The Fibonacci Quarterly 1974 (12,4): 369-372

A q-IDENTITY

L. CARLITZ*
Duke University, Durham, North Carolina 27706

1. The object of this note is to prove the following q-identity:

$$(*)\qquad \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k} (a)_k (b)_k q^{\frac12(n-k)(n+k-1)} (ab)^{n-k} = (a)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{b^k}{1-q^{n-k}a}$$

$$= (b)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{a^k}{1-q^{n-k}b},$$

where

$$(a)_k = (a,q)_k = (1-a)(1-qa)\cdots(1-q^{k-1}a), \qquad (a)_0 = 1,$$

$$(q)_k = (q,q)_k = (1-q)(1-q^2)\cdots(1-q^k), \qquad (q)_0 = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix} \qquad (0 \le k \le n)$$

and q is not a t^{th} root of unity, $1 \le t \le n$.

Since each side of

$$(1)\qquad \sum_{k=0}^{n} (-1)^{n-k} \frac{(q)_n}{(q)_k} (a)_k (b)_k q^{\frac12(n-k)(n+k-1)} (ab)^{n-k} = (b)_{n+1} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\frac12 k(k-1)} \frac{a^k}{1-q^{n-k}b}$$

is a polynomial in b of degree $\le n$, it will suffice to show that (1) holds for $b = q^{-r}$, $0 \le r \le n$.

We have

$$\left.\frac{(b)_{n+1}}{1-q^r b}\right|_{b=q^{-r}} = (1-q^{-r})\cdots(1-q^{-1})(1-q)(1-q^2)\cdots(1-q^{n-r}) = (-1)^r q^{-\frac12 r(r+1)} (q)_r (q)_{n-r}.$$

Thus the right-hand side of (1) reduces to

$$(2)\qquad (-1)^{n-r} \begin{bmatrix} n \\ n-r \end{bmatrix} q^{\frac12(n-r)(n-r-1)} (-1)^r q^{-\frac12 r(r+1)} (q)_r (q)_{n-r} a^{n-r} = (-1)^n (q)_n q^{\frac12 n(n-1)-nr} a^{n-r}.$$

As for the left-hand side, since

$$(q^{-r})_k = (1-q^{-r})(1-q^{-r+1})\cdots(1-q^{-r+k-1}) = (-1)^k q^{-rk+\frac12 k(k-1)} (1-q^r)(1-q^{r-1})\cdots(1-q^{r-k+1})$$

$$= \begin{cases} (-1)^k q^{-rk+\frac12 k(k-1)} (q)_r/(q)_{r-k} & (0 \le k \le r) \\ 0 & (k > r) \end{cases},$$

we get

$$\sum_{k=0}^{r} (-1)^{n-k} \frac{(q)_n}{(q)_k} a^{n-k} q^{\frac12(n-k)(n+k-1)} \cdot (-1)^k q^{-rk+\frac12 k(k-1)} \frac{(q)_r}{(q)_{r-k}} q^{-r(n-k)} = (-1)^n (q)_n q^{\frac12 n(n-1)-nr} a^{n-r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a$$

*Supported in part by NSF Grant GP-37924.

We shall now show that

(3) $$\sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = 1 \qquad (r = 0, 1, 2, \cdots),$$

so that the left-hand side of (1) is equal to

$$(-1)^n (q)_n q^{\frac{1}{2}n(n-1)-nr} a^{n-r}$$

in agreement with (2).

To prove (3) we take

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k \sum_{r=0}^{\infty} \frac{a^r x^r}{(q)_r}.$$

By a well known identity

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} x^k = \frac{e(x)}{e(ax)},$$

where

(4) $$e(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \prod_{n=0}^{\infty} (1 - q^n x)^{-1}.$$

Thus

$$\sum_{r=0}^{\infty} \frac{x^r}{(q)_r} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} (a)_k a^{r-k} = \frac{e(x)}{e(ax)} e(ax) = e(x)$$

and (3) follows at once.

This evidently completes the proof of (*).

