On some q-operators with applications

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1. INTRODUCTION

The expansion of differential (and difference) operators in terms of other (sometimes more elementary) operators is very old and many authors devoted a great deal of work to the subject. To give a few examples we consider the derivative operator $D=d/dx$ and the difference operator ∇, $\nabla f(x)=f(x)-$ $-f(x-1)$.

We first recall that most elementary texts on differential equations mention Boole's identity

$$x^nD^n=xD(xD-1)(xD-2)\cdots(xD-n+1). \tag{1.1}$$

Its finite difference analog is

$$(x)_n\nabla^n=x\nabla(x\nabla-1)(x\nabla-2)\cdots(x\nabla-n+1) \tag{1.2}$$

where $(x)_n=x(x-1)(x-2)\cdots(x-n+1)$ for $n=1,2,\ldots,(x)_0=1$.

Let $\circ$ represents operator composition. Then recently Carlitz [10] proved

$$\prod_{j=1}^{n}(xD-x+a+j)=x^{-a}e^{x}D^n\circ(x^{n+a}e^{-x})=n!\sum_{k=0}^{n}\frac{x^k}{k!}L_{n-k}^{(a+k)}(x)D^k. \tag{1.3}$$

W.A. Al-Salam [3] showed that if $\theta=x(1+xD)$ then

$$\left\{\begin{aligned}\theta^n\{x^\alpha e^{-x}f(x)\}&=x^{n+\alpha}e^{-x}\prod_{j=1}^{n}(xD-x+\alpha+j)f(x)\\&=x^{\alpha+n}e^{-x}n!\sum_{k=0}^{n}\frac{x^k}{k!}L_{n-k}^{(\alpha+k)}(x)D^kf(x)\end{aligned}\right. \tag{1.4}$$

Gould and Hopper gave [15]

$$(1.5)\qquad \prod_{j=1}^{n}(xD+a+j)=x^{-a}D^n\circ x^{n+a}=\sum_{k=1}^{n}\binom{n}{k}\binom{n+\alpha}{n-k}(n-k)!x^kD^k$$

Burchnall [9] proved

$$(1.6)\qquad (D-2x)^n=e^{x^2}D^n\circ e^{-x^2}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}(x)D^k.$$

Osipov [23] showed that

$$(1.7)\qquad \{D(x+\alpha)D\}^n=D^n\circ(x+\alpha)^n\circ D^n=\sum_{k=0}^{n}\binom{n}{k}\frac{n!}{k!}(x+\alpha)^kD^{k+n}.$$

This was later generalized by Al-Salam and Ismail in [6].

N. Meller constructed in [22] an operational calculus for the "Bessel" operator

$$(1.8)\qquad B=x^{-\alpha}Dx^{\alpha+1}D=x\frac{d^2}{dx^2}+(\alpha+1)\frac{d}{dx}$$

Koornwinder [21] in the course of giving an analytical proof of the addition theorem for the Jacobi polynomial used the operator

$$(1.9)\qquad \Omega_\beta=\frac{d^2}{dx^2}+\frac{2\beta+1}{x}\frac{d}{dx},$$

which is related to (1.8) by the change of variable $x^2=u$. He showed, among other results, that

$$(1.10)\qquad 2^{2n}n!(n+\alpha+1)_n(1-x^2)^\alpha P_n^{(\alpha,\beta)}(2x^2-1)=\Omega_\beta^n(1-x^2)^{2n+\alpha}.$$

and

$$(1.10\text{b})\qquad \begin{cases} (-1)^n2^{2n}n!(\alpha+\beta+1)_n(1+x^2)^{-n-\alpha-\beta-1}P_n^{(\alpha,\beta)}\left(\dfrac{1-x^2}{1+x^2}\right)= \\ =\Omega_\alpha^n(1+x^2)^{-\alpha-\beta-1}. \end{cases}$$

Relations of this type are also implied by (1.3), (1.4), and (1.6), namely,

$$(1.11)\qquad L_n^{(\alpha)}(x)=\frac{1}{n!}\prod_{j=1}^{n}(xD-x+\alpha+j)\cdot 1=\frac{x^{-\alpha-n}e^x}{n!}\theta^n\{x^\alpha e^{-x}\},$$

and

$$(1.12)\qquad H_n(x)=(2x-D)^n\cdot 1$$

In addition, Viskov [25] proved that

$$(1.13)\qquad L_n^{(\alpha)}(x)=\frac{(-1)^n}{n!}e^xB^ne^{-x}$$

and the present writer [2] that

(1.14) $$L_n^{(\alpha)}(x)=\frac{(-1)^n}{n!}\,e^{-B}x^n.$$

Recently there has been an intense interest in q-series and functions. Thus it is natural to try to obtain q-analogs for some of these operators and their expansions as well as their relations to some special function. This subject is also not new. For example Jackson [19] considered (with slightly different notation) the q-derivative operator

