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Production matrices

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Abstract

We translate the concept of succession rule and the ECO method into matrix notation, introducing the concept of a *production matrix*. Among other things, we show that certain operations on production matrices correspond to well-known operations on the numerical sequences determined by them.

1. Introduction

Infinite matrices are the forerunner of many branches of classical mathematics (infinite quadratic forms, integral equations, summability, etc.) and modern operator theory. Moreover, the idea of translating a combinatorial theory into a theory of infinite matrices is nowadays a current trend in discrete mathematics. To confirm this statement, we cite Riordan arrays [13,22,24,25,27], recursive matrices [8], Aigner's admissible matrices [1,2]. In this paper, we propose yet another instance of this bent; namely, we propose a possible translation of the concept of succession rule, and hence of the ECO method, into matrix notation.

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The ECO method, introduced by Renzo Pinzani and his collaborators, is a constructive method to produce all the objects of a given class, according to the growth of a certain parameter (the *size*) of the objects. The roots of the ECO method can be traced back to the paper [11], where the authors study Baxter permutations: for the first time, a combinatorial construction is presented which can be described by means of a generating tree (see below), as it usually happens for every ECO construction. Basically, the idea is to perform local expansions on each object of size n, thus constructing a set of objects of the successive size. This construction should induce a partition of all the objects of any given size (that is, through the ECO construction, all the objects of a given size are produced exactly once from the objects of immediately lower size). If an ECO construction is sufficiently regular, then it is often possible to describe it using a *succession rule*, whose definition is given in Section 2. This concept has been first introduced by Julian West in [29,30], and only later it has been recognized as an extremely useful tool for the ECO method. Intimately related to the concept of succession rule is the notion of *generating tree*, which is the most common way of representing a succession rule. The main applications of the ECO method are: enumeration [6], random generation [7], and exhaustive generation [3,12] of various combinatorial structures. For all these topics we refer the reader to the rich survey [5].

A significant contribution to the study of succession rules from the point of view of generating functions has been given in [4]. The authors focus on the relationship between the form and the generating function of a succession rule, and then provide a classification of rules as *rational*, *algebraic*, or *transcendental*, according to their generating function type. More recently, some algebraic properties of succession rules have been determined in [14,15,20].

The main idea of our work is to define and study the properties of at least two kinds of matrices associated with a succession rule. The first one, called *production matrix*, is directly deduced from the succession rule, whereas the second one, the *ECO matrix*, is essentially the matrix describing the distribution of the labels within the generating tree of the rule. Whereas an ECO matrix has a deep internal structure, a production matrix can be quite freely chosen among the infinite matrices with nonnegative entries. Moreover, the knowledge of the production matrix related to a succession rule allows us to easily find the associated ECO matrix. For these reasons, our attention will focus mainly on production matrices and their properties.

We wish to point out that this is not the first attempt to define a matrix counterpart of the notion of succession rule. West [29,30] first perceived the idea of a production matrix (he speaks of *transfer matrix*), but never made use of it, nor gave a precise definition. The concept of ECO matrix was first introduced in [18] (under the name of *AGT matrix*), where some properties were also studied, mainly from the point of view of Riordan matrices. We intend to investigate the relationship between the Riordan theory and production matrices in a forthcoming publication.

The main goal of our approach is to provide a representation of succession rules that is more suitable for computations. In Section 3 we define some operations on production matrices in order to reproduce well-known operations on the numerical sequences they represent. This leads to the determination of the generating functions of such sequences, often more easily than it was previously done by other methods (see [4,5,15]). Throughout the whole paper, a huge amount of examples are described or just sketched. We believe that the possibility of dealing with so many concrete cases is a peculiarity of our matrix approach and a good reason to pursue this investigation.

We will use the following more or less standard notations for the generating functions of the Catalan, large Schröder, and small Schröder numbers:

$$C = \frac{1-\sqrt{1-4z}}{2z} = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \cdots,$$

Catalan numbers (A000108);

$$R = \frac{1-z-\sqrt{1-6z+z^2}}{2z} = 1 + 2z + 6z^2 + 22z^3 + 90z^4 + 394z^5 + \cdots,$$

large Schröder numbers (A006318);

$$S = \frac{1+z-\sqrt{1-6z+z^2}}{4z} = 1 + z + 3z^2 + 11z^3 + 45z^4 + 197z^5 + \cdots,$$

small Schröder numbers (A001003);

Throughout the paper the A$******$ number between parentheses following a sequence is the identification number of that sequence in [26]. Most of the matrices we are going to consider are infinite; their lines (rows and columns) will be indexed by nonnegative integers, and we will write "line 0" to mean the first line, "line 1" to mean the second line, and so on.

2. Basic definitions

A succession rule is a formal system consisting of an *axiom* (a), $a \in \mathbf{N}^+$, and a set of *productions*:

$$\big\{(k_t) \rightsquigarrow \big(e_1(k_t)\big)\big(e_2(k_t)\big)\cdots\big(e_{k_t}(k_t)\big)\colon\ t \in \mathbf{N}\big\},$$

where $e_i : \mathbf{N}^+ \to \mathbf{N}^+$, which explains how to derive the *successors* $(e_1(k)), (e_2(k)), \ldots, (e_k(k))$ of any given label (k), $k \in \mathbf{N}^+$. In general, for a succession rule Ω, we use the more compact notation:

$$\Omega : \begin{cases} (a) \\ (k) \rightsquigarrow \big(e_1(k)\big)\big(e_2(k)\big)\cdots\big(e_k(k)\big). \end{cases} \tag{1}$$

(a), (k), $(e_i(k))$, are called the *labels* of Ω (where $a, k, e_i(k)$ are positive integers). The rule Ω can be represented by means of a *generating tree*, that is a rooted tree whose vertices are the labels of Ω; (a) is the label of the root and each node labeled (k) has k sons labeled by $e_1(k), \ldots, e_k(k)$ respectively, according to the production of (k) in (1). A succession rule Ω defines a sequence of positive integers $(a_n)_{n\geqslant 0}$, a_n being the number of the nodes at level n in the generating tree determined by Ω. By convention the root is at level 0, so $a_0 = 1$. The function $f_\Omega(x) = \sum_{n\geqslant 0} a_n x^n$ is the *generating function* determined by Ω.

