Special functions and q-commuting variables

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Abstract: This paper is mostly a survey, with a few new results. The first part deals with functional equations for q-exponentials, q-binomials and q-logarithms in q-commuting variables and more generally under q-Heisenberg relations. The second part discusses translation invariance of Jackson integrals, q-Fourier transforms and the braided line.

Last modified: August 26, 1996

Note: Report No. 1, Institut Mittag-Leffler, Djursholm, Sweden, 1995/96; q-alg/9608008; to appear in "Special Functions, q-Series and Related Topics", The Fields Institute Communications Series.

1. Introduction

Identities for special functions often involve several variables, even if the special function itself depends on only one variable. In general these variables are real or complex, so they commute with each other. The theory of quantum groups has been quite successful in producing identities for q-special funcions, in particular addition formulas, see e.g. the survey in Koelink [21, Section 1] and further references given there. Although quantum groups themselves abound of non-commuting variables satisfying certain relations, one usually does not find back a similar type of variables in the resulting q-special function identities. I want to advertize here that special function identities in non-commuting variables should be studied more extensively and systematically. They often provide more elegant formulas than the corresponding identities in commuting variables, and they may be closer to a quantum group theoretical origin and therefore have more canonical properties. Another feature (which may be evaluated in a positive or negative sense) is that such identities are often more algebraic and formal in spirit and further away from Weierstrass type analysis than the identities in commuting variables. The most interesting and challenging cases with non-commuting variables occur when formal infinite series and convergent infinite series mix with each other. One has to be extremely careful there in order to avoid paradoxes, see Section 9.

The present paper surveys (and extends a little) special function theory involving q-commuting variables x and y (i.e., satisfying the relation $xy = qyx$ with q complex, usually taken between 0 and 1).

The contents are as follows. In Section 2 we discuss Schützenberger's q-binomial formula. Sections 3 deals with various functional equations for q-exponentials, and Section 4 gives some extensions of these results to q-Heisenberg cases. Section 5 describes possible equivalence with formulas involving commuting variables, via the operational interpretation. In Section 6 we discuss the q-logarithm. The next four sections are much inspired by the paper by Kempf & Majid [19]. We discuss translation invariance under a q-commuting translation variable for Jackson integrals over a finite interval (Section 7) and over the interval $(-\infty, \infty)$ (Section 9). In Section 8 we introduce a q-Fourier transform pair in connection with discrete q-Hermite polynomials. While this is in commuting variables, it is related to two types of q-Fourier transforms involving non-commuting variables which have been studied, respectively, by Kempf & Majid [19] and Finkelstein & Marcus [12]. A deeper understanding of many of these results can be obtained by means of Majid's [27] braided quantum groups, in particular the braided line. This is the topic of Section 10. Finally, two further directions are very briefly indicated in Section 11.

Conventions $\mathbf{Z}_+$ will denote the set of nonnegative integers. ¿From Section 3 on, whenever we work with q, it is supposed that $0 < q < 1$, unless said otherwise. The notation for q-hypergeometric series follows the book by Gasper & Rahman [6].

2. The q-binomial formula

Newton's binomial formula says:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k, \quad n \in \mathbf{Z}_+. \tag{2.1}$$

Here it is implicitly understood that x and y commute: $xy = yx$. A q-analogue of (2.1) for *q-commuting variables* x, y, i.e., satisfying the relation

$$xy = qyx \tag{2.2}$$

for some $q \in \mathbf{C}$, first appeared in literature in Schützenberger [32], see also Cigler [7, (7)]:

Proposition 2.1 (*q-binomial formula*) Let $q \in \mathbf{C}$. Let $\mathbf{C}_q[x,y]$ be the complex associative algebra with 1 generated by x and y and with relation (2.2). Then the following identity is valid in the algebra $\mathbf{C}_q[x,y]$:

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} x^k, \quad n \in \mathbf{Z}_+. \tag{2.3}$$

Here we used the *q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-k+1})}{(1-q)(1-q^2)\dots(1-q^k)}, \tag{2.4}$$

while the *q-shifted factorial* is given by

$$(a;q)_k := (1-a)(1-qa)\dots(1-q^{k-1}a), \quad a \in \mathbf{C},\ k \in \mathbf{Z}_+. \tag{2.5}$$

The recurrence relations below show that the q-binomial coefficient (2.4) is a polynomial in q and therefore remains meaningful for q being a root of unity.

We will give a constructive proof of Proposition 2.1 which goes essentially back to Polya & Alexanderson [29] and which was later written down by Askey [2]. It is straightforward that the monomials $y^l x^k$ ($k, l \in \mathbf{Z}_+$) form a basis for $\mathbf{C}_q[x,y]$ considered as a linear space and that $(x+y)^n$ will have a unique expansion of the form

$$(x+y)^n = \sum_{k=0}^{n} c_{n,k}\, y^{n-k} x^k \tag{2.6}$$

with the coefficients $c_{n,k}$ also depending on q. It follows immediately from (2.6) that $c_{n,0} = 1 = c_{n,n}$. Also, expansion of $(x+y)^n = (x+y)^{n-1}(x+y)$ and $(x+y)^n = (x+y)(x+y)^{n-1}$, respectively, yields for $n > k > 0$:

$$c_{n,k} = q^k c_{n-1,k} + c_{n-1,k-1}, \qquad c_{n,k} = c_{n-1,k} + q^{n-k} c_{n-1,k-1}. \tag{2.7}$$

Elimination of $c_{n-1,k}$ from these two recurrence equations leaves us with the two-term recurrence

$$c_{n,k} = \frac{1-q^n}{1-q^k}\, c_{n-1,k-1}.$$

Iteration of this last recurrence yields the right-hand side of (2.4). This proves (2.3).

The advantage of this proof is that it is constructive. If one just wants to prove by induction with respect to n that (2.6) holds with $c_{n,k}$ being given by (2.4) then it is sufficient to have only one of the recurrences in (2.7). This is the way in which one usually works in case $q = 1$, where the q-binomial formula (2.3) by continuity becomes the binomial formula (2.1). In that case the two recurrences in (2.7) coincide and it is not possible to get a two-term recurrence formula by elimination.

A second observation, due to Andrews, and written down in Askey [3], is that the q-binomial formula (2.3) is equivalent to an identity in commuting variables. Note that, if the generators x, y of $\mathbf{C}_q[x,y]$ satisfy the q-commutation relations (2.2), then $-yx, y$ also satisfy these relations:

$$(-yx)y = qy(-yx).$$

Hence, we get from (2.3) that

$$(-yx+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k}(-yx)^k. \tag{2.8}$$

The left-hand side of (2.8) equals

$$\begin{aligned} &y(1-x)y(1-x)\ldots y(1-x) \quad (2n \text{ factors}) \\ &\quad = y^n(1-q^{n-1}x)\ldots(1-qx)(1-x) = y^n (x;q)_n. \end{aligned}$$

(Note that the definition (2.5) of q-shifted factorial remains valid for a in any complex associative algebra with 1.) As for the right-hand side of (2.8) note that

$$y^{n-k}(-yx)^k = (-1)^k q^{\frac{1}{2}k(k-1)} y^n x^k$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^k\, q^{-\frac{1}{2}k(k-1)}\, q^{nk}\, \frac{(q^{-n};q)_k}{(q;q)_k}\,. \tag{2.9}$$

Hence the identity (2.8) can be equivalently written as

$$y^n (x;q)_n = y^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} (q^n x)^k.$$

Because of the earlier observation about a basis of monomials for $\mathbf{C}_q[x,y]$ we conclude that

$$(x;q)_n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} (q^n x)^k.$$

This is still an identity in the algebra $\mathbf{C}_q[x,y]$. However, we can map it to an identity in $\mathbf{C}$ by using the algebra homomorphism $\pi\colon \mathbf{C}_q[x,y] \to \mathbf{C}$ such that, for some $z \in \mathbf{C}$, $\pi(x) = q^n z$ and $\pi(y) = 0$. This yields the formula giving the evaluation of a terminating q-binomial series:

$$(q^{-n}z;q)_n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\, z^k, \quad n \in \mathbf{Z}_+. \tag{2.10}$$

This formula is well-known, see [16, (II.4)]. It was earlier observed by Cigler [7, (8)] that formula (2.10) follows from (2.3). He used an operational interpretation of (2.3). We will discuss such interpretations in a later section. Then we will also see that (2.10) is in fact equivalent to (2.3).

3. Identities for q-exponentials with q-commuting variables

The two *q-exponentials* are defined by

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}\,, \tag{3.1}$$

$$E_q(z) := \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}\, z^n}{(q;q)_n} = (-z;q)_\infty\,. \tag{3.2}$$

Here we assume $0 < q < 1$ and $(a;q)_\infty$ is defined as the (convergent) infinite product:

$$(a;q)_\infty := \prod_{j=0}^{\infty} (1 - q^j a).$$

For convergence of the infinite series in (3.1), (3.2) with $z \in \mathbf{C}$ we need $|z| < 1$ in (3.1). However, because of its product representation, e_q has an analytic continuation to $\mathbf{C}\backslash\{q^{-k} \mid k \in \mathbf{Z}\}$. See [16, Section 1.3] for the proofs of the second equalities in (3.1) and (3.2). The two q-exponential series can also be considered as formal power series in the formal variable z. Of course, no convergence condition is needed in that case. ¿From (3.1), (3.2) we have

$$e_q(z)\, E_q(-z) = 1 \tag{3.3}$$

for $|z| < 1$. ¿From the second equalities in (3.1) and (3.2) we see that

$$e_q(qz) = (1-z)\, e_q(z), \quad E_q(z) = (1+z)\, E_q(qz). \tag{3.4}$$

In general, algebraic identities for convergent power series remain valid for the corresponding formal power series. In particular, this applies to (3.3) and (3.4).

Fix $q \in (0,1)$ and let $\mathbf{C}_q[[x,y]]$ be the complex associative algebra with 1 of formal power series

$$\sum_{k,l=0}^{\infty} c_{k,l}\, y^l x^k$$

with arbitrary complex coefficients $c_{k,l}$ and where x, y satisfy relation (2.2), i.e. $xy = qyx$. The following Proposition generalizing the classical functional equation $e^x e^y = e^{x+y}$ for commuting variables x, y, was given first by Schützenberger [32], see also Cigler [7, (10)].

Proposition 3.1 In the algebra $\mathbf{C}_q[[x,y]]$ we have the identities

$$e_q(x+y) = e_q(y)e_q(x), \tag{3.5}$$
$$E_q(x+y) = E_q(x)\,E_q(y). \tag{3.6}$$

Proof By means of the q-binomial formula (2.3) we can write the left-hand side of (3.5) as an element of $\mathbf{C}_q[[x,y]]$, and next rewrite it as the right-hand side:

$$e_q(x+y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(q;q)_n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k}x^k = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(q;q)_{n-k}(q;q)_k}\, y^{n-k}x^k$$
$$= \sum_{k,l=0}^{\infty} \frac{1}{(q;q)_l(q;q)_k}\, y^l x^k = e_q(y)e_q(x).$$

This settles (3.5). For the proof of (3.6) note that, in view of (3.3), $E_q(x+y)$ is a left and right inverse to $e_q(-x-y)$, and $E_q(x)E_q(y)$ is a left and right inverse to $e_q(-y)e_q(-x)$. Now use (3.5). $\square$

The reader is warned that the apparent symmetry in x and y in the left hand side of (3.5) does not allow to conclude that $e_q(x+y) = e_q(x)e_q(y)$, since the relation $xy = qyx$ is not symmetric in x and y. The next Proposition gives a formula for $e_q(x)e_q(y)$ in the algebra $\mathbf{C}_q[[x,y]]$, i.e. for the right-hand side of (3.5) with the order of the two factors interchanged. It is a special case of a more general result given first in operational form by Rogers [30], which we will discuss in the next section. See also Gelfand & Fairlie [17, (46)], Floreanini & Vinet [13, formula (23d)], Faddeev & Volkov [11, p.314, formula (2)], Faddeev & Kashaev [10, Section 2], A. N. Kirillov [20, Section 2.5, Lemma 9], and McDermott & Solomon [28, (10)].

Proposition 3.2 In the algebra $\mathbf{C}_q[[x,y]]$ we have the identities

$$e_q(x)\,e_q(y) = e_q(y-yx)\,e_q(x) \tag{3.7}$$
$$= e_q(x+y-yx) \tag{3.8}$$
$$= e_q(y)\,e_q(-yx)\,e_q(x) \tag{3.9}$$
$$= e_q(y)\,e_q(x-yx), \tag{3.10}$$
$$E_q(y)\,E_q(x) = E_q(x+y+yx), \tag{3.11}$$
$$= E_q(x)\,E_q(yx)\,E_q(y). \tag{3.12}$$

Proof Because $e_q(x)$ is invertible as a formal power series (cf. (3.3)), formula (3.7) can equivalently be stated as

$$e_q(x)\,e_q(y)\,e_q(x)^{-1} = e_q(y-yx). \tag{3.13}$$

For any two formal power series $f(z)$ and $g(z)$, with $f(z)$ being invertible and $g(z) = \sum_{k=0}^{\infty} c_k z^k$, we have

$$f(x)g(y)f(x)^{-1} = \sum_{k=0}^{\infty} c_k\, f(x)\,y^k\, f(x)^{-1} = \sum_{k=0}^{\infty} c_k \left(f(x)yf(x)^{-1}\right)^k = g\big(f(x)\,y\,f(x)^{-1}\big)$$

as identities in $\mathbf{C}_q[[x,y]]$. In particular,

$$e_q(x)\,e_q(y)\,e_q(x)^{-1} = e_q\big(e_q(x)\,y\,e_q(x)^{-1}\big). \tag{3.14}$$

Now, since $xy = qyx$ and by (3.4) we have

$$e_q(x)\, y\, e_q(x)^{-1} = y\, e_q(qx)\, e_q(x)^{-1} = y\,(1-x)\, e_q(x)\, e_q(x)^{-1} = y\,(1-x). \tag{3.15}$$

Together with (3.14) this settles (3.13) and hence (3.7).

