Finite Eulerian posets which are binomial, Sheffer or triangular

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Abstract

In this paper we study finite Eulerian posets which are binomial, Sheffer or triangular. These important classes of posets are related to the theory of generating functions and to geometry. The results of this paper are organized as follows:

- We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets;
- We give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases;
- In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets.

We also study Eulerian triangular posets. This paper answers questions asked by R. Ehrenborg and M. Readdy. This research is also motivated by the work of R. Stanley about recognizing the *boolean lattice* by looking at smaller intervals.

Key words: Eulerian poset, binomial poset, Sheffer poset, triangular poset

1. Introduction

The theory of binomial posets was developed in [3] by Doubilet, Rota and Stanley to formalize certain aspects of the theory of generating functions. Binomial posets can be used to unify various aspects of enumerative combinatorics and generating functions. These posets are highly regular posets since the essential requirement is that every two intervals of the same length have the same number of maximal chains. Ehrenborg and Readdy in [5] and independently Reiner in [9] generalized the notion of a binomial poset to a larger class of posets, which we call Sheffer posets.

Ehrenborg and Readdy [4] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets, where infinite posets are posets which contain an infinite chain. They introduced the open question of characterizing the finite case. This paper deals with these questions.

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A *triangular poset* is a graded poset such that the number of maximal chains in each interval $[x, y]$ depends only on $\rho(x)$ and $\rho(y)$, where $\rho(x)$ and $\rho(y)$ are ranks of the elements x and y, respectively. Here we define Sheffer posets which are special class of triangular posets. A *Sheffer poset* is a graded poset such that the number of maximal chains $D(n)$ in an n-interval $[\hat{0}, y]$ depends only on $\rho(y)$, the rank of the element y, and the number $B(n)$ of maximal chains in an n-interval $[x, y]$, where $x \neq \hat{0}$, depends only on $\rho(x, y) = \rho(y) - \rho(x)$. Two factorial functions $B(n)$ and $D(n)$ are called *binomial factorial functions* and *Sheffer factorial functions*, respectively. A *binomial poset* is a graded poset such that the number of maximal chains $B(n)$ in an n-interval $[x, y]$ depends only on $\rho(x, y) = \rho(y) - \rho(x)$.

A graded poset P is *Eulerian* if every non-singleton interval of P satisfies the *Euler-Poincaré* relation: the number of elements of even rank is equal to the number of elements of odd rank in that interval. In other words, for all $x \leq y$ in P, the Möbius function is given by $\mu(x, y) = (-1)^{\rho(y)-\rho(x)}$, where ρ is the rank function of P. Eulerian posets form an important class of posets as there are many geometric examples such as the face lattices of convex polytopes, and more generally, the face posets of regular CW-spheres.

As we mentioned above, Ehrenborg and Readdy in [4] classify the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Since we are concerned here with finite posets, we drop the requirement that binomial, Sheffer and triangular posets have an infinite chain. This paper deals with the following natural questions, as suggested by Ehrenborg and Readdy in [4].

(i) Which Eulerian posets are binomial?
(ii) Which Eulerian posets are Sheffer?

We also briefly look over Eulerian triangular posets.

We should mention that Stanley has proved that one can recognize *boolean lattices* by looking at smaller intervals (see [7], Lemma 8). Farley and Schmidt answer a similar question for *distributive lattices* in [6]. The project of studying Eulerian binomial posets and Eulerian Sheffer posets is also motivated by their works. In many cases we use the factorial function of smaller intervals to characterize the whole posets.

1.1. *Our results*

All posets considered in this paper are finite. Let us first describe the two following poset operations:

Let Q_i, $i = 1, \ldots, k$, be posets which contain a unique maximal element $\hat{1}$ and a unique minimal element $\hat{0}$. We define $\boxplus_{i=1\ldots k} Q_i$ to be the poset which is obtained by identifying all of the minimal elements as well as identifying all of the maximal elements of the posets Q_i. We define the *k-summation* of P, denoted $\boxplus^k(P)$, to be $\boxplus_{i=1\ldots k} P$.