In this paper we propose a different approach for the study of succession rules, based on linear algebra tools.

Instead of representing succession rules by generating trees, we represent them by matrices $P = (p_{k,i})_{k,i\geqslant 0}$. Assume that the set of the labels of a succession rule is $\{(l_k)\}_k$, and in particular that l_0 is the label of the axiom. Then we define $p_{k,i}$ to be the number of labels l_i produced by label l_k. We call P the *production matrix* of the given succession rule. Observe that the first row of a production matrix gives precisely the production of the axiom.

The labels do not occur explicitly in this matrix representation of the succession rule. However, they are the row sums of the matrix. In particular, the label l_0 of the axiom is the first row sum of P.

Example. To the succession rule

$$\begin{cases} (2) \\ (2) \rightsquigarrow (3)^2 \\ (k) \rightsquigarrow (3)(4)\cdots(k)(k+1)^2 \end{cases} \tag{2}$$

there corresponds the production matrix

$$P = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & 0 & \dots \\ 0 & 1 & 1 & 2 & 0 & \dots \\ 0 & 1 & 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Writing the succession rule as

$$\begin{aligned} &(2) \rightsquigarrow (2)^0(3)^2, \\ &(3) \rightsquigarrow (2)^0(3)^1(4)^2, \\ &(4) \rightsquigarrow (2)^0(3)^1(4)^1(5)^2, \\ &\dots\dots\dots\dots\dots\dots\dots \end{aligned}$$

the matrix P is nothing but the matrix of the exponents (where an exponent is zero if and only if the label it refers to does not appear in the production).

In the generating tree (see Fig. 1) at level zero we have only one node with label l_0 $(= 2)$. This is represented by the row vector

$$r_0 = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots).$$

At the next levels of the generating tree, the distribution of the labels $l_1, l_2, \dots$ is given by the row vectors

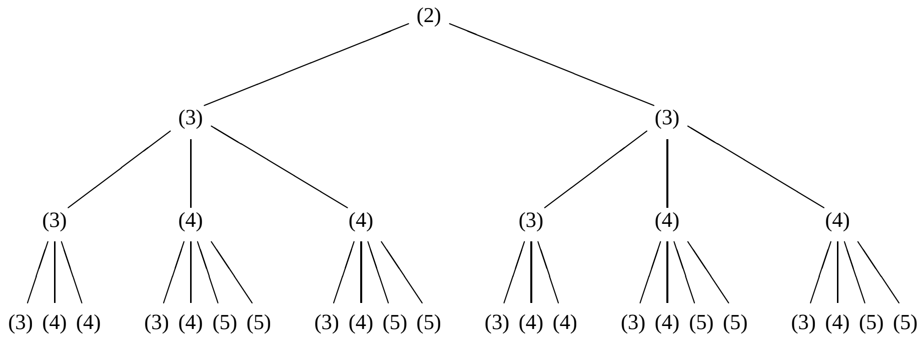

Fig. 1. The first levels of the generating tree associated with the succession rule in (2).

$$
\begin{aligned}
r_1 &= r_0P = (0 \quad 2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots),\\
r_2 &= r_1P = (0 \quad 2 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0 \quad \dots),\\
r_3 &= r_2P = (0 \quad 6 \quad 8 \quad 8 \quad 0 \quad 0 \quad 0 \quad \dots),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

Stacking these row matrices, we obtain the matrix

$$
A_P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \dots\\
0 & 2 & 0 & 0 & 0 & \dots\\
0 & 2 & 4 & 0 & 0 & \dots\\
0 & 6 & 8 & 8 & 0 & \dots\\
0 & 22 & 28 & 24 & 16 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

The row sums of the above matrix are $1, 2, 6, 22, 90, 394, 1806, \dots$, i.e. the large Schröder numbers. This is the sequence corresponding to the succession rule of our example. The enumerative properties of this succession rule have been examined in detail in [5]. We also recall that matrices like A_P (where the entry (n, k) gives the number of nodes labelled l_k at level n of the generating tree) were also studied in [18], where they have been called AGT matrices. In general, we will refer to A_P as the *ECO*[2] *matrix* induced by P.

Remarks. Let P be the production matrix of a given succession rule Ω. Throughout the whole paper we will denote by $u^\top$ the row vector $(1\ 0\ 0\ \dots)$ and by e the column vector $(1\ 1\ 1\ \dots)^\top$ of appropriate sizes. The following facts are easy to verify, so many of them will be stated without any further explanation.

[2] ECO stands for Enumeration of Combinatorial Objects (see [5]).

(i) The labels of the nodes of the corresponding generating tree are the row sums of P. If two row sums happen to be equal, then, as labels, they will be considered to be distinct. This can be achieved by using, for example, distinguishing subscripts; in the vocabulary of succession rules, these are called *colored succession rules* [14].

(ii) The distribution of the nodes having various labels at the various levels is given by the ECO matrix

$$A_P = \begin{pmatrix} u^\top \\ u^\top P \\ u^\top P^2 \\ \vdots \end{pmatrix}$$

(indeed, we have $r_0 = u^\top, r_1 = r_0 P = u^\top P, r_2 = r_1 P = u^\top P^2, \ldots$). The same fact can be expressed in a concise way by the matrix equality

$$DA_P = A_P P, \tag{4}$$

where $D = (\delta_{i,j+1})_{i,j\geqslant 0}$ (δ is the usual Kronecker delta). In some sources [19,24] the matrix P is also called the *Stieltjes transform matrix of* A_P.