(1.15) $$D_q f(x)=\frac{f(x)-f(qx)}{x}=x^{-1}\{I-\eta\}f(x)$$

where $\eta f(x)=f(qx)$. He proved a q-analog of (1.1) that may be stated in the form

(1.16) $$\begin{cases} x^n D_q^n=(I-\eta)(I-q^{-1}\eta)\cdots(I-q^{-n+1}\eta)=(q^{-n+1}\eta;\,q)_n \\ \quad =\sum_{j=0}^{n}(-1)^j\begin{bmatrix} n \\ j\end{bmatrix} q^{\frac{1}{2}j(j+1)-nj}\eta^j. \end{cases}$$

The last equality follows from a well known theorem of Euler and where we have used the familiar q-notation

$$(a;q)_0=1,(a;q)_n=(1-a)(1-qa)\cdots(1-q^{n-1}a)\ \text{for}\ n=1,2,3,\ldots,$$

$$\begin{bmatrix} n \\ j\end{bmatrix}=\begin{bmatrix} n \\ j\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_j(q;q)_{n-j}}.$$

We shall also use the notation

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\ \text{for}\ |q|<1.$$

A q-analog of (1.12) was obtained by Cigler [12]

(1.17) $$h_n(x)=(x-q^{n-1}D_q)(x-q^{n-2}D_q)\cdots(x-D_q)\cdot 1$$

and the more general formula

(1.18) $$S_n^{(\alpha)}(x)=(x-q^n b(D_q))(x-q^{n-1}b(D_q))\cdots(x-b(D_q))\cdot 1$$

where $b(u)=au+b$, a,b are constants. These polynomials are essentially the polynomial set $\{U_n^{(\alpha)}\}$ due to Al-Salam and Carlitz [5]. $h_n(x)=U_n^{(-1)}(x)$ is a q-analog of the Hermite polynomials (see [5]).

The present writer gave in [2] a q-analog of the Bessel operator

$$B=x^{-\alpha}Dx^{\alpha+1}D,$$

namely,

(1.19) $$B_q(\alpha)=x^{-\alpha}D_q x^{\alpha+1}D_q$$

and showed that

(1.20) $B_q^n(\alpha)=(q^{\alpha+1}\eta;\, q)_n D_q^n$ for $n=0,1,2,\ldots$

The results (1.3)–(1.7) are combinations of operator identities and Leibniz formulas. They, as well as (1.10)–(1.14), (1.17), and (1.18), give Rodrigues type formulas for orthogonal polynomials. In this paper we shall investigate and obtain similar results that involve sets of orthogonal polynomials that are of recent interest. In § 2 we define a q-analog of the operator x^2D, study it and apply it to the q-Laguerre polynomials. Next in § 3 we study a q-analog of the operator $1/xD$ and then, in § 4 apply it to the q-Bessel functions and q-Bessel polynomials. In § 5 we apply a fractional power of the operator introduced in § 3 to a q-analog of the ultraspherical polynomials. Finally, in § 6, we give a q-analog of the operator (1.9) that Koornwinder used and give q-analogs of some of his results.

2. SOME OPERATORS

We consider here a q-analog of the operator x^2D and the more general operator $x(\alpha+xD)$, namely,

(2.1) $$\theta_\alpha=x\{(1-q^\alpha)+q^\alpha xD_q\}=x(1-q^\alpha\eta)$$

so that

(2.2) $$\theta_\alpha^n x^\lambda=(q^{\alpha+\lambda};\, q)_n x^{\lambda+n} \quad n=0,1,2,\ldots$$

The following formulas can be proved by induction.

(2.3) $$\theta_\alpha^n\{x^\alpha f(x)g(x)\}=x^\alpha\sum_{j=0}^{n}\begin{bmatrix} n \\ j\end{bmatrix} q^{\alpha j}(\theta_\alpha^j g(x))(\theta_\alpha^{n-j}\eta^j f(x))$$

(2.4) $$\theta_\alpha^n=\sum_{k=0}^{n}\begin{bmatrix} n+\alpha-1 \\ n-k\end{bmatrix}\frac{(q;\, q)_n}{(q;\, q)_k}\, q^{k(\alpha+k-1)}x^{n+k}D_q^k$$

(2.5) $$\theta_\alpha^n=x^n(q^\alpha\eta;\, q)_n \quad (n=0,1,2,\ldots)$$

(2.6) $$\theta_0^n\{x^\alpha f(x)\}=x^\alpha\theta_\alpha^n f(x).$$

Formula (2.3) is a Leibniz formula. (2.4) is a q-analog of a formula, for $\alpha=0,1$, due to H.W. Gould [14]. Formulas (2.5) and (2.6) are consequences of the identity $F(\eta)(x^n g(x))=x^n F(q^n\eta)g(x)$.