Next it follows from Proposition 3.1 that the right-hand side of (3.7) equals (3.8), since $x(y - yx) = q(y-yx)x$. The equalities (3.9) and (3.10) also follow by application of Proposition 3.1. Finally, (3.11) and (3.12) follow from (3.8) and (3.9) in a similar way as we obtained (3.6). □

Note that the equalities (3.7)–(3.11) reduce to the classical identities $e^x e^y = e^y e^x = e^{x+y}$ if we replace x, y by $(1-q)x, (1-q)y$ and let $q \uparrow 1$ on using

$$\lim_{q\uparrow 1} e_q((1-q)z) = e^z = \lim_{q\uparrow 1} E_q((1-q)z), \tag{3.16}$$

cf. [16, (1.3.17)].

As a corollary to both Proposition 3.1 and Proposition 3.2 we have a functional equation for the *q-Gaussians*

$$g_q(x) := e_{q^2}(-x^2), \quad G_q(x) := E_{q^2}(-x^2) \tag{3.17}$$

in q-commuting variables. First note that that, for $z \in \mathbf{C}$ with $|z| < 1$, we have

$$e_{q^2}(-z^2) = \frac{1}{(-z^2;q^2)_\infty} = \frac{1}{(iz;q)_\infty(-iz;q)_\infty} = e_q(iz)\, e_q(-iz).$$

Thus the equality

$$e_{q^2}(-z^2) = e_q(iz)\, e_q(-iz) \tag{3.18}$$

is also valid for arbitrary real z or as an identity for formal power series.

Corollary 3.3 In the algebra $\mathbf{C}_q[[x,y]]$ we have the identities

$$e_{q^2}(-(x+y)^2) = e_{q^2}(-y^2)\, e_q(-yx)\, e_{q^2}(-x^2), \tag{3.19}$$
$$E_{q^2}(-(x+y)^2) = E_{q^2}(-x^2)\, E_q(-yx)\, E_{q^2}(-y^2). \tag{3.20}$$

Proof We apply first (3.18), next Proposition 3.1 (twice), next Proposition 3.2 and finally (3.18) again (twice):

$$\begin{aligned} e_{q^2}(-(x+y)^2) &= e_q(i(x+y))\, e_q(-i(x+y)) = e_q(iy)\, e_q(ix)\, e_q(-iy)\, e_q(-ix) \\ &= e_q(iy)\, e_q(-iy)\, e_q(-yx)\, e_q(ix)\, e_q(-ix) = e_{q^2}(-y^2)\, e_q(-yx)\, e_{q^2}(-x^2). \end{aligned}$$

This yields (3.19). Then formula (3.20) follows by taking the inverse on both sides and replacing x, y by ix, iy, respectively. □

Let us next discuss generalizations of the previous results in this section for the case of non-terminating q-binomial series

$$ {}_1\phi_0(a;;q,z) := \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1,\ a\in\mathbf{C}, \tag{3.21} $$

see [16, (II.3)]. Formula (3.21) can be rewritten as

$$ {}_1\phi_0(a;;q,z) = E_q(-az)\, e_q(z), \tag{3.22} $$

and this remains valid as an identity for formal power series. The termwise limit for $q\uparrow 1$ of ${}_1\phi_0(q^a;;q,z)$ is

$$ {}_1F_0(a;;z) := \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, z^k = (1-z)^{-a}. $$

The functional equation $(1-x)^{-a}(1-y)^{-a} = (1-x-y+xy)^{-a}$ in commuting variables x,y can equivalently be written as

$$ {}_1F_0(a;;x)\, {}_1F_0(a;;y) = {}_1F_0(a;;x+y-yx). \tag{3.23} $$

We now give two q-analogues of (3.23), valid in the algebra $\mathbf{C}_q[[x,y]]$.

Proposition 3.4 In the algebra $\mathbf{C}_q[[x,y]]$ (so $xy=qyx$) we have for $a\in\mathbf{C}$ the identities

$$ {}_1\phi_0(a;;q,x)\, {}_1\phi_0(a;;q,y) = {}_1\phi_0(a;;q,x+y-yx), \tag{3.24} $$
$$ {}_1\phi_0(a;;q,y)\, {}_1\phi_0(a;;q,x) = {}_1\phi_0(a;;q,x+y-ayx). \tag{3.25} $$

Proof In view of (3.22) the identities (3.24), (3.25) can be equivalently written as

$$ \begin{aligned} e_q(x)\, E_q(-ax)\, E_q(-ay)\, e_q(y) &= E_q(-a(x+y-yx))\, e_q(x+y-yx), \\ E_q(-ay)\, e_q(y)\, e_q(x)\, E_q(-ax) &= e_q(x+y-ayx)\, E_q(-a(x+y-ayx)). \end{aligned} $$

In view of (3.6), (3.8), (3.5) and (3.11) these identities are in their turn equivalent to

$$ \begin{aligned} e_q(x)\, E_q(-a(x+y))\, e_q(y) &= E_q(-a(x+y-yx))\, e_q(x)\, e_q(y), \\ E_q(-ay)\, e_q(x+y)\, E_q(-ax) &= e_q(x+y-ayx)\, E_q(-ay)\, E_q(-ax). \end{aligned} $$

Once more, these identities can be rewritten into equivalent forms:

$$ \begin{aligned} E_q(-a(x+y)) &= (e_q(x))^{-1}\, E_q(-a(x+y-yx))\, e_q(x), \\ e_q(x+y) &= e_q(ay)\, e_q(x+y-ayx)\, (e_q(ay))^{-1}. \end{aligned} $$

By a similar argument as in the proof of Proposition 3.2 these two identities will follow if we can show that

$$ x+y = (e_q(x))^{-1}\,(x+y-yx)\, e_q(x), \tag{3.26} $$
$$ x+y = e_q(ay)\,(x+y-ayx)\,(e_q(ay))^{-1}. \tag{3.27} $$

As for (3.26), its right-hand side can be rewritten as

$$ x + (e_q(x))^{-1}\, y(1-x)\, e_q(x) = x + e_q(x)^{-1}\, y\, e_q(qx) = x + e_q(x)^{-1}\, e_q(x)\, y = x+y, $$

where we used (3.4). Formula (3.27) can be proved by a similar argument. $\square$

Formula (3.25) was given by Faddeev & Volkov [11, p.315, multiplication rule for $s(\lambda,w)$], see also Kirillov [20, Section 2.5, Exercise 3]. The terminating cases of (3.24), (3.25) (i.e. $a=q^{-n}$, see Kirillov [20, Section 2.5, Lemma 10]) are:

Corollary 3.5 In the algebra $\mathbf{C}_q[x,y]$ we have for $n\in\mathbf{Z}_+$ the identities

$$(x;q)_n\,(y;q)_n=(x+y-q^nyx;q)_n,\quad (y;q)_n\,(x;q)_n=(x+y-yx;q)_n. \tag{3.28}$$

4. Identities for q-exponentials with q-Heisenberg relations

Proposition 3.2 can be generalized to an identity in the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ of formal power series

$$\sum_{k,l,m=0}^{\infty} a_{k,l,m}\,c^m y^l x^k$$

with arbitrary complex coefficients $a_{k,l,m}$ and where x,y,c satisfy the *q-Heisenberg relations*

$$xy-qyx=(1-q)c,\quad xc=cx,\quad yc=cy. \tag{4.1}$$

Here $q\in(0,1)$ is fixed as before. Note that c is a central element of $\mathbf{C}_{qHeis}[[x,y,c]]$. Note that, by adding the relation $c=0$, we can map the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ homomorphically onto the algebra $\mathbf{C}_q[[x,y]]$. It is also interesting to remark that the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ is isomorphic to the algebra of formal power series in x,y,z with relations

$$xy-yx=(1-q)z,\quad xz-qzx=0,\quad zy-qyz=0. \tag{4.2}$$

Just let z and c be related by

$$z=c-yx. \tag{4.3}$$

We need the following identity in $\mathbf{C}_{qHeis}[[x,y,c]]$, which is proved by induction with respect to n:

$$x^ny=q^nyx^n+(1-q^n)cx^{n-1}. \tag{4.4}$$

The generalization below of Proposition 3.2 was first given by Rogers [30] (in operational form, see Bowman [6] for a modern treatment). The same result was found later, independently, by Gelfand & Fairlie [17, (46)], McDermott & Solomon [28, (10)] and Kashaev [18, (4.19)].

Proposition 4.1 In the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ (i.e. with relations (4.1)) we have the identities

$$\begin{aligned} e_q(x)\,e_q(y) &= e_q(y-yx+c)\,e_q(x) && (4.5)\\ &= e_q(y)\,e_q(-yx+c)\,e_q(x) && (4.6)\\ &= e_q(y)\,e_q(x-yx+c). && (4.7)\end{aligned}$$

Proof As in the previous proof we have equality (3.14). By (4.4) we see that

$$e_q(x)\,y=\sum_{n=0}^{\infty}\frac{x^ny}{(q;q)_n}=\sum_{n=0}^{\infty}\frac{q^nyx^n}{(q;q)_n}+\sum_{n=1}^{\infty}\frac{(1-q^n)cx^{n-1}}{(q;q)_n}=y\,e_q(qx)+c\,e_q(x).$$

Hence, by (3.4),

$$e_q(x)\,y=(y-yx+c)\,e_q(x). \tag{4.8}$$

Now (4.5) follows from (3.14) and (4.8). The other two identities follow by application of Proposition 3.1. □

Remark 4.2 Because of (4.2) and (4.3), formula (4.6) can be rewritten as

$$e_q(x)\,e_q(y) = e_q(y)\,e_q\big((1-q)^{-1}[x,y]\big)\,e_q(x). \tag{4.9}$$

After rescaling we have, still as an identity in $\mathbf{C}_{qHeis}[[x,y,c]]$:

$$e_q\big((1-q)x\big)\,e_q\big((1-q)y\big) = e_q\big((1-q)y\big)\,e_q\big((1-q)[x,y]\big)\,e_q\big((1-q)x\big). \tag{4.10}$$

Replace x, y, c in (4.1) by $X, Y, (1-q)^{-1}C$, respectively, and let next $q \uparrow 1$ in (4.1) and (4.10). Then we obtain the identity

$$e^X e^Y = e^Y e^C e^X \quad \text{or equivalently} \quad e^X e^Y = e^{Y+C} e^X \tag{4.11}$$

in the algebra of formal power series in X, Y, C satisfying the *Heisenberg relations*

$$[X,Y] = C, \quad [X,C] = 0, \quad [Y,C] = 0. \tag{4.12}$$

Note that the second identity in (4.11) immediately follows from

$$e^X Y e^{-X} = \exp(\mathrm{ad}\, X)\, Y = Y + [X,Y] = Y + C,$$

where $(\mathrm{ad}\, U)\, V := [U,V] = UV - VU$. A slightly deeper identity in the Heisenberg algebra is obtained by applying the *Baker-Campbell-Hausdorff formula* (see for instance Varadarajan [35, Section 2.15]) to the case of relations (4.12). This yields

$$e^Y e^X = e^{X+Y-\frac{1}{2}C} \quad \text{or equivalently} \quad e^{X+Y} = e^Y e^{\frac{1}{2}C} e^X. \tag{4.13}$$

Formula (4.13) has often been observed in literature. A q-analogue of (4.13) in a q-Heisenberg algebra (but not precisely the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ with relations (4.1)) was found by Gelfand & Fairlie [17, (49)]. In fact, the result follows easily from (3.5):

Proposition 4.3 In the algebra of formal power series in x, w, z under relations

$$xw - qwx = (1-q)z^2, \quad xz = qzx, \quad zw = qwz \tag{4.14}$$

we have the identity

$$e_q(x+w) = e_q(w)\,e_{q^2}(z^2)\,e_q(x). \tag{4.15}$$

Proof It follows from relations (4.14) that $(x-z)(w+z) = q(w+z)(x-z)$. Hence, by repeated application of (3.5):

$$e_q(x+w) = e_q(z+w)\,e_q(x-z) = e_q(w)\,e_q(z)\,e_q(-z)\,e_q(x).$$

Now the result follows by application of (3.18). $\square$

Remark 4.4 The algebra of formal power series in x, w, v under relations

$$xw - qwx = (1-q)v, \quad xv = q^2 vx, \quad vw = q^2 wv \tag{4.16}$$

is isomorphically embedded into the algebra of formal power series in x, w, z under relations (4.14) by adding the relations $v = z^2$ to relations (4.16). This is seen by observing that the algebra of polynomials in x, w, v under relations (4.16) has a basis of elements $w^k x^l v^m$ $(k, l, m \in \mathbf{Z}_+)$ and that the algebra of polynomials in x, w, z under relations (4.14) has a basis of elements $w^k x^l z^m$ $(k, l, m \in \mathbf{Z}_+)$ (use Bergman's [4] diamond lemma). Thus it follows from Proposition 4.3 that, in the algebra of formal power series in x, w, v under relations (4.16), we have the identity

$$e_q(x + w) = e_q(w)\, e_{q^2}(v)\, e_q(x). \tag{4.17}$$

Remark 4.5 In relations (4.16) replace x, w, v by $(1-q)X, (1-q)Y, (1-q)C$ and let $q \uparrow 1$. Then X, Y, C will satisfy the Heisenberg relations (4.12) and identity (4.17) becomes the second identity in (4.13).