(iii) The sequence $(a_n)_{n\geqslant 0}$ induced by the succession rule is given by $a_n = u^\top P^n e$.

(iv) The bivariate generating function of the matrix A_P is

$$G(t,z) = u^\top (I - zP)^{-1} \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \end{pmatrix}.$$

(v) The sequence corresponding to the succession rule has generating function

$$f_P(z) = u^\top (I - zP)^{-1} e.$$

The above expression for the generating function $f_P(z)$ can also be derived with a graph-theoretic argument. Consider the directed graph whose nodes are the labels of Ω and having P as its adjacency matrix. The paths in this graph that start at the root-vertex (corresponding to the first row sum of P) are basically the walks in the combinatorial interpretation of generating trees proposed in [4] (i.e. walks on the integer half-line starting at a fixed point and such that the only allowable transitions are those specified by the rule). Then, taking into account the well-known property of the adjacency matrix of a directed graph in connection with the number of paths of a given length between two vertices, it follows at once that $u^\top P^n e$ is the number of nodes at level n in the generating tree. Namely, the $(1, j)$-entry of P^n is the number of nodes l_j at level n. Since we are interested only in walks that start at the root, we retain only the first row of the matrix P^n for $n = 0, 1, 2, \ldots$. From these rows we have formed the matrix A_P.

(vi) The sequence corresponding to the succession rule has exponential generating function

$$F_P(z) = u^\top \exp(zP)e.$$

Example. We intend to find the sequence induced by the production matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & 0 & \dots \\ 0 & 0 & 2 & 3 & 0 & \dots \\ 0 & 0 & 0 & 3 & 4 & \dots \\ 0 & 0 & 0 & 0 & 4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Denoting by P_n the upper left n by n submatrix of P, it is not difficult to compute the exponential of the matrix zP_n, since it is an upper triangular matrix. The eigenvalues of P_n are easily seen to be the nonnegative numbers $0, 1, 2, 3, \dots, n-1$, each with multiplicity 1. Thus we have immediately $\exp(zP_n) = C\exp(zD_n)C^{-1}$, where D_n is the diagonalization of P_n, so that $\exp(zD_n) = (\delta_{i,j}e^{iz})_{0\leqslant i,j\leqslant n}$, and C is a suitable invertible matrix. More precisely, simple computations show that C is an upper triangular matrix in which the (i,j)-entry is the binomial coefficient $\binom{j}{i}$. This implies that also C^{-1} is upper triangular and its (i,j)-entry has the form $(-1)^{j+i}\binom{j}{i}$. Now the computation of the first row of $\exp(zP_n)$ is immediate, and we find for it

$$(1 \quad e^z - 1 \quad (e^z-1)^2 \quad (e^z-1)^3 \quad \dots \quad (e^z-1)^{n-1}).$$

Taking the sum of these entries and letting $n \to \infty$, for the exponential generating function induced by P we obtain $G_P(z) = 1/(2-e^z)$.

The corresponding sequence is $1, 1, 3, 13, 75, 541, 4683, 47293, \dots$ (A000670; ordered Bell numbers) and counts the number of ordered partitions of a set. The succession rule corresponding to P has the form

$$\begin{cases} (1) \\ (2k+1) \rightsquigarrow (2k+1)^k(2k+3)^{k+1}. \end{cases}$$

This rule suggests a simple ECO-construction for ordered set partitions. Take a positive integer n and consider an ordered partition of $[n] = \{1, 2, \dots, n\}$, say $\pi = (B_1, \dots, B_k)$ (the B_i's are the blocks of the ordered partition). We can construct $2k+1$ ordered partitions of $[n+1]$ starting from π in the following way:

- add $n+1$ to each of the k blocks of π, so obtaining k ordered partitions of $[n+1]$;
- insert the block $B = \{n+1\}$ either between B_i and B_{i+1}, for $i < k$, or before B_1 or after B_k, so obtaining $k+1$ ordered partitions of $[n+1]$.

For instance, starting from the ordered partition $(\{1,2\},\{5\},\{3,4\})$, we get the seven ordered partitions $(\{1,2,6\},\{5\},\{3,4\})$, $(\{1,2\},\{5,6\},\{3,4\})$, $(\{1,2\},\{5\},\{3,4,6\})$, $(\{6\},\{1,2\},\{5\},\{3,4\})$, $(\{1,2\},\{6\},\{5\},\{3,4\})$, $(\{1,2\},\{5\},\{6\},\{3,4\})$, $(\{1,2\},\{5\},\{3,4\},\{6\})$.

It is immediate to verify that, if we perform this construction on all the ordered partitions of $[n]$, we obtain all the ordered partitions of $[n+1]$ exactly once. This provides a combinatorial interpretation of the production matrix P. For the ECO method and the idea of an ECO-construction, we refer the reader to [5].

We close this section by recalling a concept first defined in [15] which is closely related to that of the production matrix. Given a succession rule Ω as in (1), the *rule operator* associated with Ω is the linear operator $L = L_\Omega$ defined on the vector space of polynomials in one variable with coefficients in a given field as follows:

$$\begin{aligned} L\colon \qquad \mathbf{1} &\longmapsto x^a, \\ x^k &\longmapsto x^{e_1(k)} + \cdots + x^{e_k(k)}, \\ x^h &\longmapsto hx^h, \quad \text{if the label } (h) \text{ does not appear in } \Omega \end{aligned}$$

(then extend by linearity). It is easy to see that the production matrix P_Ω of Ω is the matrix of L_Ω with respect to the basis $\{x^k\}_k$, where k runs over the set of labels of Ω. In other words, P_Ω is the matrix of the restriction of L_Ω to the subspace generated by those powers of x whose exponents are the labels of Ω. Therefore, the theory of production matrices is a sort of concrete counterpart of the theory of rule operators. The main advantage in using matrices lies in a better possibility for computations.