2. The identity (*) can also be proved by making use of the q-analog of Gauss's theorem (see for example [1, p. 68]):

(5) $$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(q)_k (x)_k} \left(\frac{x}{ab} \right)^k = \frac{e(x)e(x/ab)}{e(x/a)e(x/b)},$$

where $e(x)$ is defined by (4).

Define the operator E by means of

$$E^n f(x) = f(q^n x) \qquad (n = 0, 1, 2, \cdots)$$

and Δ^n by means of the operational formula

$$\Delta^n = (1 - E)(q - E) \cdots (q^{n-1} - E).$$

Then it is easily verified that

$$\Delta^n = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} E^{n-r}.$$

It follows that

$$\Delta^n x^k = \sum_{r=0}^{n} (-1)^{n-r} \begin{bmatrix} n \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} q^{(n-r)k} x^k = (q^k - 1)(q^k - q) \cdots (q^k - q^{n-1}) x^k,$$

so that

(6) $$\Delta^n x^k = \begin{cases} 0 & (n > k) \\ (-1)^k q^{\frac{1}{2}k(k-1)} (q)_k x^k & (n = k) \end{cases}$$

Now multiply both sides of (5) by $(x)_n$ and apply Δ^n. Then divide by x^n and put $x = 0$. In view of (6) the LHS becomes

$$\sum_{k=0}^{n} \frac{(a)_k(b)_k}{(q)_k}(ab)^{-k}\cdot(-1)^{n-k}q^{\frac{1}{2}(n-k)(n-k-1)+k(n-k)}\cdot(-1)^n q^{\frac{1}{2}n(n-1)}(q)_n$$

(7)
$$= (-1)^n q^{\frac{1}{2}n(n-1)}\sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac{1}{2}(n-k)(n+k-1)}(ab)^{-k}\ .$$

As for the RHS, we have first

$$(x)_n\frac{e(x)e(x/ab)}{e(x/a)e(x/b)} = \frac{e(q^n x)e(x/ab)}{e(x/a)e(x/b)}$$
$$= \sum_{j=0}^{\infty}\frac{(q^{-n}/a)_j}{(q)_j}(q^n x)^j\sum_{k=0}^{\infty}\frac{(a)_k}{(q)_k}\left(\frac{x}{ab}\right)^k$$
$$= \sum_{r=0}^{\infty}\frac{x^r}{(q)_r}\sum_{k=0}^{r}\begin{bmatrix} r\\ k\end{bmatrix}(a)_k(ab)^{-k}(q^{-n}/a)_{r-k}q^{n(r-k)}\ .$$

Apply Δ^n, divide by x^n and put $x = 0$. We get

(8)
$$(-1)^n q^{\frac{1}{2}n(n-1)}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(a)_k(ab)^{-k}(q^{-n}/a)_{n-k}q^{n(n-k)}\ .$$

Since

$$(q^{-n}/a)_{n-k} = (1-q^{-n}/a)(1-q^{-n+1}/a)\cdots(1-q^{-k-1}/a) = (-1)^{n-k}a^{-n+k}q^{-\frac{1}{2}n(n+1)+\frac{1}{2}k(k+1)}$$
$$\cdot(1-q^{k+1}a)(1-q^{k+2}a)\cdots(1-q^n a)\ ,$$

(8) becomes

$$q^{\frac{1}{2}n(n-1)}(ab)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac{1}{2}n(n-1)-nk+\frac{1}{2}k(k+1)}b^{n-k}\frac{(a)_{n+1}}{1-q^k a}$$
$$= (-1)^n q^{\frac{1}{2}n(n-1)}(ab)^{-n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac{1}{2}k(k-1)}b^k\frac{(a)_{n+1}}{1-q^{n-k}a}\ .$$

Comparing this with (7) it is clear that we have proved (*).