We may also rewrite (4.15) as an identity in $\mathbf{C}_{qHeis}[[x, y, c]]$. Just put $w := y[x, y]$ in (4.2). Then x, w, z satisfy relations (4.14). Thus (4.15) becomes in terms of x, y, z:

$$e_q(x + y[x, y]) = e_q(y[x, y])\, e_{q^2}\big((1-q)^{-1}[x, y]^2\big)\, e_q(x). \tag{4.18}$$

After rescaling we have, still under relations (4.2):

$$e_q\big((1-q)(x + y[x, y])\big) = e_q\big((1-q)y[x, y]\big)\, e_{q^2}\big((1-q)[x, y]^2\big)\, e_q\big((1-q)x\big). \tag{4.19}$$

Now make substitutions of x, y, z into X, Y, C as we earlier did for (4.10). Next let $q \uparrow 1$ in (4.1) and (4.19). Then we obtain the identity

$$e^{X+YC} = e^{YC}\, e^{\frac{1}{2}C^2}\, e^X$$

under Heisenberg relations (4.12). This identity is equivalent to (4.13).

Remark 4.6 It seems somewhat arbitrary that we stipulate relations (4.14) in order to obtain identity (4.15). In fact, they arise from a much more general Ansatz. First rewrite (4.17) equivalently as

$$E_q(-w)\, e_q(x + w)\, E_q(-x) = e_{q^2}(v).$$

Now we ask more generally under which minimal set of relations for x and w we have that

$$E_q(-w)\, e_q(x + w)\, E_q(-x) = f(v) \tag{4.20}$$

for some formal power series f in one variable and some element v which is homogeneous of degree 2 in x and w, i.e., a linear combination of x^2, w^2, xw, wx. Surprisingly, the answer is that $v = (1-q)^{-1}(xw - qwx)$, $f = e_{q^2}$ and that the relations are

$$x(xw - qwx) = q^2(xw - qwx)x, \quad (xw - qwx)y = q^2 y(xw - qwx), \tag{4.21}$$

so we recover (4.17) under relations (4.16) as the unique solution of our problem. In fact, expansion of the left-hand side of (4.20) up to quadratic terms yields $1 + (xw - qwx)/(q; q)_2$. It follows from our Ansatz that all terms of homogeneous odd degree in the expansion of the left-hand side of (4.20) must vanish. The vanishing of the third degree terms precisely yields the relations (4.21).

5. Equivalence between identities in the non-commuting and the commuting case

In the previous sections we saw many examples of identities in non-commuting variables. It is sometimes possible to rewrite these identities in terms of commuting variables, usually in various different ways. The idea is always the following. Suppose our identity in non-commuting variables lives in a certain algebra $\mathcal{A}$. Let π be a representation of the algebra $\mathcal{A}$ on a suitable linear space $\mathcal{F}$ of functions (for instance the space of polynomials or formal power series in one complex variable). Suppose that there is a subset $\{f_m\}$ of $\mathcal{F}$ such that, for all $a \in \mathcal{A}$, we have the implication: $\pi(a)\, f_m = 0$ for all $m \implies a = 0$. An identity $a = b$ in $\mathcal{A}$ is then equivalent to the collection of identities $\pi(a)\, f_m = \pi(b)\, f_m$ for all m.

As an example, fix $q \in (0,1)$, let x, y be subject to the relation $xy = qyx$, and let $\mathcal{A}$ be the algebra $\mathbf{C}_q[x,y]$ of polynomials in x, y (cf. Section 2) or the algebra $\mathbf{C}_q[[x,y]]$ of formal power series in x, y (cf. Section 3). Let $\mathbf{C}_q[x,y]$ resp. $\mathbf{C}_q[[x,y]]$ act on the space $\mathcal{F}$ of polynomials resp. formal power series in one complex variable z by an algebra representation π such that

$$(\pi(x)f)(z) := qz\, f(qz), \quad (\pi(y)f)(z) := z\, f(z). \tag{5.1}$$

Then $\pi(x)\,\pi(y) = q\,\pi(y)\,\pi(x)$, so π preserves the relation $xy = qyx$.

By induction with respect to k we see that

$$(\pi(x^k)f)(z) = q^{\frac{1}{2}\,k(k+1)}\, z^k\, f(q^k z) \quad (k \in \mathbf{Z}_+),$$

so

$$(\pi(y^l x^k)f)(z) = z^l\,(\pi(x^k)f)(z) = q^{\frac{1}{2}\,k(k+1)}\, z^{k+l}\, f(q^k z) \quad (k, l \in \mathbf{Z}_+).$$

Thus, by a little abuse of notation,

$$\pi(y^l x^k)\, z^m = q^{\frac{1}{2}k(k+1)+km}\, z^{k+l+m} \quad (k, l, m \in \mathbf{Z}_+). \tag{5.2}$$

Now we let a formal power series in x, y act on a formal power series in z in a σ-additive way:

$$\begin{aligned}\pi\left(\sum_{k,l} c_{k,l}\, y^l x^k\right) \sum_m a_m z^m &= \sum_{k,l,m} c_{k,l}\, a_m\, q^{\frac{1}{2}k(k+1)+km}\, z^{k+l+m}\\ &= \sum_{n=0}^{\infty} z^n \sum_{k+l+m=n} c_{k,l}\, a_m\, q^{\frac{1}{2}k(k+1)+km}.\end{aligned}$$

We see that the result is again a well-defined formal power series in z.

The reader is warned that a σ-additive extension of a representation as we gave above, is not always possible. Then one has to work with representations by unbounded operators on a Hilbert space, and identities satisfied by formal series may no longer hold in the representation, see for instance Woronowicz [36]. (I thank S. L. Woronowicz for pointing this out to me.)

Next we show that, if $a = \sum_{k,l} c_{k,l}\, y^l x^k$ and $\pi(a)\, z^m = 0$ for all $m \in \mathbf{Z}_+$, then $a = 0$. Indeed, if

$$\sum_{k,l} c_{k,l}\, \pi(y^l x^k)\, z^m = 0 \quad \text{for all } m \in \mathbf{Z}_+$$

then

$$\sum_{k,l} c_{k,l}\, q^{\frac{1}{2}k(k+1)+km}\, z^{k+l} = 0 \quad \text{for all } m \in \mathbf{Z}_+.$$

Hence for all $n \in \mathbf{Z}_+$ we have

$$\sum_{k=0}^{n} c_{k,n-k}\, q^{\frac{1}{2}k(k+1)}\, (q^m)^k = 0 \quad \text{for all } m \in \mathbf{Z}_+.$$

This shows that the $c_{k,n-k}$ are 0.

Now we will see how we can rewrite the identities (2.3) and (3.5) involving noncommuting variables into equivalent commutative form by means of the representation π of $\mathbf{C}_q[[x,y]]$ (or $\mathbf{C}_q[x,y]$). For fixed $n \in \mathbf{Z}_+$ the q-binomial formula (2.3) is equivalent to the set of identities

$$\pi\big((x+y)^n\big)\, z^m = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \pi(y^{n-k}x^k)\, z^m \quad (m \in \mathbf{Z}_+). \tag{5.3}$$

By induction with respect to n we see that

$$\pi\big((x+y)^n\big)\, z^m = (-q^{m+1};q)_n\, z^{m+n}. \tag{5.4}$$

By also using (5.2) and (2.9) we see that (5.3) is equivalent to

$$(-q^{m+1};q)_n\, z^{m+n} = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} (-q^{m+n+1})^k\, z^{m+n} \quad (m \in \mathbf{Z}_+).$$

We can divide both sides of this last identity by z^{m+n}. Thus we have shown that (2.3) is equivalent to the terminating q-binomial sum (2.10) considered for all $z = -q^{n+m+1}$ ($m \in \mathbf{Z}_+$), which in its turn is equivalent to (2.10) considered for all $z \in \mathbf{C}$.

Let us next handle identity (3.5) in this way. Observe first that $(\pi(y)f)(z) = zf(z)$ implies that

$$\big(\pi(g(y))\, f\big)(z) = g(z)\, f(z) \tag{5.5}$$

for any two formal power series f and g in one variable. Identity (3.5) is equivalent to the set of identities

$$\pi\left(e_q(x+y)\right)\, z^m = \pi\big(e_q(y)\, e_q(x)\big)\, z^m \quad (m \in \mathbf{Z}_+).$$

Now use (5.5) and expand $e_q(x+y)$ and $e_q(x)$. We obtain that (3.5) is equivalent to

$$\sum_{n=0}^{\infty} \frac{\pi\left((x+y)^n\right)\, z^m}{(q;q)_n} = e_q(z) \sum_{k=0}^{\infty} \frac{\pi(x^k) z^m}{(q;q)_k} \quad (m \in \mathbf{Z}_+).$$

By (5.4) and (5.2) this can be rewritten as

$$\sum_{n=0}^{\infty} \frac{(-q^{m+1};q)_n\, z^{m+n}}{(q;q)_n} = e_q(z) \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}k(k+1)+km}\, z^{k+m}}{(q;q)_k} \quad (m \in \mathbf{Z}_+).$$

On dividing both sides by z^m and on using the first equality in (3.2) we see that (3.5) is equivalent to the series of identities

$${}_1\phi_0(-q^{m+1};\,;q,z) = e_q(z)\, E_q(q^{m+1}z) \quad (m \in \mathbf{Z}_+),$$

i.e., to special cases of the evaluation of non-terminating q-hypergeometric series, cf. (3.22).

I invite the reader to experiment with several other representations π of algebras $\mathbf{C}_q[x,y]$ or $\mathbf{C}_q[[x,y]]$ on the space of polynomials or formal power series in z, for instance

$$\big(\pi(x)f\big)(z) := f(qz), \quad \big(\pi(y)f\big)(z) := z\,f(z), \tag{5.6}$$

$$\big(\pi(x)f\big)(z) := (D_qf)(z), \quad \big(\pi(y)f\big)(z) := f(qz), \tag{5.7}$$

$$\big(\pi(x)f\big)(z) := f(z+\log q), \quad \big(\pi(y)f\big)(z) := e^z\,f(z), \tag{5.8}$$

$$\big(\pi(x)f\big)(z) := e^z\,f(z), \quad \big(\pi(y)f\big)(z) := f(z-\log q). \tag{5.9}$$

In (5.7) we used the notation for the *q-derivative*

$$(D_qf)(z) := \frac{f(z)-f(qz)}{(1-q)z}\,. \tag{5.10}$$

For instance, consider the q-binomial formula (2.3) in representation (5.6), with both sides acting on $e_q(z)$, in order to arrive at a special case of the q-Chu-Vandermonde sum [16, (II.6)] (one upper parameter zero). Also consider identity (3.9) in representation (5.8) or identity (3.12) in representation (5.9), with both sides acting on functions $e^{i\mu z}$, in order to arrive at an identity in commuting variables which is equivalent to the summation formula (3.21) of non-terminating q-binomial series. Here the infinite sum results from the action of the left-hand sides of (3.9) or (3.12) on $e^{i\mu z}$. Faddeev & Kashaev [10, Section 2] point out an alternative for the action of these left-hand sides. They observe, in representation (5.9), that the action of $g_1(y)g_2(x)$ (g_1 and g_2 suitable functions) on $e^{i\mu z}$ can be written as multiplication by a certain double integral involving g_1 and g_2 (the symbol of the product of the operators $\pi(g_1(y))$ and $\pi(g_2(y))$). This argument looks quite formal and the convergence of the resulting double integral is not clear.

A wealth of further results in the spirit of this section is contained in Cigler [7], [8]. He also gives applications to continuous q-Hermite polynomials and to q-Laguerre polynomials and he develops a q-analogue of Rota's umbral calculus [31].

A well-known representation π of the algebra $\mathbf{C}_{qHeis}[[x,y,c]]$ (see (4.1), (4.2), (4.3)) on the space of formal power series is given by $\pi(x) := (1-q)\,c\,D_q$, $(\pi(y)f)(t) := t\,f(t)$. See already Rogers [30] for many results using this representation, or a modern treatment by Bowman [6].