3. Operations on production matrices

In this section we will define some operations to be performed on production matrices in order to describe usual operations on numerical sequences. For instance, we will give an explicit expression of the production matrix of the sum of two sequences in terms of the production matrices of the starting sequences. The same thing will be done for many other operations. Many ideas developed in this section have been suggested by [14,20]; we provide a translation into the vocabulary of production matrices, as well as a probably more rigorous presentation, of the results proved in the above mentioned articles; we also propose some new ones.

In the sequel we will write $P \to a_0, a_1, a_2, \ldots$ to mean that $(a_n)_{n\geqslant 0}$ is the numerical sequence determined by the production matrix P. Likewise, expressions such as $P \to (a_n)_{n\geqslant 0}$, $P \to f_P(z)$, have similar meanings.

Proposition 3.1. *If $P \to a_0, a_1, a_2, \ldots,$ then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 0 & u^\top \\ 0 & P \end{pmatrix} \to 1, a_0, a_1, a_2, \ldots.$$

First proof. The powers of M can be immediately computed:

$$M^n = \begin{pmatrix} 0 & u^\top P^{n-1} \\ 0 & P^n \end{pmatrix}, \quad n > 0.$$

Hence, for the ECO matrix A_M induced by M we obtain

$$A_M = \begin{pmatrix} u^\top \\ u^\top M \\ u^\top M^2 \\ u^\top M^3 \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^\top \\ 0 & u^\top P \\ 0 & u^\top P^2 \\ \vdots & \cdots \end{pmatrix}.$$

From here it follows at once that the row sums of A_M are $1, a_0, a_1, a_2, \ldots$. □

Second proof. We have

$$(I - zM)^{-1} = \begin{pmatrix} 1 & zu^\top (I - zP)^{-1} \\ 0 & (I - zP)^{-1} \end{pmatrix}.$$

Now, $f_M(z) = 1 + zu^\top (I - zP)^{-1} e = 1 + z f_P(z)$. □

Proposition 3.2. *If $P \to f_P(z)$ and k is a positive integer, then $kP \to f_P(kz)$.*

Proof. Using generating functions, we have $f_{kP}(z) = u^\top (I - zkP)^{-1} e = f_P(kz)$. □

Proposition 3.3. *If $P \to (a_n)_{n\geqslant 0}$, then*

$$M \overset{\text{def}}{=} P + I \to \left(\sum_{k=0}^{n} \binom{n}{k} a_k \right)_{n\geqslant 0},$$

the binomial transform of $(a_n)_{n\geqslant 0}$.

Proof. The nth term of the sequence determined by M is $u^\top (P + I)^n e$. Expanding the binomial, we obtain

$$u^\top (P + I)^n e = \sum_{k=0}^{n} \binom{n}{k} u^\top P^k e = \sum_{k=0}^{n} \binom{n}{k} a_k. \qquad \square$$

Example. We intend to find the sequence induced by the production matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 0 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We remark that the succession rule corresponding to Q has been studied in [4,15]. Let us consider the production matrix

$$P = Q - I = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to find the matrix $(I - zP)^{-1}$. Its first row is $(0! \quad 1!z \quad 2!z^2 \quad 3!z^3 \quad \dots)$ and, consequently, the induced generating function is the formal power series $\sum_{n \geqslant 0} n!z^n$.

Clearly, the corresponding sequence $(n!)_{n \geqslant 0}$ has exponential generating function $1/(1-z)$. Applying the above proposition, we conclude that the exponential generating function induced by the production matrix $Q = P + I$ is $e^z/(1-z)$. Here we use the fact that, if the exponential generating function of a sequence is $F(z)$, then the exponential generating function of the binomial transform is $e^z F(z)$.

Therefore the nth term of the sequence determined by the production matrix Q is the total number of injections into an n-set, also called *arrangements*. The statistic determined by the matrix A_Q associated with Q gives the *falling factorials* $(n)_k = k!\binom{n}{k}$ (number of injections of a k-set into an n-set).

Proposition 3.4. *If* $P \to a_0, a_1, a_2, \dots,$ *then* $P^q \to a_0, a_q, a_{2q}, a_{3q}, \dots.$ *In particular,* $P^2 \to a_0, a_2, a_4, \dots.$

Proof. Let A and B be the matrices induced by P and P^q, respectively. Then the rows of B are $u^\top, u^\top P^q, u^\top P^{2q}, \dots,$ which are rows $0, q, 2q, \dots$ of the matrix A. Consequently, the row sums of B are $a_0, a_q, a_{2q}, a_{3q}, \dots.$ □

Example. We take the production matrix $P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which induces the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... (A000045). This example was first considered in [29,30]. Then $P^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ induces the odd-subscripted Fibonacci numbers 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, ... (A001519).

Proposition 3.5. *If* $P \to a_0, a_1, a_2, \dots,$ *then*

$$\begin{pmatrix} 0 & u^\top P \\ 0 & P^2 \end{pmatrix} \to 1, a_1, a_3, a_5, \dots.$$

Proof. This is a straightforward consequence of Propositions 3.1 and 3.4, since the block matrix in this proposition is the square of the block matrix of Proposition 3.1. □

Example. Starting again with the production matrix P of the Fibonacci sequence (see the previous example), we obtain that the production matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

induces the sequence 1, 1, 3, 8, 21, 55, 144, 377, 987, 2584, ... (A001906), i.e. the sequence of the even-subscripted Fibonacci numbers preceded by a 1.