3. We have

$$\sum_{n=0}^{\infty}\frac{x^n}{(q)_n}\sum_{k=0}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k}(a)_k(b)_k q^{\frac{1}{2}(n-k)(n+k-1)}(ab)^{n-k}$$
$$= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(q)_k}x^k\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}n(n-1)}(q^k abx)^n\ .$$

Also, since

$$(a)_{n+1} = (a)_{n-k}(1-q^{n-k}a)(q^{n-k+1}a)_k\ ,$$

$$\sum_{n=0}^{\infty}\frac{(a)_{n+1}}{(q)_n}x^n\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}q^{\frac{1}{2}k(k-1)}\frac{b^k}{1-q^{n-k}a} = \sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{2}k(k-1)}\frac{b^k x^k}{(q)_k}\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}(q^{n+1}a)_k x^n$$
$$= \sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac{1}{2}k(k-1)}\frac{(q^{n+1}a)_k}{(q)_k}(bx)^k\ .$$

Thus (*) is equivalent to the identity

$$\sum_{n=0}^{\infty}(-1)^n q^{\frac12 n(n-1)}(abx)^n\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(q)_k}(q^n x)^k=\sum_{n=0}^{\infty}\frac{(a)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac12 k(k-1)}\frac{(q^{n+1}a)_k}{(q)_k}(bx)^k$$

$$(9)\qquad =\sum_{n=0}^{\infty}\frac{(b)_n}{(q)_n}x^n\sum_{k=0}^{\infty}(-1)^k q^{\frac12 k(k-1)}\frac{(q^{n+1}b)_k}{(q)_k}(ax)^k,$$

where now $|q|<1$.

4. The following special cases of (*) may be noted. For $b=q$ we have

$$\sum_{k=0}^{n}(-1)^{n-k}(a)_k q^{\frac12 k(n-k)(n+k+1)}a^{n-k}=\frac{(a)_{n+1}}{(q)_n}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\frac12 k(k+1)}}{1-q^{n-k}a}$$

$$(10)\qquad =(1-q^{n+1})\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\frac12 k(k-1)}\frac{a^k}{1-q^{n-k+1}}.$$

For $a=q$ this reduces to

$$(11)\qquad \sum_{k=0}^{n}(-1)^{n-k}(q)_k q^{\frac12(n-k)(n+k+3)}=(1-q^{n+1})\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{q^{\frac12 k(k+1)}}{1-q^{n-k+1}}.$$

Since

$$\begin{bmatrix}n+1\\k\end{bmatrix}=\frac{1-q^{n+1}}{1-q^{n-k+1}}\begin{bmatrix}n\\k\end{bmatrix}$$

and

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n+1\\k\end{bmatrix}q^{\frac12 k(k+1)}=(q)_{n+1}+(-1)^n q^{\frac12(n+1)(n+2)},$$

(11) becomes

$$(12)\qquad \sum_{k=0}^{n}(-1)^{n-k}(q)_k q^{\frac12(n-k)(n+k+3)}=(q)_{n+1}+(-1)^n q^{\frac12(n+1)(n+2)}.$$

Somewhat more generally, it follows from (10) that

$$(13)\qquad \sum_{k=0}^{n}(-1)^{n-k}(a)_k q^{\frac12(n-k)(n+k+1)}a^{n-k}=(a)_{n+1}+(-1)^n q^{\frac12 n(n+1)}a^{n+1}.$$

We shall give a direct proof of (13). The formula evidently holds for $n=0$. Assuming that it holds up to and including the value n, we replace a by qa and multiply both sides by $1-a$. Thus

$$\sum_{k=0}^{n}(-1)^{n-k}(a)_{k+1}q^{\frac12(n-k)(n+k+3)}a^{n-k}=(a)_{n+2}+(-1)^n q^{\frac12(n+1)(n+2)}a^{n+1}(1-a).$$

Hence

$$\sum_{k=0}^{n+1}(-1)^{n-k+1}(a)_k q^{\frac12(n-k+1)(n+k+2)}a^{n-k+1}=(a)_{n+2}+(-1)^n q^{\frac12(n+1)(n+2)}a^{n+1}(1-a)+(-1)^{n+1}$$

$$\cdot\, q^{\frac12(n+1)(n+2)}a^{n+1}=(a)_{n+2}+(-1)^{n+1}q^{\frac12(n+1)(n+2)}a^{n+2}.$$

REFERENCE

1. W.N. Bailey, *Generalized Hypergeometric Series,* Cambridge, 1935.

****