The operator θ_α is particularly useful in dealing with the q-Laguerre polynomials

$$L_n^{(\alpha)}(x\,|\,q)=\frac{(q^{\alpha+1};\, q)_n}{(q;\, q)_n}\sum_{k=0}^{n}\frac{(q^{-n};\, q)_k q^{\frac{1}{2}k(k+1)+k(\alpha+n)}}{(q;\, q)_k(q^{\alpha+1};\, q)_k}\, x^k.$$

We get

$$
(2.7)\qquad \begin{cases} L_n^{(\alpha)}(x|q) = \dfrac{x^{-\alpha-n-1}(-x;q)_\infty}{(q;q)_n}\,\theta_0^n\left(\dfrac{x^{\alpha+1}}{(-x;q)_\infty}\right) \\[2ex] \qquad = \dfrac{x^{-n}(-x;q)_\infty}{(q;q)_n}\,\theta_{\alpha+1}^n\left(\dfrac{1}{(-x;q)_\infty}\right). \end{cases}
$$

This is a q-analog of (1.11) and may be compared with formulas (2.9) and (2.11) of [2]. These formulas, as is the case with (4.5) and (6.12) but not with (6.6), have the advantage that they express a special function by a Rodrigues' type formulas in which the nth power of the respective operators acts on an elementary function which is independent of n.

Formulas (2.7) and (2.3) imply

$$
(2.8)\qquad L_n^{(\alpha+\beta+1)}(x|q) = q^{n(\beta+1)} \sum_{k=0}^{n} \begin{bmatrix} \beta+k \\ k \end{bmatrix} q^{-k(\beta+1)} L_{n-k}^{(\alpha)}(x|q).
$$

A more general formula than (2.7) is

$$
(2.9)\qquad \theta_0^n\left(\frac{x^{\alpha+1}f(x)}{(-x;q)_\infty}\right) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q;q)_k}{(-x;q)_\infty}\, x^{\alpha+k+1} L_n^{(\alpha)}(x|q)\theta_0^{n-k}\eta^k f(x).
$$

3. THE OPERATOR $\dfrac{1}{x} D_q$

We consider now a q-analog of the operator $\dfrac{1}{x}\dfrac{d}{dx}$, namely,

$$
\delta_q = \frac{1}{x} D_q = \frac{1}{x^2}(1-\eta).
$$

We show that

$$
(3.1)\qquad \delta_q^n f(x) = \frac{q^{-n(n-1)}}{x^{2n}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} q^{(n-k)(n-k-1)}\eta^k f(x)
$$

$$
(3.2)\qquad = q^{-\frac{1}{2}n(n-1)} x^{-2n} \sum_{k=0}^{n} \frac{(q^{-n+1};q)_k (q^n;q)_k}{(q;q)_k(-q;q)_k}\, q^k x^{n-k} D_q^{n-k} f(x).
$$

The fractional version of (3.1) is

$$
(3.3)\qquad \delta_q^\alpha f(x) = x^{-2\alpha} \sum_{k=0}^{\infty} \frac{(q^{-2\alpha};q^2)_k}{(q^2;q^2)_k}\, q^{2k}\eta^k f(x)
$$

where α is any complex number. When $\alpha=n$ a non-negative integer then (3.3) reduces to (3.1). We remark that fractional powers of the operator δ are related to the fractional powers of the q-derivative operator $D_q = 1/x\,(1-\eta)$, which were considered earlier in several works (see for example [1] and [4]). More exactly if (see [1])

$$
I_q^{-\alpha} f(x) = \frac{x^{-\alpha}}{(1-q)^\alpha} \sum_{k=0}^{\infty} q^k \frac{(q^{-\alpha};q)_k}{(q;q)_k} f(xq^k),
$$

then

$$\delta_q^\alpha f(x) = (1-q^2)^\alpha T_x I_{q^2}^{-\alpha} f(\sqrt{x}),$$

where $T_u f(u) = f(u^2)$.