Until now we have considered one numerical sequence and we have performed some "manipulations" on it, leading to another numerical sequence. For each of these "manipulations" we have determined the corresponding algebraic operation to be performed on the production matrix of the sequence. In the sequel we will deal with two numerical sequences, and we would like to describe what happens to production matrices when we consider usual algebraic operations on the sequences (like, e.g., sum, various products, and so on). To do so, we need to tackle a technical problem. If the production matrices of the sequences under consideration are both infinite, it could be meaningless to consider block matrices in which some of the blocks are the production matrices above. For example, if P and Q are infinite production matrices, then the expression

$$\begin{pmatrix} 0 & P \\ 0 & Q \end{pmatrix} \tag{5}$$

does not define a matrix (not even an infinite one), because of the presence of the infinite matrix P as a block in the upper part of the array. We will make up for this predicament by reshuffling the lines of the two production matrices. Observe that a sequence defined by a given production matrix P is determined up to a permutation of its rows, provided that

(i) the first row remains fixed,
(ii) every permutation of the rows is followed by the same permutation of the columns.

Indeed, the first row must not be moved, since its sum denotes the label of the axiom of the associated succession rule or, equivalently, the second term of the associated sequence (after the starting 1). Moreover, since the lines (rows and columns) of a production matrix are indexed by the labels of the associated succession rule, it is clear that we can list them in any order.

Therefore, every time we will be faced with an expression like (5), it will be tacitly understood that we consider the matrix obtained by suitably shuffling the lines of P and Q. More precisely, we define a matrix as in (5) to be the one obtained by alternating the lines of P and Q.

After these considerations we can start dealing with binary operations.

Proposition 3.6. *If* $P \to 1, a_1, a_2, \dots$ *and* $Q \to 1, b_1, b_2, \dots$ *then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 0 & u^\top P & u^\top Q \\ 0 & P & 0 \\ 0 & 0 & Q \end{pmatrix} \to 1, a_1 + b_1, a_2 + b_2, \dots. \tag{6}$$

Remark. Observe that, in general, the matrix in (6) makes sense only thanks to the above considerations. More precisely, M is constructed as follows. The first line of M starts with a zero, then the (possibly infinite) row vectors $u^\top P$ and $u^\top Q$ are shuffled. Next, the submatrix $\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)$ is constructed by stacking 2×2 blocks of the form $\left(\begin{smallmatrix} p_{ij} & 0 \\ 0 & q_{ij} \end{smallmatrix}\right)$, where p_{ij} and q_{ij} are the (i, j)-entries of the matrices P and Q, respectively.

First proof. Taking into account that $u^\top M^k = (0 \quad u^\top P^k \quad u^\top Q^k)$, for the matrix A_M induced by M we obtain

$$A_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u^\top P & u^\top Q \\ 0 & u^\top P^2 & u^\top Q^2 \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

From here it follows at once that the row sums of A_M are $1, a_1 + b_1, a_2 + b_2, \dots$. □

Second proof. We have

$$(I - zM)^{-1} = \begin{pmatrix} 1 & u^\top[(I - zP)^{-1} - I] & u^\top[(I - zQ)^{-1} - I] \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix},$$

where the entries not shown are irrelevant. Now, taking the first row sum of this matrix, we obtain at once that

$$f_M(z) = u^\top (I - zP)^{-1} e + u^\top (I - zQ)^{-1} e - 1 = f_P(z) + f_Q(z) - 1. \qquad \square$$

Example. Consider the production matrices

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

from the previous two examples, inducing the sequences 1, 2, 5, 13, 34, 89, 233, 610, 1597, ... of odd-subscripted Fibonacci numbers and 1, 1, 3, 8, 21, 55, 144, 377, 987, ... of even-subscripted Fibonacci numbers preceded by a 1. Observe that, in this particular case, both P and Q are finite matrices. So it is not necessary to shuffle the lines of P and Q, and the block matrix M in (6) can be interpreted as usual, without further assumptions. Therefore we have immediately that the production matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

induces the sequence $1, 3, 8, 21, 55, 144, \ldots$ of the even-subscripted Fibonacci numbers (that is, almost the same as the sequence induced by Q). The row sums of this matrix are 3, 2, 3, 1, 2, 3 and, therefore, the labels are $(3), (2), (\bar{3}), (1), (\bar{2}), (\bar{\bar{3}})$. Note that nodes of label 1 are not produced at all (the corresponding column, i.e. column 3, contains only 0's) and, consequently, we can delete row 3 and column 3 of the above matrix.

In this very simple case, however, we can easily show that the succession rule associated with the production matrix is unnecessarily complicated. In fact, the form of the rule is the following:

$$\begin{cases} (3) \\ (2) \rightsquigarrow (2)(\bar{3}), \qquad (\bar{2}) \rightsquigarrow (\bar{2})(\bar{\bar{3}}), \\ (3) \rightsquigarrow (2)(\bar{3})(\bar{\bar{3}}), \quad (\bar{3}) \rightsquigarrow (2)(\bar{3})(\bar{3}), \quad (\bar{\bar{3}}) \rightsquigarrow (2)(\bar{\bar{3}})(\bar{\bar{3}}). \end{cases} \tag{7}$$

Therefore, it is clear that we can avoid the use of colors, thus obtaining the rule

$$\begin{cases} (3) \\ (2) \rightsquigarrow (2)(3), \quad (3) \rightsquigarrow (2)(3)(3), \end{cases}$$

corresponding to the production matrix $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$.

Proposition 3.7. *If $P \to a_0, a_1, a_2, \ldots$ and $Q \to b_0, b_1, b_2, \ldots$ then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} P & eu^\top Q \\ 0 & Q \end{pmatrix} \to c_0, c_1, c_2, \ldots,$$

where $(c_n)_{n\geqslant 0}$ is the convolution of the sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$.