6. The q-logarithm

Euler's dilogarithm is defined by

$$Li_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}\,, \quad |z|<1.$$

A. N. Kirillov [20, (2.52)] defines the following q-analogue ($0<q<1$ as before):

$$Li_2(z;q) := \sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}\,, \quad |z|<1. \tag{6.1}$$

Formally we have the termwise limit

$$\lim_{q\uparrow 1}(1-q)\,Li_2(z;q) = Li_2(z). \tag{6.2}$$

Kirillov [20, Section 2.5, Lemma 8] observes the following remarkable formula:

$$Li_2(z;q) = \log\bigl(e_q(z)\bigr), \quad |z| < 1. \tag{6.3}$$

For the proof note that

$$Li_2(qz;q) = Li_2(z;q) + \log(1-z)$$

(substitute the corresponding power series). Hence

$$\exp\bigl(Li_2(z;q)\bigr) = \frac{\exp\bigl(Li_2(qz;q)\bigr)}{1-z} = \ldots = \frac{\exp\bigl(Li_2(q^k z;q)\bigr)}{(z;q)_k}.$$

On taking limits for $k \to \infty$, the right-hand side tends to

$$\frac{\exp\bigl(Li_2(0;q)\bigr)}{(z;q)_\infty} = \exp(0)\, e_q(z) = e_q(z).$$

A more precise formulation of (6.2) going back to Ramanujan (see Berndt [5, Ch. 27, Entry 6]) gives an asymptotic series for $Li_2(z;q)$ (or $\log(e_q(z))$ in rising powers of $-\log q$ as $q \uparrow 1$. Kirillov [20, Section 2.5, Corollary 10] and Ueno & Nishizawa [34] derive this asymptotic series by using Euler-Maclaurin's summation formula.

Faddeev & Kashaev [10, S2], see also Kirillov [20, Theorem I], indicate that Rogers' five-term identity for Euler's dilogarithm can be obtained as a limit case as $q \uparrow 1$ of (3.9) or (3.12). This uses (6.3) and (6.2). Faddeev and Kashaev also use the representation (5.9) of the left-hand side of (3.12) by means of a double integral (see the end of Section 5). Their arguments are quite formal.

Next we consider a q-analogue of

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad |z| < 1.$$

We define and notate it as

$$\log_q(z) := \sum_{n=1}^{\infty} \frac{z^n}{1-q^n} \tag{6.4}$$

and we consider it either as a convergent power series for $|z| < 1$ or as a formal power series. Note that we have formally the termwise limit

$$\lim_{q\uparrow 1} (1-q)\, \log_q(z) = -\log(1-z). \tag{6.5}$$

It follows from (6.1) and (6.4) that

$$\log_q(z) = z\, Li_2'(z;q).$$

Hence, by (6.3):

$$\log_q(z) = \frac{z\, e_q'(z)}{e_q(z)}\,. \tag{6.6}$$

Another interesting formula is

$$\log_q(z) = -\frac{\partial}{\partial a}\, {}_1\phi_0(a;;q,z)\Big|_{a=1}. \tag{6.7}$$

It is the q-analogue of

$$-\log(1-z)=\frac{\partial}{\partial a}(1-z)^{-a}\Big|_{a=0}.$$

For the proof of (6.7) note that

$$\frac{\partial}{\partial a}\left(\frac{(az;q)_\infty}{(z;q)_\infty}\right)=\frac{\partial}{\partial a}\left(\frac{e_q(z)}{e_q(az)}\right)=-\,\frac{z\,e_q(z)\,e_q'(az)}{e_q(az)^2}\,.$$

Hence

$$\frac{\partial}{\partial a}\left(\frac{(az;q)_\infty}{(z;q)_\infty}\right)\Bigg|_{a=1}=-\,\frac{z\,e_q'(z)}{e_q(z)}\,.$$

Now apply (3.21) and (6.6).

Note also the q-derivative (see (5.10)) of $\log_q$:

$$(1-q)\,(D_q\log_q)(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}\,. \tag{6.8}$$

The formulas (6.3), (6.6), (6.7) and (6.8) are of hybrid nature because they use classical objects (the logarithm, the classical derivative and the function $z\mapsto(1-z)^{-1}$, respectively) in a context of q-special functions.

It is a natural question to look for the inverse function to $\log_q$, analogous to the function $x\mapsto 1-e^{-x}$ being inverse to the function $y\mapsto-\log(1-y)$. However, the inverse function to $\log_q$ does not seem to have a nice explicit expression. (I thank C. Krattenthaler for checking this by use of Maple.) Some alternative way to find a q-analogue is as follows. By the chain rule $f(g(x))=x$ implies $f'(g(x))g'(x)=1$. Now the following q-analogue holds:

$$(D_qf)(g(x))\,(D_qg)(x)=1,\quad g(x):=1-e_q(-(1-q)x),\quad f(y):=(1-q)\,\log_q(y). \tag{6.9}$$

Note however that $(D_qg)(f(y))\,(D_qf)(y)\neq 1$ for f and g as in (6.9).

The following Proposition gives a q-analogue of the classical functional equation

$$\log(xy)=\log x+\log y,\ \text{ or equivalently }\ -\log\big((1-x)(1-y)\big)=-\log(1-x)-\log(1-y),$$

for commuting variables x,y. This Proposition was independently found by A. N. Kirillov [20, Section 2.5, Exercise 11] and by the author.

Proposition 6.1 In the algebra $\mathbf{C}_q[[x,y]]$ we have the identity

$$\log_q(x+y-yx)=\log_q(x)+\log_q(y). \tag{6.10}$$

Proof Since $(x-yx)y=qy(x-yx)$ we get by the q-binomial formula (2.3) that

$$\begin{aligned}(x+y-yx)^n&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q y^{n-k}(x-yx)^k\\&=\sum_{k=0}^{n}y^{n-k}\,(y;q)_k\,x^k.\end{aligned}$$

Hence

$$\log_q(x+y-yx) = \sum_{n=1}^\infty \left(\frac{1}{1-q^n}\, y^n + \sum_{k=1}^n \frac{1}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k}\, (y;q)_k\, x^k \right)$$
$$= \log_q(y) + \sum_{k=1}^\infty \left(\sum_{n=k}^\infty \frac{1}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} \right) (y;q)_k\, x^k.$$

So we are done if we can show that

$$\left(\sum_{n=k}^\infty \frac{1-q^k}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} \right) (y;q)_k = 1. \tag{6.11}$$

It is sufficient to prove (6.11) for complex y with $|y|<1$. Then

$$\sum_{n=k}^\infty \frac{1-q^k}{1-q^n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} = \sum_{m=0}^\infty \frac{1-q^k}{1-q^{m+k}} \begin{bmatrix} m+k \\ k \end{bmatrix}_q y^m$$
$$= \sum_{m=0}^\infty \frac{1-q^k}{1-q^{m+k}}\, \frac{(q^{k+1};q)_m}{(q;q)_m}\, y^m = \sum_{m=0}^\infty \frac{(q^k;q)_m}{(q;q)_m}\, y^m = \frac{(q^k y;q)_\infty}{(y;q)_\infty} = \frac{1}{(y;q)_k}\,,$$

where we used in the forelast identity the evaluation (3.21) of a non-terminating q-binomial series. □

A somewhat formal, but very quick proof of Proposition 6.1 is obtained from (3.24). Just differentiate both sides of (3.24) with respect to a, put $a=1$ and use (6.7).

Yet another proof is obtained from (3.5), which we use in the form $e_q(t(x+y)) = e_q(ty)\, e_q(tx)$ with $t\in\mathbf{R}$. Differentiate both sides with respct to t, put $t=1$ and use (6.6) in order to obtain

$$e_q(x+y)\, \log_q(x+y) = e_q(y)\, (\log_q(y) + \log_q(x))\, e_q(x).$$

Replace next $e_q(x+y)$ by $e_q(y)\, e_q(x)$ and pull the $e_q(x)$ factor through $\log_q(x+y)$, on using (3.15).

7. Jackson integral in q-commuting variables

This section reviews results from Kempf & Majid [19, Section 1]. We start with a q-analogue of Taylor series for q-commuting variables.

Fix $q\in(0,1)$. The *q-derivative* $D_q f$ of a function f on $\mathbf{R}$ was already defined in (5.10). If $f(x)=x^n$ then $(D_q f)(x) = \dfrac{1-q^n}{1-q}\, x^{n-1}$, so we can let D_q and its iterates D_q^k act on polynomials or formal power series in x. We have

$$D_q^k x^n = \frac{(q^n;q^{-1})_k}{(1-q)^k}\, x^{n-k}.$$

Hence, in the algebra $\mathbf{C}_q[x,y]$ we can rewrite the q-binomial formula (2.3) as

$$(x+y)^n = \sum_{k=0}^n \frac{(1-q)^k}{(q;q)_k}\, y^k\, D_q^k x^n. \tag{7.1}$$

Now let $f(x) := \sum_{n=0}^{\infty} c_n x^n$ be a formal power series. Then, in the algebra $\mathbf{C}_q[[x,y]]$ of formal power series in x,y under relation $xy = qyx$, we have

$$
\begin{aligned}
f(x+y) &= \sum_{n=0}^{\infty} c_n (x+y)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_n \frac{(1-q)^k}{(q;q)_k} \, y^k \, D_q^k x^n \\
&= \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q;q)_k} \, y^k \sum_{n=k}^{\infty} c_n D_q^k x^n = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q;q)_k} \, y^k \, D_q^k \, f(x).
\end{aligned}
$$

Thus we have proved:

Proposition 7.1 Let f be a formal power series in one variable. Let $xy = qyx$. Then, in the algebra $\mathbf{C}_q[[x,y]]$ we have for each $m \in \mathbf{Z}_+$ the identity

$$
\begin{aligned}
f(x+y) &= \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} \, y^k \, ((1-q)D_q)^k \, f(x) && (7.2) \\
&= \sum_{k=0}^{m-1} \frac{1}{(q;q)_k} \, y^k \, ((1-q)D_q)^k \, f(x) + y^m \, g_m(x,y) && (7.3)
\end{aligned}
$$

for a suitable element $g_m(x,y)$ of $\mathbf{C}_q[[x,y]]$.

If we write $\mathcal{O}(y^m)$ instead of $y^m \, g_m(x,y)$ then (7.3) implies in particular, for $m = 2$, that

$$
f(x+y) = f(x) + y \, D_q \, f(x) + \mathcal{O}(y^2) \quad (xy = qyx), \tag{7.4}
$$

as a q-analogue in q-commuting variables of the classical formula

$$
f(x+y) = f(x) + y f'(x) + \mathcal{O}(y^2) \quad (xy = yx).
$$

Fix $q \in (0,1)$. For a function f on $\mathbf{R}$, the *Jackson integral* is defined by

$$
\int_0^x f(t) \, d_q t := (1-q) \sum_{k=0}^{\infty} f(q^k x) \, q^k x, \tag{7.5}
$$

where $x \in \mathbf{R}$, provided the sum on the right-hand side converges absolutely, for instance if f is bounded near zero. The Jackson integral avant la lettre of $f(t) := t^n$ was already computed by Fermat:

$$
\int_0^x t^n \, d_q t = \frac{1-q}{1-q^{n+1}} \, x^{n+1}. \tag{7.6}
$$

Thus, if

$$
f(x) := \sum_{n=0}^{\infty} c_n x^n \tag{7.7}
$$

is a formal power series in a formal (not necessarily real or complex) variable x then we obtain its Jackson integral

$$
\int_0^x f(t) \, d_q t = \sum_{n=0}^{\infty} c_n \, \frac{1-q}{1-q^{n+1}} \, x^{n+1} \tag{7.8}
$$

also as a formal power series.

Let now $xy = qyx$. Then we obtain in the algebra $\mathbf{C}_q[x,y]$ by (7.6) and a twofold appication of the q-binomial formula (2.3):

$$\begin{aligned}\int_0^{x+y} t^n\,d_qt &= \frac{1-q}{1-q^{n+1}}\,(x+y)^{n+1}\\ &= \frac{1-q}{1-q^{n+1}}\,y^{n+1} + \frac{1-q}{1-q^{n+1}}\sum_{l=0}^{n}\begin{bmatrix} n+1\\ l\end{bmatrix}_q y^l x^{n+1-l}\\ &= \frac{1-q}{1-q^{n+1}}\,y^{n+1} + (1-q)\sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q \frac{1}{1-q^{n-l+1}}\,y^l x^{n-l+1}\\ &= \frac{1-q}{1-q^{n+1}}\,y^{n+1} + (1-q)\sum_{l=0}^{n}\sum_{k=0}^{\infty}\begin{bmatrix} n\\ l\end{bmatrix}_q y^l (q^kx)^{n-l+1}\\ &= \frac{1-q}{1-q^{n+1}}\,y^{n+1} + (1-q)\sum_{k=0}^{\infty}(q^kx+y)^n\,q^kx\\ &= \int_0^y t^n\,d_qt + \int_0^x (t+y)^n\,d_qt,\end{aligned}$$

where we use the definition

$$\int_0^x f(t+y)\,d_qt := (1-q)\sum_{k=0}^{\infty} f(q^kx+y)\,q^kx.$$

Thus we have shown that

$$\int_0^{x+y} t^n\,d_qt = \int_0^y t^n\,d_qt + \int_0^x (t+y)^n\,d_qt \quad (xy = qyx)$$

and therefore also for formal power series (7.7):

$$\int_0^{x+y} f(t)\,d_qt = \int_0^y f(t)\,d_qt + \int_0^x f(t+y)\,d_qt \quad (xy = qyx). \tag{7.9}$$

Now recall the definition

$$\int_a^b f(t)\,d_qt := \int_0^a f(t)\,d_qt - \int_0^b f(t)\,d_qt, \tag{7.10}$$

where the two Jackson integrals on the right-hand side are defined by (7.5). Definition (7.10) can also be used for formal variables a, b. Thus by (7.9) and (7.10) we have the following Proposition:

Proposition 7.2 Let f be a formal power series in one variable. Let $xy = qyx$. Then, in the algebra $\mathbf{C}_q[[x,y]]$ we have

$$\int_y^{x+y} f(t)\,d_qt = \int_0^x f(t+y)\,d_qt. \tag{7.11}$$

Thus the translation invariance of the Riemann integral, which seemed to be destroyed when q-deforming it to the Jackson integral, can be preserved when we work with q-commuting variables.