It is easy to verify that

$$\delta_q^\alpha \delta_q^\beta = \delta_q^{\alpha+\beta} \tag{3.4}$$

$$\delta_q^\alpha x^\lambda = \frac{(q^{\lambda-2\alpha+2}; q^2)_\infty}{(q^{\lambda+2}; q^2)_\infty} x^{\lambda-2\alpha} = \frac{\Gamma_{q^2}(\frac{1}{2}\lambda+1)(1+q)^\alpha}{\Gamma_{q^2}(\frac{1}{2}\lambda+1-\alpha)} x^{\lambda-2\alpha} \tag{3.5}$$

$$\delta_q^\alpha \{f(x)g(x)\} = \sum_{k=0}^{\infty} \begin{bmatrix} \alpha \\ k \end{bmatrix}_{q^2} \{\eta^k \delta_q^{n-k} f(x)\}\{\delta_q^k g(x)\} \tag{3.6}$$

where $\Gamma_q(z)$ is a q-analog of the gamma function which maybe defined by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}. \tag{3.7}$$

All the formulas above can be proved directly. As a sample we give a proof of (3.4).

Indeed operating on (3.3) by δ_q^β we get

$$\delta_q^\beta \delta_q^\alpha f(x) = \sum_{k=0}^{\infty} \frac{(q^{-2\alpha}; q^2)_k}{(q^2; q^2)_k} q^{2k} \delta_q^\beta \{x^{-2\alpha} f(xq^k)\}$$

$$= x^{-2\alpha-2\beta} \sum_{n=0}^{\infty} \frac{q^{2n} f(xq^n)}{(q^2;q^2)_n} \sum_{k+j=n} \frac{(q^{-2\alpha};q^2)_k (q^{-2\beta};q^2)_j}{(q^2;q^2)_j (q^2;q^2)_k} q^{-2\alpha j}$$

$$= x^{-2\alpha-2\beta} \sum_{n=0}^{\infty} \frac{q^{2n} f(xq^n)(q^{-2\alpha};q^2)_n}{(q^2;q^2)_n} {}_2\phi_1\left[\begin{matrix} q^{-2\beta}, q^{-2n}; q^2, q^2 \\ q^{2\alpha-2n+2} \end{matrix}\right].$$

The ${}_2\phi_1$ can be summed by the q-analog of Gauss' theorem

$${}_2\phi_1\left[\begin{matrix} q^{-n}, b; q, q \\ c \end{matrix}\right] = \frac{(c/b;q)_n b^n}{(c;q)_n}$$

and thus we get (3.4).

Another formula that follows from (3.5) is the q-analog of the shift operator

$$E_{q^2}(t\delta_q) x^\lambda = \sum_{k=0}^{\infty} (-1)^k \frac{q^{k(k-1)}}{(q^2;q^2)_k} t^k \delta_q^k x^\lambda = x^\lambda \frac{(t/x^2;q^2)_\infty}{(tq^\lambda/x^2;q^2)_\infty}. \tag{3.8}$$

4. THE q-BESSEL POLYNOMIALS AND FUNCTIONS

Ismail [18] introduced an interesting q-analog of the Bessel polynomials. He defined them as

$$y_n(x|q^2) = q^{\frac{1}{2}n(n-1)} {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{n+1}; q, -2qx \\ -q \end{matrix}\right]. \tag{4.1}$$

Let us recall Jackson's q-Bessel functions [18], [20], [24]

$$J_\nu^{(1)}(z;q)=\frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{(q;q)_k(q^{\nu+1};q)_k}. \tag{4.2}$$

q-Analogs of the sine and cosine functions can be defined in terms of the q-Bessel functions.

$$\frac{\sin(x;q^2)}{x}=\frac{1}{\sqrt{2}}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\left\{\frac{J_{\frac12}^{(1)}(x;q^2)}{x^{\frac12}}\right\} \tag{4.3}$$

$$\cos(x;q^2)=\frac{1}{\sqrt{2}}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\left\{\frac{J_{-\frac12}^{(1)}(x;q^2)}{x^{-\frac12}}\right\} \tag{4.4}$$

Ismail showed that the q-Bessel polynomials are orthogonal and bear the same relationship to the q-Bessel function (4.2) as the ordinary Bessel polynomials are related to the ordinary Bessel functions. It is easy to show that

$$\lim_{q\to1} y_n\left(\frac{x}{1-q};q\right)=y_n(x)$$

where $y_n(x)$ are Bessel polynomials.