Let P be a poset with $\hat{0}$. We define the *dual suspension* of P, denoted $\Sigma^*(P)$, to be the poset P with two new elements a_1 and a_2. $\Sigma^*(P)$ has the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$.

Let Q be a poset of odd rank. If Q is an Eulerian Sheffer poset then so is $\boxplus^k(Q)$. Moreover, if P is an Eulerian binomial poset, then $\Sigma^*(P)$ is an Eulerian Sheffer poset.

For Eulerian binomial posets P of rank n, we describe their structure depending on the value of n as follows:

(i) $n = 3$. $P = \boxplus_{i=1\dots k} P_{q_i}$ for some $q_1, \dots, q_r$ such that $q_i \geq 2$, where we denote by P_q, the face lattice of a q-gon.

(ii) n is even. P is either isomorphic to B_n, the boolean lattice of rank n, or T_n, the butterfly poset of rank n (defined in Definition 2.8).

(iii) n is odd. P is either isomorphic to $\boxplus^\alpha(B_n)$ or $\boxplus^\alpha(T_n)$ for some positive integer α.

For Eulerian Sheffer posets P of rank n, we describe their structure and factorial functions depending on the value of n:

(i) $n = 3$. $P = \boxplus_{i=1\dots k} P_{q_i}$ for some $q_1, \dots, q_r$ such that $q_i \geq 2$.

(ii) $n = 4$. The complete classification of factorial functions of the poset P follows from Lemma 4.4.

(iii) n is odd and $n \geq 4$. Then one of the following is true:
  - (a) $B(3) = D(3) = 6$. Then $P = \boxplus^\alpha(B_n)$ for some α.
  - (b) $B(3) = 6, D(3) = 8$. This case is open.
  - (c) $n = 5$, $B(3) = 6, D(3) = 10$. This case remains open.
  - (d) $B(3) = 6, D(3) = 4$. Then $P = \boxplus^\alpha(\Sigma^*(B_{n-1}))$ for some α.
  - (e) $B(3) = 4$. The classification follows from Theorems 3.11 and 3.13 in [4].

(iv) n is even and $n \geq 6$. Then one of the following is true:
  - (a) $B(3) = D(3) = 6$. Then $P = B_n$.
  - (b) $B(3) = 6, D(3) = 8$. The poset P has the same factorial function as the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$ and $B(k) = k!$.
  - (c) $B(3) = 6, D(3) = 4$. Then $P = \Sigma^*(\boxplus^\alpha(B_{n-1}))$ for some α.
  - (d) $B(k) = 2^{k-1}$, for $1 \leq k \leq 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$ for some $\alpha > 1$. In this case P is isomorphic to $\Sigma^* \boxplus^\alpha (T_{2m+1})$.
  - (e) $B(k) = 2^{k-1}$, $1 \leq k \leq 2m+1$. The classification follows from Theorems 3.11 and 3.13 in [4]

The paper is structured as follows. In Section 2 we cover some basic definitions. In Section 3 we completely classify the structure of Eulerian binomial posets. See Lemma 3.6, Theorems 3.11 and 3.12. These results, coupled with Ehrenborg and Readdy's classification in the infinite case, complete the classification of Eulerian binomial posets. In section 4, we give an almost complete classification of the factorial functions of Eulerian Sheffer posets. In fact, in most of above cases we completely identify the structure of the finite Eulerian Sheffer posets, a result which is stronger than merely classifying the factorial functions. In Section 5 we review *triangular posets.* We classify Eulerian triangular posets such that the factorial functions of all of their 3-intervals is equal to 6. Finally, in Section 6 we provide some conclusions and remarks.

2. Definitions and background

We encourage readers to consult Chapter 3 of [12] for basic poset terminology. All the posets which are considered in this paper are finite.

We begin by recalling that a graded interval satisfies the *Euler-Poincaré relation* if it has the same number of elements of even rank as of odd rank.

Definition 2.1. A graded poset is *Eulerian* if every non-singleton interval satisfies the Euler-Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x)-\rho(y)}$ for all $x \leq y$ in P, where ρ denotes the rank function of P.

