $x_k = (aq^k + a^{-1}q^{-k})/2$ . (Be aware that  $(1 - aq^{2k})/(1 - a)$  should be replaced by  $(1 - a^2q^{2k})/(1 - a^2)$  in [8, (2.10)].)

Both the big and the little *q*-Jacobi polynomials can be obtained as limit cases of Askey-Wilson polynomials. In order to formulate these limits, let  $r_n$  be as in (2.60). Then

$$\begin{aligned} \lim_{\lambda \rightarrow 0} r_n \left( \frac{q^{1/2}x}{2\lambda(cd)^{1/2}}; \lambda a(qd/c)^{1/2}, \lambda^{-1}(qc/d)^{1/2}, -\lambda^{-1}(qd/c)^{1/2}, -\lambda b(qc/d)^{1/2} \mid q \right) \\ = P_n(x; a, b, c, d; q) \end{aligned}$$

and

$$\lim_{\lambda \rightarrow 0} r_n \left( \frac{q^{1/2}x}{2\lambda^2}; q^{1/2}\lambda^2 a, q^{1/2}\lambda^{-2}, -q^{1/2}, -q^{1/2}b \mid q \right) = \frac{(qb; q)_n}{(q^{-n}a^{-1}; q)_n} p_n(x; b, a; q).$$

See Koornwinder [23, §6]. As  $\lambda$  becomes smaller in these two limits, the number of mass points in the orthogonality measure grows, while the support of the continuous part shrinks. In the limit we have only infinitely many mass points and no continuous mass left.

An important special class of Askey-Wilson polynomials are the *Al-Salam-Chihara polynomials*  $p_n(x; a, b, 0, 0 \mid q)$ , cf. [6, Ch. 3]. Both these polynomials and the continuous

$q$ -ultraspherical polynomials have the *continuous  $q$ -Hermite polynomials*  $p_n(x; 0, 0, 0, 0 \mid q)$  (cf. Askey & Ismail [5]) as a limit case.

The Askey-Wilson polynomials are eigenfunctions of a kind of  $q$ -difference operator. Write  $P_n(e^{i\theta})$  for the expression in (2.61) and put

$$A(z) := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

Then

$$\begin{aligned} A(z)P_n(qz) - (A(z) + A(z^{-1}))P_n(z) + A(z^{-1})P_n(q^{-1}z) \\ = -(1-q^{-n})(1-q^{n-1}abcd)P_n(z). \end{aligned}$$

The operator on the left hand side can be factorized as a product of two shift operators. See Askey & Wilson [8, §5].

An analytic continuation of the Askey-Wilson polynomials gives  *$q$ -Racah polynomials*

$$R_n(q^{-x} + q^{x+1}\gamma\delta) := {}_4\phi_3 \left[ \begin{matrix} q^{-n}q^{n+1}\alpha\beta, q^{-x}, q^{x+1}\gamma\delta \\ \alpha q, \beta\delta q, \gamma q \end{matrix} ; q, q \right],$$

where one of  $\alpha q$ ,  $\beta\delta q$  or  $\gamma q$  is  $q^{-N}$  for some  $N \in \mathbb{Z}_+$  and  $n = 0, 1, \dots, N$  (see Askey & Wilson [7]). These satisfy an orthogonality of the form

$$\sum_{x=0}^N R_n(\mu(x)) R_m(\mu(x)) w(x) = \delta_{m,n} h_n, \quad m, n = 0, 1, \dots, N,$$

where  $\mu(x) := q^{-x} + q^{x+1}\gamma\delta$ . They are eigenfunctions of a second order difference operator in  $x$ .

There is nice characterization theorem of Leonard [26] for the  $q$ -Racah polynomials. Let  $N \in \mathbb{Z}_+$  or  $N = \infty$ . Let the polynomials  $p_n$  ( $n \in \mathbb{Z}_+$ ,  $n < N + 1$ ) be orthogonal with respect to weights on distinct points  $\mu_k$  ( $k \in \mathbb{Z}_+$ ,  $k < N + 1$ ). Let the polynomials  $p_n^*$  be similarly orthogonal with respect to weights on distinct points  $\mu_k^*$ . Suppose that the two systems are dual in the sense that

$$p_n(\mu_k) = p_k^*(\mu_n^*).$$

Then the  $p_n$  are  $q$ -Racah polynomials or one of their limit cases.

## Exercises to §2

2.1 Show that (at least formally) the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

for Hermite polynomials (cf. [14, 10.13(19)]) follows from the generating function

$$(1-t)^{-\alpha-1} e^{-tx/(1-t)} = \sum_{n=0}^{\infty} L_n^\alpha(x) t^n, \quad |t| < 1,$$

for Laguerre polynomials (cf. [14, 10.12(17)]) by the limit transition

$$H_n(x) = (-1)^n 2^{n/2} n! \lim_{\alpha \rightarrow \infty} \alpha^{-n/2} L_n^\alpha((2\alpha)^{1/2}x + \alpha)$$

given in (2.7).

2.2 Show that (at least formally) the above generating function for Laguerre polynomials follows from the generating function for Jacobi polynomials

$$\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) t^n = 2^{\alpha+\beta} R^{-1} (1-t+R)^{-\alpha} (1+t+R)^{-\beta}, \quad |t| < 1,$$

where

$$R := (1 - 2xt + t^2)^{1/2},$$

(cf. [14, 10.8 (29)]) by the limit transition

$$L_n^\alpha(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)}(1 - 2x/\beta)$$

given in (2.5).

2.3 Show that (at least formally) the above generating function for Hermite polynomials follows from the specialization  $\alpha = \beta$  of the above generating function for Jacobi polynomials by the limit transition

$$H_n(x) = 2^n n! \lim_{\alpha \rightarrow \infty} \alpha^{-n/2} P_n^{(\alpha, \alpha)}(\alpha^{-1/2} x)$$

given in (2.6).

2.4 Prove that the Charlier polynomials

$$C_n(x; a) := {}_2F_0(-n, -x; -; -a^{-1}), \quad a > 0,$$

satisfy the recurrence relation

$$x C_n(x; a) = -a C_{n+1}(x; a) + (n+a) C_n(x; a) - n C_{n-1}(x; a).$$

2.5 Prove that

$$\lim_{a \rightarrow \infty} (-(2a)^{1/2})^n C_n((2a)^{1/2} x + a) = H_n(x)$$

by using the above recurrence relation for Charlier polynomials and the recurrence relation

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$$

for Hermite polynomials.

2.6 Prove the following generating function for Al-Salam-Chihara polynomials (notation as for Askey-Wilson polynomials in (2.60), (2.61)):

$$\frac{(zc, zd; q)_\infty}{(ze^{i\theta}, ze^{-i\theta}; q)_\infty} = \sum_{m=0}^{\infty} z^m \frac{p_m(\cos \theta; 0, 0, c, d | q)}{(q; q)_m}, \quad |z| < 1.$$

2.7 Use the above generating function in order to derive the following transformation formula from little  $q$ -Jacobi polynomials (notation and definition by (2.45)) to Askey-Wilson polynomials by means of a summation kernel involving Al-Salam-Chihara polynomials:

$$\begin{aligned} & \frac{a^n p_n(\cos \theta; a, b, c, d | q)}{(ab, ac, ad; q)_n} \frac{(ac, ad; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}; q)_\infty} \\ &= \sum_{m=0}^{\infty} p_n(q^m; q^{-1}ab, q^{-1}cd; q) a^m \frac{p_m(\cos \theta; 0, 0, c, d | q)}{(q; q)_m}. \end{aligned}$$

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