Proof. Set $(I - zM)^{-1} = \left(\begin{smallmatrix} X & Y \\ \star & \star \end{smallmatrix}\right)$. From the simple equality $(I - zM)^{-1}(I - zM) = I$, it follows that $X(I - zP) = I$, $-zXeu^\top Q + Y(I - zQ) = 0$. Solving these equations, we obtain $X = (I - zP)^{-1}$, and

$$\begin{aligned} Y &= z(I - zP)^{-1}eu^\top Q(I - zQ)^{-1} = (I - zP)^{-1}eu^\top(zQ - I + I)(I - zQ)^{-1} \\ &= (I - zP)^{-1}eu^\top\big[(I - zQ)^{-1} - I\big]. \end{aligned}$$

Now,

$$\begin{aligned}
f_M(z) &= u^\top (I - zM)^{-1} e \\
&= (\,u^\top \quad 0\,) \begin{pmatrix} (I - zP)^{-1} & (I - zP)^{-1} e u^\top [(I - zQ)^{-1} - I] \\ \star & \star \end{pmatrix} \begin{pmatrix} e \\ e \end{pmatrix} \\
&= u^\top (I - zP)^{-1} e u^\top (I - zQ)^{-1} e = f_P(z) f_Q(z),
\end{aligned}$$

the generating function of the convolution of $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$. □

Example. Taking for both P and Q the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, corresponding to the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., we obtain the production matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

that induces the convolution of the Fibonacci sequence with itself, i.e. 1, 2, 5, 10, 20, 38, 71, 130, 235, 420, ... (A001629).

Proposition 3.8. *If* $P \to a_0, a_1, a_2, \ldots,$ *then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 1 & u^\top P \\ 0 & P \end{pmatrix} \to a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots,$$

the sequence of the partial sums of $(a_n)_{n\geqslant 0}$.

Proof. This statement can be viewed as a corollary of Proposition 3.7. Indeed, the sequence of the partial sums of a sequence a_n is the convolution of that sequence with the sequence $(1, 1, 1, \ldots)$, the latter having (1) as its production matrix. □

Example. We take the production matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which induces the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... (A000045). Then

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

induces the sequence of partial sums 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, ... (A000071). Note that these are the Fibonacci numbers minus 1, as it is well known.

Proposition 3.9. *If* $P \to a_0, a_1, a_2, \ldots$ *and* $Q \to b_0, b_1, b_2, \ldots,$ *then* $P \otimes Q \to a_0 b_0, a_1 b_1, a_2 b_2, \ldots,$ *where* $\otimes$ *denotes Kronecker product.*

Proof. Once again, observe that the Kronecker product is well-defined only if at least Q is finite. Otherwise we have to reshuffle the lines of M as we have already done for sum and convolution. In terms of rule operators, if L and N are the rule operators associated with

P and Q, respectively, then M is the matrix of the Kronecker product $L \otimes N$ defined on the tensor product $K[x] \otimes K[x]$ as follows:

$$\begin{aligned} L \otimes N : K[x] \otimes K[x] &\longrightarrow K[x] \\ x^h \otimes x^k &\longmapsto L(x^h)N(x^k). \end{aligned}$$

For the proof we recall some simple properties of the Kronecker product, namely that $(U \otimes V)^n = U^n \otimes V^n$ and that the first row sum of a Kronecker product $U \otimes V$ is the product of the first row sum of U and the first row sum of V (both these facts are easy to show, and have a counterpart in terms of linear operators). Now, if $(c_n)_{n \geqslant 0}$ is the sequence induced by $P \otimes Q$, then

$$c_n = u^\top (P \otimes Q)^n e = u^\top (P^n \otimes Q^n) e = \bigl(u^\top P^n e\bigr)\bigl(u^\top Q^n e\bigr) = a_n b_n. \qquad \square$$

Example. Taking for both P and Q the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, corresponding to the Fibonacci sequence, we obtain the production matrix

$$P \otimes P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

of the sequence 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, ... (A007598) of the squared Fibonacci numbers. However, we can perform a sort of "contraction" on this 4×4 matrix to obtain an equivalent 3×3 one (to mean that, as production matrices, they induce the same sequence). More precisely, we can proceed as follows. Given any production matrix P, consider two rows having equal sum. In terms of succession rules, this means that we have two different labels denoted by the same positive integer but having different productions, say (k) and $(\bar{k})$ (in other words, we are dealing with a colored rule; see [14] for a detailed description of colored rules). Next sum up the two columns of P corresponding to (k) and $(\bar{k})$. In this way we obtain a matrix which describes a rule identical to the previous one, except that now (k) and $(\bar{k})$ become indistinguishable when they appear in the production of any other label. Now, if it happens that, in the modified matrix, the rows corresponding to (k) and $(\bar{k})$ are identical, it means that they are indistinguishable even as "fathers," so it is possible to delete one of the two rows from the matrix. In the present example, rows 1 and 2 have equal sum, so we sum up columns 1 and 2, obtaining the matrix

$$P' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Now, since rows 1 and 2 of P' happen to be identical, we can delete one of them. This leads us to the matrix

$$P'' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

which is smaller than but equivalent to $P \otimes P$.

Theorem 3.1. *If $P \to f_P(z)$ and k is an integer, then*

$$M = P + keu^\top \to \frac{f_P(z)}{1 - kzf_P(z)}.$$

Remark. Here P is modified by adding k to each entry in the first column; it is assumed that in the case that k is negative, $P + keu^\top$ is still a nonnegative matrix.