By way of example we give a fourth proof of the functional equation (6.10) for the q-logarithm in q-commuting variables. First observe, by straightforward application of (7.8) and (6.4), that

$$\log_q(x) = \int_0^x (1-t)^{-1}\, d_q t. \tag{7.12}$$

Now let us work in $\mathbf{C}_q[[x,y]]$. Then

$$\begin{aligned}
&\log_q(x+y-xy;q) - \log(y;q) = \int_0^{x+y-yx} (1-t)^{-1}\, d_q t - \int_0^y (1-t)^{-1}\, d_q t \\
&= \int_0^{x-yx} (1-(t+y))^{-1}\, d_q t = (1-q)\sum_{k=0}^\infty \bigl(1-q^k(x-yx)-y\bigr)^{-1} q^k(x-yx) \\
&= (1-q)\sum_{k=0}^\infty \bigl((1-y)(1-q^kx)\bigr)^{-1}(1-y)q^kx = (1-q)\sum_{k=0}^\infty (1-q^kx)^{-1} q^kx \\
&= \int_0^x (1-t)^{-1}\, d_q t = \log_q(x),
\end{aligned}$$

where we applied (7.11) in the second equality. Thus we have given a new proof of (6.10).

8. q-Hermite polynomials and a q-Fourier transform pair

The *discrete q-Hermite I polynomials* (see Koekoek & Swarttouw [22, Section 3.28] and references given there) are given by

$$\begin{aligned}
h_n(x;q) :=& x^n\, {}_2\phi_0(q^{-n}, q^{-n+1};;q^2, q^{2n-1}x^{-2}) \\
=& (q;q)_n \sum_{k=0}^{[n/2]} \frac{(-1)^k\, q^{k(k-1)}\, x^{n-2k}}{(q^2;q^2)_k\,(q;q)_{n-2k}}.
\end{aligned} \tag{8.1}$$

They are orthogonal polynomials satisfying the orthogonality relations

$$\int_{-1}^1 h_m(x;q)\, h_n(x;q)\, E_{q^2}(-q^2x^2)\, d_q x = b_q\, q^{\frac12 n(n-1)}\,(q;q)_n\,\delta_{m,n}, \tag{8.2}$$

where

$$b_q := (1-q)\,(q,-q,-1;q)_\infty. \tag{8.3}$$

There is the following generating function:

$$\frac{(t^2;q^2)_\infty}{(xt;q)_\infty} = E_{q^2}(-t^2)\, e_q(xt) = \sum_{n=0}^\infty \frac{h_n(x;q)}{(q;q)_n}\, t^n \quad (|xt|<1). \tag{8.4}$$

Several useful formulas can be derived from this generating function. First we can expand a monomial:

$$x^n = (q;q)_n \sum_{k=0}^{[\frac12 n]} \frac{h_{n-2k}(x;q)}{(q^2;q^2)_k\,(q;q)_{n-2k}}. \tag{8.5}$$

For the proof, multiply both sides of (8.4) with $e_{q^2}(t^2)$ and next compare coefficients of t.

Next we have

$$\sum_{k=0}^{m} \frac{(q^{-m};q)_k}{(q;q)_k}\, q^k\, h_k(x;q)\, x^{m-k} = \begin{cases} (-1)^n\, q^{-n^2} (q;q^2)_n & \text{if } m=2n, \\ 0 & \text{if } m=2n+1. \end{cases} \tag{8.6}$$

For the proof, multiply both sides of (8.4) with $E_q(-xt)$ and next compare coefficients of t.

¿From the generating function (8.4) together with the orthogonality relations (8.2) we obtain

$$\int_{-1}^{1} e_q(-ixt)\, h_n(x;q)\, E_{q^2}(-q^2x^2)\, d_qx = b_q\, q^{\frac{1}{2}n(n-1)}\, i^{-n}\, t^n\, e_{q^2}(-t^2). \tag{8.7}$$

For the proof, replace t by $-it$ in (8.4), multiply both sides with $h_n(x;q)\, E_{q^2}(-q^2x^2)\, e_{q^2}(-t^2)$ and q-integrate both sides over $[-1,1]$.

The *discrete q-Hermite II polynomials* (see Koekoek & Swarttouw [22, Section 3.29] and references given there) are given by

$$\widetilde{h}_n(x;q) := x^n\, {}_2\phi_1(q^{-n}, q^{-n+1}; 0; q^2, -q^2x^{-2}) \tag{8.8}$$

$$= (q;q)_n \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k\, q^{-2nk}\, q^{k(2k+1)}\, x^{n-2k}}{(q^2;q^2)_k\, (q;q)_{n-2k}}\,. \tag{8.9}$$

(There is a slight error in [22, (3.29.1)], which is corrected in formula (8.8) above.) They are related to the discrete q-Hermite I polynomials by

$$h_n(ix;q^{-1}) = i^n\, \widetilde{h}_n(x;q). \tag{8.10}$$

They are orthogonal polynomials satisfying orthogonality relations given by a Jackson integral over $(-\infty,\infty)$. Let us use the notation

$$\int_0^{\gamma.\infty} f(t)\, d_qt := (1-q) \sum_{k=-\infty}^{\infty} f(q^k\gamma)\, q^k\gamma \tag{8.11}$$

for the Jackson integral over $(0,\infty)$ of a function defined on $\{q^k\gamma \mid k \in \mathbf{Z}\}$ for some $\gamma \in (0,\infty)$, where we suppose that the sum on the right hand side of (8.11) absolutely converges. So the definition is dependent on γ but invariant under the transform $\gamma \mapsto q\gamma$. For f defined on $\{\pm q^k\gamma \mid k \in \mathbf{Z}\}$ we can also define the Jackson integral of f over $(-\infty,\infty)$, again depending on γ and invariant under $\gamma \mapsto q\gamma$:

$$\int_{-\gamma.\infty}^{\gamma.\infty} f(t)\, d_qt := \left(\int_0^{\gamma.\infty} f(t)\, d_qt - \int_0^{-\gamma.\infty} f(t)\, d_qt \right) \tag{8.12}$$

$$= (1-q) \sum_{k=-\infty}^{\infty} \big(f(q^k\gamma) + f(-q^k\gamma) \big) q^k\gamma. \tag{8.13}$$

We again suppose that the infinite sums are absolutely convergent.

The orthogonality relations for the discrete q-Hermite II polynomials are:

$$\int_{-\gamma.\infty}^{\gamma.\infty} \widetilde{h}_m(x;q)\, \widetilde{h}_n(x;q)\, e_{q^2}(-x^2)\, d_qx = c_q(\gamma)\, q^{-n^2}\, (q;q)_n\, \delta_{m,n}, \tag{8.14}$$

where

$$c_q(\gamma) := \frac{2(1-q)\,(q^2,-q\gamma^2,-q\gamma^{-2};q^2)_\infty\,\gamma}{(-\gamma^2,-q^2/\gamma^2,q;q^2)_\infty}\,. \tag{8.15}$$

Note that the orthogonality measure is not uniquely determined. There is the generating function

$$\frac{(-xt;q)_\infty}{(-t^2;q^2)_\infty} = e_{q^2}(-t^2)\,E_q(xt) = \sum_{n=0}^{\infty} \frac{q^{\frac12 n(n-1)}}{(q;q)_n}\,\widetilde{h}_n(x;q)\,t^n. \tag{8.16}$$

Now we can obtain formulas analogous to (8.5)–(8.7), either by using (8.16) just as (8.4) was used in the discrete q-Hermite I case, or by using (8.10):

$$x^n = (q;q)_n \sum_{k=0}^{[\frac12 n]} \frac{q^{-2nk+3k^2}}{(q;q)_{n-2k}\,(q^2;q^2)_k}\,\widetilde{h}_{n-2k}(x;q), \tag{8.17}$$

$$\sum_{k=0}^{m} \frac{(q^{-m};q)_k}{(q;q)_k}\,q^{mk}\,\widetilde{h}_k(x;q)\,x^{m-k} = \begin{cases} (-1)^n\,(q;q^2)_n & \text{if } m=2n, \\ 0 & \text{if } m=2n+1, \end{cases} \tag{8.18}$$

$$\int_{-\gamma.\infty}^{\gamma.\infty} E_q(iqxt)\,\widetilde{h}_n(x;q)\,e_{q^2}(-x^2)\,d_qx = c_q(\gamma)\,q^{-\frac12 n(n-1)}\,i^n\,t^n\,E_{q^2}(-q^2t^2). \tag{8.19}$$

Now we can combine formulas for cases I and II of the discrete q-Hermite polynomials. It follows from (8.5), (8.7) and (8.9) that

$$\int_{-1}^{1} e_q(-ixt)\,x^n\,E_{q^2}(-q^2x^2)\,d_qx = b_q\,q^{\frac12 n(n-1)}\,i^{-n}\,\widetilde{h}_n(t;q)\,e_{q^2}(-t^2) \tag{8.20}$$

and it follows from (8.17), (8.19) and (8.1) that

$$\int_{-\gamma.\infty}^{\gamma.\infty} E_q(iqxt)\,x^n\,e_{q^2}(-x^2)\,d_qx = c_q(\gamma)\,q^{-\frac12 n(n-1)}\,i^n\,h_n(t;q)\,E_{q^2}(-q^2t^2). \tag{8.21}$$

By comparing (8.7) with (8.21) or (8.20) with (8.19), we arrive at the following pair of q-Fourier transforms, which are inverse to each other when acting on suitable functions:

Theorem 8.1 Let

$$(\mathcal{F}_q f)(y) := \frac{1}{b_q}\int_{-1}^{1} e_q(-ixy)\,f(x)\,d_qx, \tag{8.22}$$

$$(\widetilde{\mathcal{F}}_{q,\gamma} g)(x) := \frac{1}{c_q(\gamma)}\int_{-\gamma.\infty}^{\gamma.\infty} E_q(iqxy)\,g(y)\,d_qy. \tag{8.23}$$

Then the following two statements are equivalent:
(a) $f(x) = \text{polynomial}(x) \times E_{q^2}(-q^2x^2)$ and $g = \mathcal{F}_q f$;
(b) $g(y) = \text{polynomial}(y) \times e_{q^2}(-y^2)$ and $f = \widetilde{\mathcal{F}}_{q,\gamma} g$.

In particular:

$$f(x) = h_n(x;q)\, E_{q^2}(-q^2x^2) \iff g(y) = q^{\frac{1}{2}n(n-1)}\, i^{-n}\, y^n\, e_{q^2}(-y^2); \tag{8.24}$$

$$f(x) = x^n\, E_{q^2}(-q^2x^2) \iff g(y) = q^{\frac{1}{2}n(n-1)}\, i^{-n}\, \widetilde{h}_n(y;q)\, e_{q^2}(-y^2). \tag{8.25}$$

By q-integration by parts and by using (3.4) we can see how $\mathcal{F}_q$ and $\widetilde{\mathcal{F}}_{q,\gamma}$ send a q-derivative operator to a multiplication operator and a multiplication operator to a q-derivative operator. For this purpose we also need a variant D_q^+, called *forward q-derivative*, of the (*backward*) q-derivative $D_q = D_q^-$ as defined in (5.10):

$$(D_q^- f)(x) := \frac{f(x) - f(qx)}{(1-q)x}, \quad (D_q^+ f)(x) := \frac{f(q^{-1}x) - f(x)}{(1-q)x}. \tag{8.26}$$

Proposition 8.2 If f is continuous at 0 and $f(q^{-1}) = 0 = f(-q^{-1})$ then

$$(1-q)\,(\mathcal{F}_q(D_q^+ f))(y) = iy\,(\mathcal{F}_q f)(y).$$

If g is continuous at 0 and $\lim_{n\to\infty} E_q(\pm ix\gamma q^{-n+1})\, g(\pm\gamma q^{-n}) = 0$ then

$$(1-q)\,(\widetilde{\mathcal{F}}_{q,\gamma}(D_q^- g))(x) = -ix\,(\widetilde{\mathcal{F}}_{q,\gamma} g)(x).$$

It follows from (8.24), (8.25) and Proposition 8.2 that (with $D_q^\pm$ acting on functions of x):

$$(1-q)\, D_q^+\big(h_n(x;q)\, E_{q^2}(-q^2x^2)\big) = -q^{-n}\, h_{n+1}(x;q)\, E_{q^2}(-q^2x^2), \tag{8.27}$$

$$(1-q)\, D_q^-\big(\widetilde{h}_n(x;q)\, e_{q^2}(-x^2)\big) = -q^{n}\, \widetilde{h}_{n+1}(x;q)\, e_{q^2}(-x^2). \tag{8.28}$$

These two formulas can also be proved independently. Thus, if (8.27) and (8.28) are given then the general case of (8.24) and (8.25) follows from the special case $n = 0$ by means of Proposition 8.2.