Using our δ_q operator we can immediately prove the following

THEOREM. *Let $y_n(x;q)$ be the q-Bessel polynomials defined above, we have*

$$\delta_q^n\left\{\frac{e_q(\alpha x)}{x}\right\}=q^{-n^2}\frac{e_q(\alpha x)}{x^{n+1}}\alpha^n y_n\left(\frac{-1}{2\alpha x}\middle| q^2\right) \tag{4.5}$$

$$y_{n+1}(x;q^2)=2x(1-q^{2n+1})y_n(x;q^2)+q^{2n-1}y_{n-1}(x;q^2) \tag{4.6}$$

$$\sum_{n=0}^{\infty}\frac{t^n}{(q^2;q^2)_n}y_n(x;q^2)=\left(\frac{-1}{2x};q\right)_\infty\sum_{j=0}^{\infty}\frac{(2tqx;q^2)_\infty}{(q;q)_j(2xtq^j;q^2)_\infty}\left(\frac{-1}{2x}\right)^j. \tag{4.7}$$

where $e_q(x)$ is a q-analog of the exponential function e^x and is defined by $e_q(x)=\{(x;q)_\infty\}^{-1}$. We note that (4.5) is a q-analog of the following formula of Hadwiger [16]

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{e^{\alpha x}}{x}=\left(-\frac{1}{2}\right)^n e^{\alpha x}x^{-2n-1}L_n^{(-2n-1)}(-2\alpha x) \tag{4.8}$$

to which it goes as $q\to1$. Formulas (4.5) and (4.7) are new. Formula (4.6) was obtained by Ismail [18] using different methods.

We first prove (4.5). Consider the left hand side of (4.5), expand $e_q(\alpha x)$ and $1/e_q(\alpha x)$ then use (3.5) and (1.16), we get

$$\begin{aligned}&\frac{x^{n+1}}{e_q(\alpha x)}\delta_q^n\left\{\frac{e_q(\alpha x)}{x}\right\}=\\&=\sum_{j=0}^{\infty}\frac{(-1)^j\alpha^j x^{j+n+1}q^{\frac12 j(j-1)}}{(q;q)_j}\sum_{k=0}^{\infty}\frac{\alpha^k}{(q;q)_k}\delta_q^n x^{k-1}\end{aligned}$$

$$= \sum_{k,j=0}^{\infty} \frac{(-1)^j \alpha^{k+j} (q^{k+1-2n}; q^2)_\infty}{(q;q)_j (q;q)_k (q^{k+1}; q^2)_\infty} x^{k+j-n}$$

$$= (-\alpha)^n q^{-n^2} \sum_{m=0}^{\infty} \frac{(\alpha x)^{m-n}}{(q;q)_m} \sum_{j=0}^{\infty} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} q^{\frac{1}{2}j(j-1)+(m-j)n} (q^{1-m+j}; q^2)_n$$

$$= (-\alpha)^n q^{-n^2} \sum_{m=0}^{\infty} \frac{q^{mn} (\alpha x)^{m-n}}{(q;q)_m} \{D_q^m x^{m-n-1} (q^{1-m}x; q^2)_n\}_{x=1}$$

$$= (-\alpha)^n q^{-n^2} \sum_{m=0}^{\infty} \frac{q^{mn} (\alpha x)^{m-n}}{(q;q)_m} (q^{-n}; q)_m \frac{(q^{1+n}; q)_{n-m}}{(-q;q)_{n-m}}$$

$$= \alpha^n q^{-n^2} y_n\left(\frac{-1}{2\alpha x}; q^2\right).$$

Since $x^{m-n-1}(q^{1-m}x; q^2)_n$ is a polynomial of degree $m-1$ then the fourth equality shows that the infinite series in m stops after $m=n$. The next equality is then calculated by term by term q-differentiation then putting $x=1$. The resulting series can then be summed by the use of the q-analog of Gauss' theorem.

Now to prove (4.6) we see that

$$\delta_q^n\{e_q(\alpha x)\} = \delta_q^{n-1}\left\{\frac{1}{x} D_q e_q(\alpha x)\right\} = \alpha \delta_q^{n-1}\left\{\frac{e_q(\alpha x)}{x}\right\}. \tag{4.9}$$

Next we consider

$$\delta_q^{n+1}\left\{x^2 \frac{e_q(\alpha x)}{x}\right\} = \alpha q \delta_q^n\{e_q(\alpha x)\} + (1-q)\delta_q^n\left\{\frac{e_q(\alpha x)}{x}\right\}. \tag{4.10}$$

Now (4.6) follows if we use Leibniz formula (3.6) on the left hand side of (4.10) and the relations (3.15) and (3.12).

Finally to prove (4.7) we use (3.5) to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{(q^2;q^2)_k} t^k \delta_q^k\left\{\frac{e_q(\alpha x)}{x}\right\} = \sum_{j=0}^{\infty} \frac{\alpha^j x^{j-1}}{(q;q)_j} \frac{(t/x^2; q^2)_\infty}{(tq^{j-1}/x^2; q^2)_\infty}.$$

This and (4.5) imply (4.7).