Proof. Denoting $X = (I - zP)^{-1}$, $Y = (I - zP - kzeu^\top)^{-1}$, we have $f_P(z) = u^\top Xe$, $f_M(z) = u^\top Ye$. It is easy to see that $X(I - zP) = I$, $I + kzeu^\top Y = (I - zP)Y$. Now, we have successively

$$\begin{aligned} kzf_P(z)f_M(z) &= kzu^\top Xeu^\top Ye = u^\top X\big[(I - zP)Y - I\big]e \\ &= u^\top X(I - zP)Ye - u^\top Xe = u^\top Ye - u^\top Xe, \end{aligned}$$

i.e. $kzf_P(z)f_M(z) = f_M(z) - f_P(z)$, which is equivalent to the assertion of the theorem. □

Remark. Using the previous theorem and making use of the identities

$$R = \frac{S}{1 - zS}, \qquad \frac{S - 1}{z} = \frac{R}{1 - zR}, \qquad \frac{C - 1}{z} = \frac{C}{1 - zC},$$

one can easily derive production matrices inducing the left-hand sides of these relations from production matrices for S, R, and C, respectively.

Proposition 3.10. *If $P \to f_P(z)$ then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} 0 & u^\top \\ 0 & P + eu^\top \end{pmatrix} \to \frac{1}{1 - zf_P}.$$

Proof. This follows at once from the previous theorem (for $k = 1$) and Proposition 3.1. Indeed, the previous theorem gives a production matrix inducing $f_P/(1 - zf_P)$ and then Proposition 3.1 gives the production matrix M, inducing $1 + zf_P/(1 - zf_P)$. □

Example. We take the production matrix $P = (1)$ which induces the sequence $1, 1, 1, \dots$ having generating function $1/(1-z)$. Applying Proposition 3.10 to P, we find that

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \to \frac{1}{1 - \frac{z}{1-z}}.$$

By iteration we obtain that

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \frac{1}{1 - \frac{z}{1-\frac{z}{1-z}}}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \frac{1}{1 - \frac{z}{1-\frac{z}{1-\frac{z}{1-z}}}},$$

and so on. In the limit,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \to C(z) \stackrel{\text{def}}{=} \frac{1 - \sqrt{1-4z}}{2z},$$

the generating function of the Catalan numbers 1, 1, 2, 5, 14, 42, 132, 429, ... (A000108). The generating functions induced by the finite production matrices of order $h \geqslant 2$ (displayed above for $h \leqslant 4$) count the number of ordered trees of height at most h (see, for example, [10,23]), or of Dyck paths of height at most h (see [21]). For an interpretation of the Catalan function $C(z)$ as a continued fraction, see [16].

The next result uses the techniques developed throughout the paper to provide a new class of operations on production matrices (and so also on succession rules).

Theorem 3.2. *Let b, c, and r be nonnegative integers. If $P \to f_P(z)$ then*

$$M \stackrel{\text{def}}{=} \begin{pmatrix} b & ru^\top \\ ce & P \end{pmatrix} \to \frac{1 + rzf_P(z)}{1 - bz - rcz^2 f_P(z)}.$$

Proof. Let $(I - zM)^{-1} = \left(\begin{smallmatrix} \alpha & y^\top \\ \star & \star \end{smallmatrix}\right)$, where the entries not shown are irrelevant. By considering the trivial equality $(I - zM)^{-1}(I - zM) = I$, we obtain:

$$\alpha(1 - bz) - czy^\top e = 1, \qquad -\alpha rzu^\top + y^\top (I - zP) = 0,$$

from where

$$\alpha = \frac{1}{1 - bz - rcz^2 f_P(z)}, \qquad y^\top = \frac{rz}{1 - bz - rcz^2 f_P(z)} u^\top (I - zP)^{-1}.$$

Now,

$$f_M(z) = \alpha + y^\top e = \frac{1 + rzf_P(z)}{1 - bz - rcz^2 f_P(z)}. \qquad \square$$

The above theorem has numerous applications.

Example. Consider the production matrix

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & \dots \\ 2 & 1 & 1 & 1 & \dots \\ 2 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It can be written as $M = \left(\begin{smallmatrix} 2 & u^\top \\ 2e & P \end{smallmatrix}\right)$, where

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

However, it is known that P induces the generating function $(C(z) - 1)/z$ of the Catalan numbers 1, 2, 5, 14, 42, ..., where $C(z) = (1 - \sqrt{1-4z})/(2z)$. Now, taking $b = 2$, $c = 2$, $r = 1$, and $f_P(z) = (C(z) - 1)/z$ in Theorem 3.2, after some elementary computations we obtain

$$f_M(z) = \frac{1 - \sqrt{1-4z}}{2z\sqrt{1-4z}},$$

the generating function of the sequence $\binom{2n-1}{n}$ of half the central binomial coefficients.

Example. We consider again the production matrix Q of the example after Proposition 3.3. We can write

$$Q = \begin{pmatrix} 1 & u^\top \\ 0 & Q_1 \end{pmatrix}, \quad \text{with}$$

$$Q_1 = \begin{pmatrix} 1 & 2u^\top \\ 0 & Q_2 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 1 & 3u^\top \\ 0 & Q_3 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 1 & 4u^\top \\ 0 & Q_4 \end{pmatrix}, \quad \dots.$$

Applying Theorem 3.2 to these matrices, we obtain

$$f_Q = \frac{1 + zf_{Q_1}}{1-z}, \qquad f_{Q_1} = \frac{1 + 2zf_{Q_2}}{1-z}, \qquad f_{Q_2} = \frac{1 + 3zf_{Q_3}}{1-z}, \quad \dots.$$

From here we obtain easily

$$f_Q(z) = \frac{1}{1-z} + \frac{1!z}{(1-z)^2} + \frac{2!z^2}{(1-z)^3} + \cdots.$$

It follows that the sequence induced by Q is the binomial transform of the sequence $1!, 2!, 3!, \ldots,$ i.e. the sequence of the arrangements (A000522).