Note that iteration of (8.27), (8.28) yields Rodrigues type formulas

$$h_n(x;q) = (-1)^n\, q^{\frac{1}{2}n(n-1)}\, e_{q^2}(q^2x^2)\,(1-q)^n\,(D_q^+)^n E_{q^2}(-q^2x^2), \tag{8.29}$$

$$\widetilde{h}_n(x;q) = (-1)^n\, q^{-\frac{1}{2}n(n-1)}\, E_{q^2}(x^2)\,(1-q)^n\,(D_q^-)^n\, e_{q^2}(-x^2). \tag{8.30}$$

All formulas, given in this section, involving discrete q-Hermite polynomials until now are analogues of formulas for classical Hermite polynomials: just take limits as $q \uparrow 1$ (after possibly some rescaling). However, there are some other nice formulas for classical Hermite polynomials for which the q-analogue can be better given with q-commuting variables. For instance, in the algebra $\mathbf{C}_q[x,y]$ $(xy = qyx)$ we have:

$$h_n(x+y;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k}\, h_k(x;q) \tag{8.31}$$

For the proof, let t be scalar and use the generating function (8.4) and the functional equation (3.5):

$$\sum_{n=0}^{\infty} \frac{h_n(x+y;q)}{(q;q)_n}\, t^n = e_q((x+y)t)\, E_{q^2}(-t^2) = e_q(yt)\, e_q(xt)\, E_{q^2}(-t^2)$$

$$= \sum_{l=0}^{\infty} \frac{y^l}{(q;q)_l}\, t^l \sum_{k=0}^{\infty} \frac{h_k(x;q)}{(q;q)_k}\, t^k = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k}\, h_k(x;q) \right) \frac{t^n}{(q;q)_n}.$$

Another example of a formula for Hermite polynomials without q-analogue in commuting variables is the formula expanding $H_n(\lambda x)$ in terms of a series of $H_m(x)$:

$$H_n(\lambda x) = n! \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k\,(1-\lambda^2)^k\,\lambda^{n-2k}}{(n-2k)!\,k!}\,H_{n-2k}(x). \tag{8.32}$$

This formula can be proved by use of the generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)\,t^n}{n!}\,.$$

Just write

$$e^{2\lambda xt-t^2} = e^{2\lambda xt-\lambda^2t^2}\,e^{-(1-\lambda^2)t^2},$$

expand all factors in terms of powers of t and compare coefficients of t^n on both sides. Note that we used that $e^{-t^2} = e^{-\lambda^2t^2}\,e^{-(1-\lambda^2)t^2}$, which strongly suggests to use q-commuting variables for a q-analogue.

Suppose now that λ, μ satisfy the relation $\lambda\mu = q^{\frac{1}{2}}\mu\lambda$ and let x and t be scalar. Then $\lambda^2\mu^2 = q^2\mu^2\lambda^2$, so, by (3.6):

$$e_q(\lambda xt)\,E_{q^2}(-(\lambda^2+\mu^2)t^2) = \big(e_q(\lambda xt)\,E_{q^2}(-\lambda^2t^2)\big)E_{q^2}(-\mu^2t^2).$$

Now expand factors in terms of powers of t by using the generating function (8.4) and compare coefficients of t^n on both sides. Then we obtain a q-analogue of (8.32):

$$\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k\,q^{k(k-1)}\,\lambda^{n-2k}\,(\lambda^2+\mu^2)^k\,x^{n-2k}}{(q;q)_{n-2k}\,(q^2;q^2)_k} = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k\,q^{k(k-1)}\,\lambda^{n-2k}\,\mu^{2k}\,h_{n-2k}(x;q)}{(q;q)_{n-2k}\,(q^2;q^2)_k}\,. \tag{8.33}$$

The left-hand side of (8.33), after multiplication by $(q;q)_n$, can be considered as $h_n(x;q)$ being kind of rescaled by means of λ and μ. Formula (8.33) can be used in order to arrive at a q-analogue of the Fourier transform sending $H_n(x)\,e^{-\frac{1}{2}x^2}$ to a constant multiple of this, or more generally $H_n(x)\,e^{-ax^2}$ to a constant multiple of $H_n((4a(1-a))^{-\frac{1}{2}}x)\,e^{-x^2/(4a)}$ (see [9, (1.10.8), (2.10.10)]). Just combine (8.21) with (8.33). I do not give details, but the reader should compare with very related results in Finkelstein & Marcus [12], where the quantum group $SU_q(2)$ is also brought into the game.

The q-Fourier transform $\widetilde{\mathcal{F}}_{q,\gamma}$ (see (8.23)) also occurs in Kempf & Majid [19, Section VIB]. However, the Fourier kernel is written there as $e_q(iyx)$ with x and y q-commuting. Their discussion is tied up very much with the notion of translation invariant Jackson integral over $(-\infty,\infty)$ and with $\mathbf{C}_q[x]$ considered as a braided Hopf algebra, see the next two sections for a few more details.

As far as I know, a purely analytic approach to the q-Fourier transform pair in Theorem 8.1 has not earlier appeared in literature. Of course, there are many natural further questions, e.g. extension of the transforms to a bigger class of functions and continuity properties of the transforms.

9. Translation invariance of Jackson integral over $(-\infty,\infty)$

The results of this section are essentially due to Kempf & Majid [19, Section IV], but the approach is different.

In (8.12), (8.13) we gave the definition of a Jackson integral over $(-\infty,\infty)$. We cannot extend this definition to the case $\int_{-x.\infty}^{x.\infty} f(t)\,d_qt$ where f is a formal power series and x is a formal variable (compare with (7.8) for the Jackson integral from 0 to x), since the Jackson integral of $f(t) := t^n$ over $(-\infty,\infty)$ is not well-defined.

Still, in view of the classical formula

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} f(t+y)\,dt \quad (y \in \mathbf{R}),$$

valid for absolutely convergent integrals, we would like to find an extension of the Jackson integral translation invariance (7.11) on finite intervals for q-commuting variables to the case of an infinite interval. So we would like to have that

$$\int_{-x.\infty}^{x.\infty} f(t)\,d_qt = \int_{-x.\infty}^{x.\infty} f(t+y)\,d_qt \quad (xy = qyx) \tag{9.1}$$

for suitable formal power series f for which both sides of (9.1) have meaning.

Let me first give a completely formal proof of (9.1), as a limit case of (7.11). For $r \in \mathbf{Z}_+$ it follows from (7.11) that

$$\int_{-q^{-r}(x+q^ry)}^{q^{-r}(x+q^ry)} f(t)\,d_qt = \int_{-q^{-r}x}^{q^{-r}x} f(t+y)\,d_qt \quad (xy = qyx).$$

Now let $r \to \infty$. Then equality (9.1) is obtained as a formal limit case.

Evidently this argument gives only heuristic evidence for the validity of (9.1). Let me give next a still formal, but more satisfactory proof of (9.1) for suitable functions f. By formal substitution of (8.13) in the right-hand side of (9.1) and by (7.2) we have

$$\begin{aligned}\int_{-x.\infty}^{x.\infty} f(t+y)\,d_qt &= (1-q)\sum_{k=-\infty}^{\infty}\big(f(q^kx+y)+f(-q^kx+y)\big)q^kx\\ &= \sum_{m=0}^{\infty} y^m\,(1-q)\sum_{k=-\infty}^{\infty}\big(f_m(q^kx)+f_m(-q^kx)\big)\,q^kx\\ &= \sum_{m=0}^{\infty} y^m \int_{-x.\infty}^{x.\infty} f_m(t)\,d_qt,\end{aligned} \tag{9.2}$$

where

$$f_m(z) = \frac{1}{(q;q)_m}\,((1-q)D_q)^m\,f(z). \tag{9.3}$$

If f is a function on $\mathbf{R}$ then so is f_m. So the Jackson integrals $\int_{-x.\infty}^{x.\infty} f_m(t)\,d_qt$ will have concrete meaning for $x \in \mathbf{R}$ and if the sums defining the Jackson integral are convergent. However, x must q-commute with y, so x cannot be in $\mathbf{R}$. We can circumvent this dilemma by passing to a suitable representation of the relation $xy = qyx$. Let us take a slight extension of the representation (5.6), now using the dot notation instead of π:

$$x.g(z) := \gamma\,g(qz), \quad y.g(z) := z\,g(z). \tag{9.4}$$

Here $\gamma \in \mathbf{R}\backslash\{0\}$ is fixed and $g(z)$ is a formal power series. Thus

$$x.z^k = \gamma q^k z^k \quad (k \in \mathbf{Z}_+),$$

which we will formally extend to

$$f(x).z^k = f(\gamma q^k)\, z^k \quad (k \in \mathbf{Z}_+)$$

if f is a function on $\mathbf{R}$. Hence

$$\left(\int_{-x.\infty}^{x.\infty} f(t)\, d_q t\right).\, z^k = \left(\int_{-\gamma q^k.\infty}^{\gamma q^k.\infty} f(t)\, d_q t\right) z^k = \left(\int_{-\gamma.\infty}^{\gamma.\infty} f(t)\, d_q t\right) z^k$$

provided that the sums defining the Jackson integral

$$I_f(\gamma) := \int_{-\gamma.\infty}^{\gamma.\infty} f(t)\, d_q t$$

converge absolutely. Note that $I_f(\gamma) = I_f(q\gamma)$. For such f we conclude that, for any formal power series g:

$$\left(\int_{-x.\infty}^{x.\infty} f(t)\, d_q t\right).\, g(z) = I_f(\gamma)\, g(z). \tag{9.5}$$

Now take up (9.2) again in the representation (9.4). We find

$$\left(\int_{-x.\infty}^{x.\infty} f(t+y)\, d_q t\right).\, g(z) = \sum_{m=0}^{\infty} I_{f_m}(\gamma)\, y^m.g(z) = \sum_{m=0}^{\infty} I_{f_m}(\gamma)\, z^m\, g(z),$$

while

$$\left(\int_{-x.\infty}^{x.\infty} f(t)\, d_q t\right).\, g(z) = I_f(\gamma)\, g(z).$$

So formula (9.1) in the representation (9.4) is equivalent with the vanishing of $I_{f_m}(\gamma)$ $(m = 1, 2, \ldots)$. It is easy to find a class of functions f for which these numbers vanish. Note that each f_m $(m \in \mathbf{Z}_+)$ is a q-derivative of another function.

Lemma 9.1 Let $\gamma \in \mathbf{R}\backslash\{0\}$ and let f be a function on $\{\pm\gamma q^k \mid k \in \mathbf{Z}\}$ such that $\lim_{k\to\infty} f(q^k\gamma) = \lim_{k\to\infty} f(-q^k\gamma)$ and $\lim_{k\to\infty} f(\pm q^{-k}\gamma) = 0$. Then $\int_{-\gamma.\infty}^{\gamma.\infty} (D_q f)(t)\, d_q t = 0$.

Proof It follows by summation by parts that

$$\int_{-\gamma.\infty}^{\gamma.\infty} (D_q f)(t)\, d_q t = \lim_{m,n\to\infty} \big(f(q^{-m}\gamma) - f(q^n\gamma) - f(-q^{-m}\gamma) + f(-q^n\gamma)\big). \qquad \square$$

Proposition 9.2 Let $\gamma \in \mathbf{R}\backslash\{0\}$ and let f be a function on $\{\pm\gamma q^k \mid k \in \mathbf{Z}\} \cup \{0\}$ such that, for all $m \in \mathbf{Z}_+$, $D_q^m f$ is continuous at 0 and

$$|(D_q^m f)(\pm q^{-k}\gamma)| = \mathcal{O}(q^{(1+\varepsilon)k}) \quad \text{as } k \to \infty \tag{9.6}$$

for certain $\varepsilon > 0$. Then f satisfies the translation invariance (9.1) in the representation (9.4). If, moreover, the estimate (9.6) is satisfied for all $\varepsilon > 0$ (so f and all its q-derivatives are *rapidly decreasing* on the domain of definition), then f multiplied with any polynomial also satisfies the translation invariance (9.1) in the representation (9.4).

Proof Because of (9.3) and the estimate on $D_q^m f$, the Jackson integrals defining the $I_{f_m}(\gamma)$ converge absolutely. For each $m > 0$, f_m is a q-derivative, so $I_{f_m}(\gamma) = 0$ by Lemma 9.1 and the estimate for $D_q^{m-1} f$. For the proof of the last statement use the q-Leibniz rule $(D_q(fg))(x) = f(x)\,(D_q g)(x) + (D_q f)(x)\, g(qx)$. $\square$

Note that the continuity at 0 in the Proposition is satisfied if f is the restriction of a function which is C^∞ on a neighbourhood of 0.

As an example we consider Jackson integrals involving the q-Gaussian $g_q(x) = e_{q^2}(-x^2)$ (cf. (3.17)). This function satisfies the conditions of the Proposition (including the rapid decreasing property). In fact, it follows by induction with respect to m that $(D_q^m g_q)(x) = p_m(x)\, e_{q^2}(-x^2)$ with p_m a polynomial of degree m (more concretely a discrete q-Hermite II polynomial, see (8.30)). Now observe that

$$|e_{q^2}(-x^2)| = \prod_{k=0}^{\infty} |1 + q^{2k} x^2|^{-1} \le (1 + q^{2n} x^2)^{-n} = \mathcal{O}(|x|^{-2n}) \quad \text{as } x \to \pm\infty$$

for all $n \in \mathbf{Z}_+$. Thus we know that (9.1) in the representation (9.4) is valid for $f(x) := x^m\, e_{q^2}(-x^2)$. Let us see the implications of this result for q-special functions, by which we can make contact with the results of Section 8.