For the q-sine and q-cosine functions we have from (4.3), (4.4), and (3.5)

THEOREM

$$\delta_q^n\left\{\frac{\sin(x; q^2)}{x}\right\} = (-1)^n \frac{(q^2;q^2)_\infty}{(q,q^2)_\infty} \left\{\frac{J_{n+\frac{1}{2}}^{(1)}(x;q^2)}{(2x)^{n+\frac{1}{2}}}\right\} \tag{4.11}$$

and

$$\delta_q^n\left\{\frac{\cos(x; q^2)}{x}\right\} = \frac{1}{2^{n+\frac{1}{2}}} \frac{(q^2;q^2)_\infty}{(q,q^2)_\infty} \left\{\frac{J_{-n-\frac{1}{2}}^{(1)}(x;q^2)}{(2x)^{n+\frac{1}{2}}}\right\} \tag{4.12}$$

Formulas (4.11) and (4.12) reduce, when $q=1$, to similar formulas for the ordinary sine and cosine functions in terms of the Bessel function [26, p. 364], [16].

5. A q-ANALOG OF BILODEAU'S FORMULA

Another interesting application of the δ_q-operator is to find a q-analog of the following formula for the ultraspherica polynomials due to Bilodeau [8]

$$(5.1) \qquad P_n^\lambda(x)=\frac{(-1)^{n+1}x^{(n+3)}\sqrt{\pi}}{2^{\lambda-\frac{1}{2}}(n+1)!\Gamma(\lambda)}\left(\frac{1}{x}D_x\right)^{\lambda+n+\frac{1}{2}}[(1-x)^{n+1}x^{\lambda+\frac{1}{2}n-1}]$$

valid for $\lambda>-1/2$, $\lambda\neq 0$ and where D_x^λ is the fractional differentiation operator which, in this case, may be defined as

$$D_x^\lambda x^\beta=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\lambda+1)}x^{\beta-\lambda}, \quad \left(\frac{1}{x}\frac{d}{dx}\right)^\lambda=2^\lambda, \quad \frac{d}{du}\, u=x^2.$$

We show in this section that a q-analog of this formula exists. It is

$$(5.2) \qquad \begin{cases} P_n^{(\lambda)}(x;q)=(-1)^{n+1}\dfrac{(q^{2\lambda};q^2)_\infty q^{\frac{1}{2}n(n+2\lambda)+1}}{(q;q^2)_\infty(q;q)_{n+1}}x^{n+3}\delta_q^{n+\lambda+\frac{1}{2}} \\ \qquad [x^{n+2\lambda-2}(xq^{-n-\lambda+\frac{1}{2}};q)_{n+1}] \end{cases}$$

where $P_n^{(\lambda)}(x;q)$ is a q-analog of the ultraspherical polynomials due to Andrews and Askey [7].

To prove the assertion stated after formula (5.2) we consider

$$x^{n+3}\delta_q^{n+\lambda+\frac{1}{2}}[x^{n+2\lambda-2}(xq^{-n-\lambda+\frac{1}{2}};q)_{n+1}]$$

which appears on the right hand side of (5.2). Expand $(xq^{-n-\lambda+\frac{1}{2}};q)_{n+1}$ using Euler's formula

$$(a;q)_n=\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}q^{\frac{1}{2}k(k-1)}a^k,$$

then operate term by term by $\delta_q^{n+\lambda-\frac{1}{2}}$ and use (3.5) we get that the above expression is

$$=\sum_{k=0}^{n+1}(-1)^k\begin{bmatrix} n+1 \\ k \end{bmatrix}q^{\frac{1}{2}k(k-2n-2\lambda)}\frac{(q^{-1-n+k};q^2)_\infty}{(q^{k+n+2\lambda};q^2)_\infty}x^k$$

$$=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}q^{\frac{1}{2}(n-k)(-k-n-2\lambda)}\frac{(q^{-1-k};q^2)_\infty}{(q^{2n+2\lambda-k};q^2)_\infty}x^{n-k}.$$

The infinite product in the numerator vanishes unless k is even. After some simplifications and reductions we can now show that

$$(5.3) \qquad P_n^{(\lambda)}(x;q)=\frac{(q^{2\lambda};q^2)_n}{(q;q)_n}x^n{}_2\phi_1\left[\begin{matrix} q^{-n},q^{-n+1};q^2,1/x^2 \\ q^{2-2n-2\lambda} \end{matrix}\right].$$

This polynomial satisfy the three term recurrence relation

$$(5.4)\quad \begin{cases} (1-q^{n+1})P_{n+1}^{(\lambda)}(x;q) = (1-q^{2\lambda+2n})xP_n^{(\lambda)}(x;q) - \\ \qquad\qquad -(1-q^{2\lambda+n-1})q^{2\lambda+n-2}P_{n-1}^{(\lambda)}(x;q) \\ \qquad\qquad P_0^{(\lambda)}(x;q)=1,\ P_1^{(\lambda)}(x;q) = \dfrac{1-q^{2\lambda}}{1-q}\,x. \end{cases}$$

This can be verified by equating coefficients of x^m in (5.4).