Remark. It can be interesting to rephrase Theorem 3.2 in terms of succession rules. Consider the rule

$$\Omega : \begin{cases} \big(k(0)\big) \\ (k(n)) \rightsquigarrow \big(k(i_1)\big)^{j_1}\big(k(i_2)\big)^{j_2}\cdots\big(k(i_m)\big)^{j_m}, \end{cases} \tag{8}$$

having $f_\Omega(z)$ as its generating function. Then the succession rule

$$\Theta : \begin{cases} (b+r) \\ (b+r) \rightsquigarrow (b+r)^b\big(c+k(0)\big)^r \\ \big(c+k(n)\big) \rightsquigarrow (b+r)^c\big(c+k(i_1)\big)^{j_1}\cdots\big(c+k(i_m)\big)^{j_m} \end{cases} \tag{9}$$

has

$$f_\Theta(z) = \frac{1+rzf_\Omega(z)}{1-bz-rcz^2 f_\Omega(z)}$$

as its generating function.

Theorem 3.2 has two important immediate corollaries.

Corollary 3.1. *Let P be an infinite production matrix of the form $P = \begin{pmatrix} b & ru^\top \\ ce & P \end{pmatrix}$, where b, c, r are nonnegative integers. Then the induced generating function $f_P(z)$ satisfies the quadratic equation $rcz^2 f_P^2 - (1-bz-rz)f_P + 1 = 0$.*

Examples.

$b=0,\ c=1,\ r=1$ yields $1, 1, 2, 4, 9, 21, \ldots$ (the Motzkin numbers; A001006);

$b=1,\ c=1,\ r=1$ yields $1, 2, 5, 14, 42, 132, \ldots$ (the Catalan numbers; A000108);

$b=3,\ c=3,\ r=1$ yields $1, 4, 19, 100, 562, 3304, \ldots$ (see [9]; A007564);

$b=4,\ c=4,\ r=1$ yields $1, 5, 29, 185, 1257, 8925, \ldots$ (number of Dyck-like paths, using steps of the form (k, k) and $(k, -k)$, for any positive integers k; A059231).

Corollary 3.2. *If the generating function $f(z)$ of a sequence satisfies an equation of the form*

$$rcz^2f^2 - (1 - bz - rz)f + 1 = 0 \tag{10}$$

with b, c and r nonnegative integers then

$$P = \begin{pmatrix} b & ru^\top \\ ce & P \end{pmatrix} \tag{11}$$

defines recursively a production matrix P inducing $f(z)$.

Example. The sequence 1, 3, 10, 36, 137, 543, ... (A002212) that counts restricted hexagonal polyominoes [17] satisfies the equation $f = 1 + 3zf + z^2f^2$. This agrees with (10) when $rc = 1$ and $b + r = 3$, which has a solution $r = c = 1$, $b = 2$. Consequently, a production matrix inducing f is

$$P = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 1 & 1 & 2 & 1 & \dots \\ 1 & 1 & 1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example. The generating function $f(z)$ of the sequence 1, 3, 11, 45, 197, ... (A001003) of the little Schröder numbers satisfies the equation $f = 1+3zf +2z^2f^2$. This agrees with (10) when $rc = 2$ and $b + r = 3$, which has two solutions: $b = 1$, $c = 1$, $r = 2$ and $b = 2$, $c = 2$, $r = 1$. Consequently, we obtain two production matrices

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots \\ 1 & 1 & 2 & 0 & \dots \\ 1 & 1 & 1 & 2 & \dots \\ 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & \dots \\ 2 & 2 & 2 & 1 & \dots \\ 2 & 2 & 2 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollaries 3.1 and 3.2 have higher-order analogues. Here we give only one counterpart of Corollary 3.1.

Corollary 3.3. *Let P be an infinite production matrix of the form*

$$P = \begin{pmatrix} b_0 & r_1 & 0 & 0 \\ c_1 & b_1 & r_2 & 0 \\ c_1 & c_2 & b_2 & r_3u^\top \\ c_1e & c_2e & c_3e & P \end{pmatrix},$$

where $b_0, b_1, b_2, c_1, c_2, c_3, r_1, r_2, r_3$ are nonnegative integers. Then the induced generating function $f_P(z)$ is obtained by eliminating g and h from the following three relations:

$$f = \frac{1 + r_1 z g}{1 - b_0 z - r_1 c_1 z^2 g}, \qquad g = \frac{1 + r_2 z h}{1 - b_1 z - r_2 c_2 z^2 h}, \qquad h = \frac{1 + r_3 z f}{1 - b_2 z - r_3 c_3 z^2 f}.$$

Proof. We can write

$$P = \begin{pmatrix} b_0 & r_1 u^\top \\ c_1 e & Q \end{pmatrix}, \quad \text{with } Q = \begin{pmatrix} b_1 & r_2 u^\top \\ c_2 e & R \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} b_2 & r_3 u^\top \\ c_3 e & P \end{pmatrix},$$

and now Theorem 3.2 yields the three equalities of the corollary. □

Example. We intend to find the generating function induced by the production matrix

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can write

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & u^\top \\ e & 0 & 0 & P \end{pmatrix}$$

and now we have to eliminate g and h from the relations

$$f = \frac{1 + zg}{1 - z - z^2 g}, \qquad g = 1 + zh, \qquad h = 1 + zf,$$

leading to $z^4 f^2 - (1 - 2z)(1 + z + z^2) f + 1 + z + z^2 = 0$.

The function f obtained from here is the generating function of the sequence 1, 2, 4, 8, 17, 37, 82, 185, 423, … (A004148) enumerating secondary structures of RNA molecules [28].

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