It follows from (8.1) that

$$h_m(0;q) = \begin{cases} (-1)^n\, q^{n(n-1)}\, (q;q^2)_n & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1. \end{cases} \tag{9.7}$$

Hence the case $t = 0$ of (8.21) yields

$$\int_{-\gamma.\infty}^{\gamma.\infty} t^m\, e_{q^2}(-t^2)\, d_q t = \begin{cases} c_q(\gamma)\, q^{-n^2} (q;q^2)_n & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1 \end{cases} \tag{9.8}$$

where $c_q(\gamma)$ is given by (8.15). In fact, the case $m = 2n$ of (9.8) is a consequence of Ramanujan's ${}_1\psi_1$ summation formula, see [16, (II.29)].

Consider (9.1) with $f(x) := x^m\, e_{q^2}(-x^2)$. The left-hand side of (9.1), when acting as an operator on a formal power series $g(z)$ in the representation (9.4), can be evaluated as

$$\left(\int_{-x.\infty}^{x.\infty} t^m\, e_{q^2}(-t^2)\, d_q t\right) . g(z) = \begin{cases} c_q(\gamma)\, q^{-n^2} (q;q^2)_n\, g(z) & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1 \end{cases} \tag{9.9}$$

We expand the right-hand side of (9.1), still formally, as

$$\begin{aligned} &\int_{-x.\infty}^{x.\infty} e_{q^2}(-(t+y)^2)\, (t+y)^m\, d_q t \\ &= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q e_{q^2}(-y^2) \int_{-x.\infty}^{x.\infty} e_q(-yt)\, e_{q^2}(-t^2)\, y^{m-k}\, t^k\, d_q t \\ &= \sum_{k=0}^{m} q^{-(k+1)(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_q e_{q^2}(-y^2) \left(\int_{-x.\infty}^{x.\infty} e_q(-yt)\, e_{q^2}(-t^2)\, t^k\, d_q t\right) y^{m-k}, \end{aligned} \tag{9.10}$$

where we used (2.3) and (3.19) for the first equality, while the second equality follows from (8.13) and the q-commutation of x and y. Now we give meaning to the right-hand side of (9.10) as an operator acting on a formal power series $g(z)$ in the representation (9.4). Consider first:

$$\left(\int_{-x.\infty}^{x.\infty} e_q(-yt)\, e_{q^2}(-t^2)\, t^k\, d_q t\right) . g(z)$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} \frac{(-1)^l q^{\frac12 l(l-1)} y^l}{(q;q)_l} \left(\int_{-x.\infty}^{x.\infty} e_{q^2}(-t^2)\, t^{k+l}\, d_q t \right) . \, g(z) \\
&= \sum_{l=0}^{\infty} \frac{(-1)^l q^{\frac12 l(l-1)} z^l}{(q;q)_l} \left(\int_{-\gamma.\infty}^{\gamma.\infty} e_{q^2}(-t^2)\, t^{k+l}\, d_q t \right) g(z) \\
&= \left(\int_{-\gamma.\infty}^{\gamma.\infty} E_q(-zt)\, e_{q^2}(-t^2)\, t^k\, d_q t \right) g(z) \\
&= c_q(\gamma)\, q^{-\frac12 k(k-1)}\, i^k\, h_k(iq^{-1}z;q)\, E_{q^2}(z^2)\, g(z).
\end{aligned} \tag{9.11}
$$

Here we used (9.5) and (8.21). After substitution of this result in (9.10) we obtain that the right-hand side of (9.10) acting on $g(z)$ becomes:

$$
c_q(\gamma) \sum_{k=0}^{m} \frac{(q^{-m};q)_k}{(q;q)_k}\, q^k\, i^{-m}\, h_k(iq^{-1}z;q)\, (iq^{-1}z)^{m-k}\, g(z). \tag{9.12}
$$

We know, at least formally, that (9.9) and (9.12) must be equal to each other. But this equality can equivalently be written as (8.6), which we had already proved in an elementary way.

Let us next consider whether (9.1) holds when f equals the second q-Gaussian (cf. (3.17))

$$
G_q(x) := E_{q^2}(-x^2) = (x^2;q^2)_\infty .
$$

Note that $G_q(\pm q^{-m}) = 0$ for $m \in \mathbf{Z}_+$. So $G_q(\pm\gamma q^{-m}) = 0$ for m a sufficiently large integer if γ is an integer power of q. However, if γ is not an integer power of q then $|G_q(\gamma q^{-m})|$ increases faster than C^m for any $C > 1$ as $m \to \infty$. Indeed, take $n \in \mathbf{Z}$ such that $\gamma^2 q^{-2n-2} - 1 \ge C$. Then, for $m \ge n$:

$$
|G_q(\gamma q^{-m})| \ge (\gamma^2 q^{-2n-2} - 1)^{m-n}\, |G(\gamma q^{-n})|.
$$

So the Jackson integral $\int_{-\gamma.\infty}^{\gamma.\infty} G_q(t)\, t^m\, d_q t$ (the analogue of (9.8)) only converges absolutely for $\pm\gamma$ being an integer power of q and then it turns down to computing the Jackson integral over $[-q,q]$.

It follows from (8.9) that

$$
\widetilde{h}_m(0;q) = \begin{cases} (q;q^2)_n\, (-1)^n\, q^{n-2n^2} & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1. \end{cases} \tag{9.13}
$$

Hence the case $t = 0$ of (8.20) yields

$$
\int_{-q}^{q} E_{q^2}(-t^2)\, t^m\, d_q t = \begin{cases} b_q\, q^{2n+1}\, (q;q^2)_n & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1, \end{cases} \tag{9.14}
$$

where b_q is given by (8.3). Alternatively, formula (9.14) can also be obtained by a completely elementary computation.

Since G_q is a C^∞-function and since it vanishes on the set $\{\pm q^{-m} \mid m \in \mathbf{Z}_+\}$, it clearly satisfies all conditions of Proposition 9.2 for $\gamma = 1$. So the function $f(x) := x^m\, E_{q^2}(-x^2)$ will satisfy (9.1) in the representation (9.4) for $\gamma = 1$, i.e. in the representation (5.6). In order to see the implications of this, we can imitate what we did for the other q-Gaussian g_q. The left-hand side of (9.1), when acting as an operator on a formal power series $g(z)$ in the representation (5.6), can be evaluated as

$$
\left(\int_{-x.\infty}^{x.\infty} t^m\, E_{q^2}(-t^2)\, d_q t \right) . \, g(z) = \begin{cases} b_q\, q^{2n+1} (q;q^2)_n\, g(z) & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1 \end{cases} \tag{9.15}
$$

by using (9.14). We expand the right-hand side of (9.1), still formally, as

$$\begin{aligned}
&\int_{-x.\infty}^{x.\infty} (t+y)^m \, E_{q^2}(-(t+y)^2) \, d_q t \\
&= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q y^{m-k} \int_{-x.\infty}^{x.\infty} t^k \, E_{q^2}(-t^2) \, E_q(-yt) \, E_{q^2}(-y^2) \, d_q t \\
&= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q y^{m-k} \left(\int_{-x.\infty}^{x.\infty} t^k \, E_{q^2}(-t^2) \, E_q(-yt) \, d_q t \right) E_{q^2}(-q^{-2}y^2). \qquad (9.16)
\end{aligned}$$

Here we used (2.3) and (3.20). Now we give meaning to the right-hand side of (9.16) as an operator acting on a formal power series $g(z)$ in the representation (5.6). First derive, analogous to the proof of (9.11) but now using (8.20), that

$$\begin{aligned}
\left(\int_{-x.\infty}^{x.\infty} t^k \, E_{q^2}(-t^2) \, E_q(-yt) \, d_q t \right) . \, g(z) &= \left(\int_{-q}^{q} t^k \, E_{q^2}(-t^2) \, e_{q^2}(-q^{-2}zt) \, d_q t \right) g(z) \\
&= q b_q \, i^{-k} \, q^{\frac{1}{2}k(k+1)} \, e_{q^2}(q^{-2}z^2) \, \widetilde{h}_k(-iq^{-1}z; q) \, g(z).
\end{aligned}$$

After substitution of this result in (9.16) we obtain that the right-hand side of (9.16) acting on $g(z)$ becomes

$$q b_q \sum_{k=0}^{m} \frac{(q^{-m}; q)_k}{(q; q)_k} \, q^{(m+1)k} \, i^k \, z^{m-k} \, \widetilde{h}_k(-iq^{-1}z; q) \, g(z). \qquad (9.17)$$

The right-hand side of (9.15) must be equal to (9.17). But this equality can equivalently be written as (8.17), which we had already proved in an elementary way.

Thus we have seen in this section that the translation invariance (9.1) for the case that $f(t) = t^m g_q(t)$ or $t^m G_q(t)$ turns down to the identities (8.6) and (8.17) for discrete q-Hermite polynomials.

10. Braided Hopf algebras

In this section we introduce braided Hopf algebras and show the relevance of this structure for the results of Section 8. First we recall the notion of an ordinary Hopf algebra (see for instance Abe [1], Sweedler [33], Koornwinder [25, Section 1]).

We will work over the field of complex numbers, so a *linear space* will mean a complex linear space. If V and W are linear spaces then the *tensor product* $V \otimes W$ will be the linear space which is the algebraic tensor product of V and W, so $V \otimes W$ will be spanned by the elements $v \otimes w$ $(v, w \in V)$. The tensor products $V \otimes \mathbf{C}$ and $\mathbf{C} \otimes V$ will be naturally identified with V.

By an *algebra* we will mean a complex associative algebra with identity element 1. The field of complex numbers is an algebra in an evident way. If $\mathcal{A}$ is an algebra then the mapping $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ will denote the linear extension of the bilinear mapping $(a_1, a_2) \mapsto a_1 a_2 : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are algebras then an *algebra homomorphism* $\phi: \mathcal{A} \to \mathcal{B}$ is a linear mapping satisfying $\phi(a_1 a_2) = \phi(a_1)\phi(a_2)$ and $\phi(1) = 1$.

If V is a linear space then the *flip operator* is the linear operator $\sigma: V \otimes V \to V \otimes V$ such that $\sigma(v \otimes w) = w \otimes v$. If $\mathcal{A}$ is an algebra then we give an algebra structure to $\mathcal{A} \otimes \mathcal{A}$ by putting $(a \otimes b)(c \otimes d) := ac \otimes bd$ and by extending this to a bilinear mapping of $(\mathcal{A} \otimes \mathcal{A}) \times (\mathcal{A} \otimes \mathcal{A})$ to $\mathcal{A} \otimes \mathcal{A}$. Let $m_{\mathcal{A} \otimes \mathcal{A}}: \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the linear operator m corresponding to this algebra structure. Thus $m_{\mathcal{A} \otimes \mathcal{A}}$ acting on $a \otimes b \otimes c \otimes d$ can be written as the composition of two operators:

$$a \otimes b \otimes c \otimes d \longrightarrow a \otimes c \otimes b \otimes d \longrightarrow ac \otimes bd,$$

so

$$m_{\mathcal{A} \otimes \mathcal{A}} = (m_{\mathcal{A}} \otimes m_{\mathcal{A}}) \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}). \qquad (10.1)$$

Definition 10.1 A *Hopf algebra* is an algebra $\mathcal{A}$ equipped with three additional operators $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (*comultiplication*), $\varepsilon: \mathcal{A} \to \mathbf{C}$ (*counit*) and $S: \mathcal{A} \to \mathcal{A}$ (*antipode*), where Δ and ε are algebra homomorphisms and S is a linear mapping, and where the following additional properties are satisfied:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \quad (\textit{coassociativity}), \tag{10.2}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{10.3}$$

$$\big(m \circ (S \otimes \mathrm{id}) \circ \Delta\big)(a) = \varepsilon(a)\, 1 = \big(m \circ (\mathrm{id} \otimes S) \circ \Delta\big)(a) \quad (a \in \mathcal{A}). \tag{10.4}$$

It can be shown as a consequence of this definition that the antipode is anti-multiplicative and anti-comultiplicative:

$$\begin{aligned} &S(ab) = S(b)S(a), \quad S(1) = 1, \\ &(S \otimes S) \circ \sigma \circ \Delta = \Delta \circ S, \quad \varepsilon \circ S = \varepsilon. \end{aligned} \tag{10.5}$$

In the definition of Hopf algebra the flip operator σ entered in the specification (10.1) of the algebra structure of $\mathcal{A} \otimes \mathcal{A}$, and this algebra structure is needed since $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is required to be an algebra homomorphism. In a *braided Hopf algebra* the role of σ is taken over by some other bijective linear mapping $\Psi: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, the so-called *braiding*. The multiplication in $\mathcal{A} \otimes \mathcal{A}$ is now defined by

$$(a \otimes b)(c \otimes d) := (m \otimes m)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(a \otimes b \otimes c \otimes d). \tag{10.6}$$

The braiding Ψ has to satisfy some further axioms such that multiplication in $\mathcal{A} \otimes \mathcal{A}$ is associative and comultiplication in $\mathcal{A} \otimes \mathcal{A}$ is coassociative, which I will not give here. The definition of Hopf algebra is now precisely as in Definition 10.1, but with modified multiplication rule in $\mathcal{A} \otimes \mathcal{A}$. Braided Hopf algebras were introduced by Majid, see [27] and references given there. I refer to his papers for further details.