It is now easy to see that these polynomial set is a special case of the big q-Jacobi polynomials of Andrews and Askey [7]. Indeed we have

$$(5.5)\qquad P_n^{(\lambda)}(x;q) = \frac{q^{-n}(q^{2\lambda};q)_n}{(q;q)_n}\,{}_3\phi_2\left[\begin{matrix} q^{-n}, q^{n+2\lambda}, qx; q, q \\ q^{\lambda+\frac12}, -q^{\lambda+\frac12} \end{matrix}\right]$$

which also satisfy (5.4) and the initial values.

I am indebted to a referee for pointing out that they can also be given in terms of the little q-Jacobi polynomials (6.4). In fact we have

$$P_{2n}^{\lambda}(x;q) = \frac{(q^{2\lambda};q^2)_n q^{n(n+2\lambda-2)}}{(q;q^2)_n}\,p_n(x^2q^{3-2\lambda}, \lambda-\tfrac12, -\tfrac12; q^2)$$

$$P_{2n+1}^{\lambda}(x;q) = \frac{(q^{2\lambda};q^2)_{n+1} q^{n(n+2\lambda-2)}}{(q;q^2)_{n+1}}\,xp_n(x^2q^{3-2\lambda}, \lambda-\tfrac12, \tfrac12; q^2).$$

For orthogonal polynomials that are built in a similar fashion from two other sets of orthogonal polynomials see [11, pp. 40–43].

6. THE q-JACOBI POLYNOMIALS

We get now a q-analog of Koornwinder's formula (1.10). We start from our formula (1.20) which is a q-analog of (1.8). We define

$$(6.1)\qquad \Lambda_q(c) = \left\{q^{2c+1}D_q^2 + \frac{1-q^{2c+1}}{x}\,D_q\right\} = \frac{1}{x}\,(1-q^{2c+1}\eta)D_q.$$

So that

$$(6.2)\qquad \Lambda_q(c)\{x^n\} = (1-q^n)(1-q^{2c+n})x^{n-2}.$$

To realize the relationship between Λ_q and B_q operators we note that, in particular, $T_q\eta = \eta^2$. We also have $T_xD_{q^2} = 1/xD_qT_x$ and $T_x\eta^2 = \eta T_x$.

Thus

$$\Lambda_q(c)T_x = \frac{1}{x}\,(1-q^{2c+1}\eta)D_qT_x = (1-q^{2c+2}\eta)\,\frac{1}{x}\,D_qT_x$$
$$= T_x(1-q^{2c+2}\eta^2)D_{q^2} = T_xT_qB_q(c).$$

Using this relation we get

$$(6.3)\qquad \Lambda_q^n(c)T_x = T_x \prod_{k=1}^{n} (1 - q^{2c+2k}\eta^2)D_{q^2}^n.$$

We next recall the little q-Jacobi polynomials [17]

$$(6.4)\qquad p_n(x, \alpha, \beta; q) = \frac{(-1)^n(q^{\beta+1}; q)_n}{(q; q)_n}\, {}_2\Phi_1\left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}; q, x \\ q^{\beta+1} \end{matrix}\right].$$

This is a q-analog of the Jacobi polynomial $P_n^{(\alpha,\beta)}(2x-1)$.

We now give the following formulas for the little q-Jacobi polynomials.

$$(6.5)\qquad \begin{cases} \Lambda_q(\beta)p_n(x^2, \alpha, \beta; q^2) = \\ = q^{-2n}(1-q^{2n+2\beta})(1-q^{2n+2\beta+2\alpha+2})p_{n-1}(x^2, \alpha+2, \beta; q^2) \end{cases}$$

$$(6.6)\qquad \begin{cases} \Lambda_q(\beta)\eta^{-2}\left(\dfrac{(x^2; q^2)_\infty}{(x^2q^{2\alpha+4}; q^2)_\infty}\, p_{n-1}(x^2, \alpha+2, \beta; q^2)\right) \\ = q^{-2n-2}(1-q^{2n})(1-q^{2n+2\alpha+2})\, \dfrac{(x^2; q^2)_\infty}{(x^2q^{2\alpha}; q^2)_\infty}\, p_n(x^2, \alpha, \beta; q^2) \end{cases}$$