By way of example consider the *braided line* $\mathcal{A} := \mathbf{C}_q[x]$ (see Koornwinder [23, Section 6.8], Majid [26]). As an algebra it is just the algebra $\mathbf{C}[x]$ of polynomials in one variable x, generated by the element x, and with basis $1, x, x^2, \ldots$. Now let $q \in (0,1)$ and introduce the braiding Ψ by specifying it on a basis of $\mathcal{A} \otimes \mathcal{A}$:

$$\Psi(x^k \otimes x^l) := q^{kl}\, x^l \otimes x^k \quad (k, l \in \mathbf{Z}_+).$$

So for multiplication in $\mathcal{A} \otimes \mathcal{A}$ we will have:

$$\begin{aligned} (x^k \otimes 1)(1 \otimes x^l) &= x^k \otimes x^l, \\ (1 \otimes x^k)(x^l \otimes 1) &= q^{kl}\, x^l \otimes x^k, \\ (x^{k_1} \otimes x^{k_2})\,(x^{l_1} \otimes x^{l_2}) &= q^{k_2 l_1}\, x^{k_1+l_1} \otimes x^{k_2+l_2}. \end{aligned}$$

Hence

$$\begin{aligned} \big((x^{k_1} \otimes x^{k_2})\,(x^{l_1} \otimes x^{l_2})\big)\,(x^{m_1} \otimes x^{m_2}) &= q^{k_2 l_1 + k_2 m_1 + l_2 m_1}\, x^{k_1+l_1+m_1} \otimes x^{k_2+l_2+m_2} \\ &= (x^{k_1} \otimes x^{k_2})\,\big((x^{l_1} \otimes x^{l_2})\,(x^{m_1} \otimes x^{m_2})\big), \end{aligned}$$

which proves the associativity.

Note that $\mathcal{A} \otimes \mathcal{A}$ can be considered as the algebra with generators $1 \otimes x$ and $x \otimes 1$ and with relation $(1 \otimes x)(x \otimes 1) = q(x \otimes 1)(1 \otimes x)$, so it is isomorphic with the algebra $\mathbf{C}_q[x, y]$ under the isomorphisms $x^l \otimes x^k \mapsto y^l x^k$.

More generally we can make the n-fold tensor product $\otimes^n\mathcal{A}$ into an algebra by the rule

$$(x^{k_1}\otimes\cdots\otimes x^{k_n})(x^{l_1}\otimes\cdots\otimes x^{l_n})=q^{\sum_{i>j}k_il_j}\,x^{k_1+l_1}\otimes\cdots\otimes x^{k_n+l_n}.$$

A simple computation shows that the multiplication is associative. Futhermore, the linear subspace of $\otimes^n\mathcal{A}$ spanned by the elements

$$1\otimes\cdots\otimes 1\otimes x^{k_1}\otimes 1\otimes\cdots\otimes 1\otimes x^{k_2}\otimes 1\otimes\cdots\cdots\otimes 1\otimes x^{k_r}\otimes 1\otimes\cdots\otimes 1$$

(non-zero powers of x only allowed at positions $i_1,i_2,\ldots,i_r$) is a subalgebra of $\otimes^n\mathcal{A}$ isomorphic to $\otimes^r\mathcal{A}$.

Since the comultiplication $\Delta\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ has to be an algebra homomorphism, it is sufficient to define it on the generator x of $\mathcal{A}$:

$$\Delta(x):=x\otimes 1+1\otimes x.$$

Then

$$\big((\Delta\otimes\mathrm{id})\circ\Delta\big)(x)=x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x=\big((\mathrm{id}\otimes\Delta)\circ\Delta\big)(x),$$

so the coassociativity, being valid on the generator $x\in\mathcal{A}$, will be valid in general. If f is a polynomial in one variable then

$$\Delta(f(x))=f(x\otimes 1+1\otimes x). \tag{10.7}$$

For the comultiplication applied to a general basis element $x^n\in\mathcal{A}$ we find

$$\Delta(x^n)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^{n-k}\otimes x^k. \tag{10.8}$$

Indeed, by rewriting the two sides of (10.8) we have to prove that

$$(x\otimes 1+1\otimes x)^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q (x\otimes 1)^{n-k}\,(1\otimes x)^k,$$

and this is true by the q-binomial formula (2.3) since $(1\otimes x)(x\otimes 1)=q(x\otimes 1)(1\otimes x)$.

In order to find the counit $\varepsilon\colon\mathcal{A}\to\mathbf{C}$ we appply the first identity of (10.3) to x^n and we use (10.8):

$$x^n=(\varepsilon\otimes\mathrm{id})(\Delta(x^n))=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q \varepsilon(x^{n-k})\,x^k,$$

which yields

$$\varepsilon(x^n)=\begin{cases}0 & \text{if } n>0,\\ 1 & \text{if } n=0.\end{cases} \tag{10.9}$$

Finally we look for the existence of an antipode $S\colon\mathcal{A}\to\mathcal{A}$. It turns out that $S(x^n)$ is uniquely found by letting one the identitities of (10.4) (say the second one) act on the basis elements x^n of $\mathcal{A}$. We obtain

$$S(x^n):=(-1)^n\,q^{\frac12 n(n-1)}\,x^n. \tag{10.10}$$

Indeed, the second identity of (10.4) yields

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}\, S(x^k) = \big(m \circ (\mathrm{id} \otimes S) \circ \Delta\big)(x^n) = \varepsilon(x^n)\, 1 = \delta_{n,0}\, 1, \tag{10.11}$$

so $S(x^n)$ is obtained by recurrence with respect to n. Now the left-hand side of (10.11) with $S(x^k)$ given by (10.10) becomes

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k\, q^{\frac{1}{2}k(k-1)}\, x^n = \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k}\, q^{nk}\, x^n = {}_1\phi_0(q^{-n};\,;q,q^n)\, x^n = (1;q)_n\, x^n,$$

which equals the right-hand side of (10.11). The proof that the first identity of (10.4) acting on x^n also holds, amounts to the same computation as we just gave.

Observe that

$$\begin{aligned} S(x^{m+n}) &= q^{mn}\, S(x^m)\, S(x^n), \\ \Delta(S(x^n)) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(n-k)}\, S(x^{n-k}) \otimes S(x^k). \end{aligned}$$

Hence

$$S \circ m = m \circ (S \otimes S) \circ \Psi, \quad \Delta \circ S = (S \otimes S) \circ \Psi \circ \Delta. \tag{10.12}$$

This can be considered as an analogue of (10.5) for the braided case, where the flip σ is replaced by the braiding Ψ.

We can reformulate formula (8.31) by using the above comultiplication:

$$\Delta(h_n(x;q)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k} \otimes h_k(x;q). \tag{10.13}$$

When we apply $m \circ (S \otimes \mathrm{id}) \circ \Delta$ to both sides of (10.13) then we obtain

$$h_n(0;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k}\, q^{\frac{1}{2}(n-k)(n-k-1)}\, x^{n-k}\, h_k(x;q).$$

In view of (9.7) this is just (8.6).

We can extend the above braided Hopf algebra structure of $\mathbf{C}_q[x]$ to the algebra $\mathbf{C}_q[[x]]$ of formal power series in x. Then Δ, ε and S are well-defined in a termwise way on $\mathbf{C}_q[[x]]$, by using (10.8), (10.9) and (10.10). For $\Delta(f(x))$, f being a formal power series, we can still use (10.7). In particular, for Δ, ε and S acting on $e_q(x)$ we obtain

$$\Delta(e_q(x)) = e_q(x) \otimes e_q(x), \quad \varepsilon(e_q(x)) = 1, \quad S(e_q(x)) = E_q(-x). \tag{10.14}$$

The first identity, which was already observed in Koornwinder [23, Section 6.8], follows from (and is equivalent to) (3.5) since $\Delta(e_q(x)) = e_q(x \otimes 1 + 1 \otimes x)$ and $(1 \otimes x)(x \otimes 1) = q(x \otimes 1)(1 \otimes x)$. Now it follows from (10.4) and (10.14) that

$$1 = \varepsilon(e_q(x))1 = \big(m \circ (S \otimes \mathrm{id}) \circ \Delta\big)(e_q(x)) = S(e_q(x))\, e_q(x) = E_q(-x)\, e_q(x),$$

i.e., we have reobtained (3.3).

Formula (9.1) (the translation invariance of the Jackson integral over $(-x.\infty, x.\infty)$) can also be rephrased in terms of the above comultiplication. We will work in a very formal way. Let $\int$ be the linear operator defined by

$$\int(f(x)) := \int_{-x.\infty}^{x.\infty} f(t)\,d_qt.$$

If y is an element with the property that $xy = q^k yx$ for some integer k, then y commutes with $\int(f(x))$, so $\int(f(x))$ may be considered as a scalar. Now we can rewrite (9.1) as

$$\big((\mathrm{id}\otimes \textstyle\int)\circ\Delta\big)(f(x)) = \textstyle\int(f(x))\;1. \tag{10.15}$$

Indeed, the left-hand side can formally be written as

$$\int_{-x.\infty}^{x.\infty} f(x\otimes 1 + 1\otimes t)\,d_qt$$

and $(1\otimes t)(x\otimes 1) = q(x\otimes 1)(1\otimes t)$ if $t = q^k x$ for some integer k.

Next we will consider the q-Fourier transform $\mathcal{F}_q$ defined by (8.22) from the point of view of this comultiplication. We fix $q\in(0,1)$ and we rewrite (8.22) very formally as

$$\mathcal{F}_y(f(x)) := \int_{-x.\infty}^{x.\infty} e_q(-ity)\,f(t)\,d_qt \quad (y\in\mathbf{R}).$$

Then the linear operator $\mathcal{F}_y$, like $\int$, will map to a space of scalars. It follows from (10.15) and (3.5) that

$$\begin{aligned}\mathcal{F}_y(f(x))\;1 &= (\mathrm{id}\otimes \textstyle\int)\big(\Delta(e_q(-ixy))\,\Delta(f(x))\big)\\ &= (\mathrm{id}\otimes \textstyle\int)\Big(\big(e_q(-ixy)\otimes e_q(-ixy)\big)\,\Delta(f(x))\Big)\\ &= (\mathrm{id}\otimes \mathcal{F}_y)(\Delta(f(x))\;e_q(-ixy)\end{aligned}$$

Now multiply the left-hand side and the right-hand side both by $E_q(ixy)$. Then we obtain

$$(\mathrm{id}\otimes\mathcal{F}_y)(\Delta(f(x)) = \mathcal{F}_y(f(x))\;E_q(ixy). \tag{10.16}$$

This formula is a q-analogue of the well-known property of the Fourier transform that

$$\int_{-\infty}^{\infty} e^{-ity}\,f(x+t)\,dt = e^{ixy}\int_{-\infty}^{\infty} e^{-ity}\,f(t)\,dt.$$

Formula (10.16) can also be written as

$$(\mathrm{id}\otimes \textstyle\int)\big((1\otimes e_q(-ixy))\;\Delta(f(x))\big) = (\mathrm{id}\otimes \textstyle\int)\Big(\big((S\otimes\mathrm{id})\circ\Delta\big)(e_q(-ixy))\,(1\otimes f(x))\Big).$$

More generally we have

$$(\mathrm{id}\otimes \textstyle\int)\big((1\otimes g(x))\;\Delta(f(x))\big) = (\mathrm{id}\otimes \textstyle\int)\Big(\big((S\otimes\mathrm{id})\circ\Delta\big)(g(x))\;(1\otimes f(x))\Big), \tag{10.17}$$

which is a q-analogue of

$$\int_{-\infty}^{\infty} g(t)\,f(x+t)\,dt = \int_{-\infty}^{\infty} g(t-x)\,f(t)\,dt.$$

Formula (10.17) is given by Kempf & Majid in [19, (139)] with diagrammatic proof in their Figure 2(b). See this paper also for many further results about Jackson integral and q-Fourier transform in connection with the braided line.

11. Further results

In this final section I very briefly mention three further results.

1. Special function identities involving non-commuting variables satisfying relations which may be more complicated than q-commutation occur very naturally as addition formulas obtained from quantum groups. A prototype of this is the addition formula for little q-Legendre polynomials, in the context of $SU_q(2)$. See Koornwinder [24], where the formula with non-commuting variables is next equivalently rewritten in commuting variables. In the context of $U_q(n)$ a similar but more complicated addition formula in non-commuting variables was obtained by Floris [14] for q-disk polynomials. Next Floris & Koelink [15] found an equivalent form in commuting variables of this addition formula. See Koelink [21] for other examples involving $SU_q(2)$ of rewriting addition formulas from non-commutative form into commutative form.
2. Some q-hypergeometric series which are not summable when parameters and argument commute, may suddenly become summable when these variables do not commute but satisfy certain relations. G. Andrews (private communication) showed me a surprising example of this involving a ${}_{m+1}\phi_m$. Many further results in this direction, often in operational form, can be found in Bowman [6].
3. The following was communicated to me by A. Yu. Volkov. The second part of formula (3.28) nicely generalizes to an identity in x, y, c with q-Heisenberg relations (4.1):

$$(y;q)_n\,(x;q)_n = \prod_{k=0}^{n-1}(1 - q^k(x + y - yx + c) + q^{2k}c).$$

The proof is by induction with respect to n. On letting $n \to \infty$ and taking the inverse on both sides, we obtain an addition to Proposition 4.1.

Acknowledgements This paper was written while the author was a guest at the Institute Mittag-Leffler, Djursholm, Sweden during October, November 1995. I am grateful to the director of the Institute and to the organizers of the special year "Analysis on Lie Groups" for hospitality. I also thank G. Andrews, R. Askey, D. Bowman, A. N. Kirillov, C. Krattenthaler, A. Yu. Volkov and S. L. Woronowicz for helpful remarks.

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