Iterating (6.6) we get

$$(6.7)\qquad \begin{cases} q^{n(n-1)}(q^2; q^2)_n(q^{2n+2\alpha+2}; q^2)_n\, \dfrac{(x^2; q^2)_\infty}{(q^{2\alpha}x^2; q^2)_\infty}\, p_n(x^2, \alpha, \beta, q^2) \\ = \eta^{-2n}\Lambda_q^n(\beta)\left\{\dfrac{(x^2; q^2)_\infty}{(q^{2\alpha+4n}x^2; q^2)_\infty}\right\}. \end{cases}$$

Formula (6.7) is a q-analog of (1.10). Formulas (6.5)–(6.6) are q-analogs of the formulas [21, (2.4), (2.5)] that Koornwinder also gave.

Next we note that since

$$(1+x^2)^n P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n, -n-\beta; -x^2 \\ \alpha+1 \end{matrix}\right],$$

we may take for its q-analog the polynomial

$$(6.8)\qquad \begin{cases} f_n(x; \alpha, \beta, q) = \dfrac{(q^{2\alpha+2}; q^2)_\infty}{(q^2; q^2)_n(q^{2\alpha+2n+2}; q^2)_\infty}\, {}_2\Phi_1\left[\begin{matrix} q^{-2n}, q^{-2n-2\beta}; q^2, -x^2 \\ q^{2\alpha+2} \end{matrix}\right] \\ \qquad = \dfrac{(-1)^n}{(q^{2\alpha+2n+2}; q^2)_\infty}\, p_n(-x^2, -2n-\alpha-\beta-1, \alpha; q^2). \end{cases}$$

From this we immediately obtain

$$(6.9)\quad \begin{cases} \Lambda_q(\alpha) f_n(x; \alpha, \beta, q) = \\ = -q^{-4n-2\beta}(1-q^{2n+2\alpha})(1-q^{2n+2\beta}) f_{n-1}(x; \alpha, \beta, q), \end{cases}$$

which is a q-analog of [21, (2.10)]. To get a q-analog of

$$(6.10)\quad (1+x^2)^{-n-\alpha-\beta-1} P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right)$$

we first transform the ${}_2\phi_1$ in (6.8) by means of the Heine's transformation

$${}_2\phi_1\left[\begin{matrix} a, b; q, z \\ c \end{matrix}\right] = \frac{(abz/c; q)_\infty}{(z; q)_\infty}\, {}_2\phi_1\left[\begin{matrix} c/a, c/b; q, abz/c \\ c \end{matrix}\right].$$

We thus get a q-analog of the function (6.10)

$$\frac{(-x^2; q^2)_\infty}{(-x^2 q^{-4n-2\alpha-2\beta-2}; q^2)_\infty} f_n(x; \alpha, \beta, q)$$
$$= \frac{(q^{2\alpha+2}\ q^2)_\infty}{(q^2; q^2)_n (q^{2\alpha+2n+2}; q^2)_\infty}$$
$${}_2\phi_1\left[\begin{matrix} q^{2\alpha+2n+2}, q^{2\alpha+2\beta+2n+2}; q^2, -x^2 q^{-4n-2\alpha-2\beta-2} \\ q^{2\alpha+2} \end{matrix}\right].$$

This formula now leads to

$$(6.11)\quad \begin{cases} (\Lambda_q(\alpha)\eta^{-2}) \dfrac{(-x^2; q^2)_\infty}{(-x^2 q^{-4n-2\alpha-2\beta+2}; q^2)_\infty} f_{n-1}(x; \alpha, \beta, q) \\ = -q^{-4n-2\alpha-2\beta-2}(1-q^{2n})(1-q^{2n+2\alpha+2\beta}) \\ \dfrac{(-x^2; q^2)_\infty}{(-x^2 q^{-4n-2\alpha-2\beta-2}; q^2)_\infty} f_n(x; \alpha, \beta, q). \end{cases}$$

which is a q-analog of [21, (2.11)].

Lastly if we iterate (6.11) we get the q-analog of (1.10b), namely,

$$(6.12)\quad \begin{cases} (\Lambda_q(\alpha)\eta^{-2})^n \dfrac{(-x^2; q^2)_\infty}{(-x^2 q^{-2-2\alpha-2\beta}; q^2)_\infty} \\ = (-1)^n (q^2; q^2)_n (q^{2\alpha+2\beta+2}; q^2)_n q^{-2n(n+\alpha+\beta+2)} \\ \dfrac{(-x^2; q^2)_\infty}{(-x^2 q^{-4n-2\alpha-2\beta-2}; q^2)_\infty} f_n(x; \alpha, \beta, q). \end{cases}$$

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