Definition 2.2. A finite poset P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ is called a *(finite) binomial poset* if it satisfies the following two conditions:

(i) Every interval $[x, y]$ is graded; in particular P has rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval.*

(ii) For all $n \in \mathbb{N}$, $n \leq \operatorname{rank}(P)$, any two n-intervals have the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* or *binomial factorial function* of the poset P.

Next, we define the *atom function* $A(n)$ to be the number of coatoms in a binomial interval of length n. Therefore, $A(n) = \frac{B(n)}{B(n-1)}$ and $B(n) = A(n) \cdots A(1)$.

Consider a binomial poset P. The number of maximal chains passing through each element of rank k in any interval of rank n is $B(k)B(n-k)$, for $1 \leq k \leq n$. The total number of chains in this interval is $B(n)$. Hence, the number of elements of rank k in any interval of rank n is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{1}$$

Sheffer posets were defined by Ehrenborg and Readdy [5] and independently defined by Reiner [9].

Definition 2.3. A finite poset P with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ is called a *(finite) Sheffer poset* if it satisfies the following three conditions:

(i) Every interval $[x, y]$ is graded; in particular, P has a rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an *n-interval.*

(ii) Two n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$ have the same number $D(n)$ of maximal chains.

(iii) Two n-interval $[x, y]$ and $[u, v]$ such that $x \neq \hat{0}$ and $u \neq \hat{0}$ have the same number $B(n)$ of maximal chains.

Let us consider a Sheffer poset P. An interval $[\hat{0}, y]$, where $y \neq \hat{0}$, is called a *Sheffer interval* whereas an interval $[x, y]$ with $x \neq \hat{0}$ is called a *binomial interval.* $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function* of P, respectively. Next we define $A(n)$ and $C(n)$ to be the number of coatoms in a binomial interval of length n and a Sheffer interval of length n. $A(n)$ and $C(n)$ are called the *atom function* and *coatom function* of P, respectively. It is not hard to see that $A(n) = \frac{B(n)}{B(n-1)}$ and $B(n) = A(n) \cdots A(1)$, as well as $C(n) = \frac{D(n)}{D(n-1)}$ and $D(n) = C(n)C(n-1) \cdots C(1)$.

The number of elements of rank k in a Sheffer interval of rank n is

$$\frac{D(n)}{D(k)B(n-k)}. \tag{2}$$

Moreover, for a binomial interval $[x, y]$ of rank n in this Sheffer poset, the number of elements of rank k is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{3}$$

The *dual suspension* of a poset P is defined in [4] as follows.

Definition 2.4. Let P be a poset with $\hat{0}$. We define the *dual suspension* of P, denoted $\Sigma^*(P)$, to be the poset P with two new elements a_1 and a_2. $\Sigma^*(P)$ has the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2B(n-1)$, for $n \geq 2$.

Definition 2.5. Let P be a poset with $\hat{1}$. We define the *suspension* of P, denoted by $\Sigma(P)$, to be the poset P with two new elements a_1 and a_2. $\Sigma(P)$ has the following order relation:
$\hat{1} >_{\Sigma(P)} a_i >_{\Sigma(P)} y$, for all $y < \hat{1}$ in P and $i = 1, 2$.

Definition 2.6. Let P be a poset with $\hat{0}$ and $\hat{1}$, and let k be a positive integer. We define the *k-summation* of P, denoted $\boxplus^k(P)$, to be the poset which is obtained by identifying all minimal elements and all maximal elements of k copies of P.

The *dual* of poset P, denoted P^*, is defined as follows: P^* has the same set of elements as P and the following order relation, $x <_{P^*} y$ if and only if $y <_P x$.

Definition 2.7. The *boolean lattice* B_n of rank n is the poset of subsets of $[n] = \{1, \cdots, n\}$ ordered by inclusion.

Definition 2.8. The *butterfly poset* T_n of rank n consists of the elements of $\hat{0} \cup (D_{n-1} \times \{1, 2\}) \cup \hat{1}$, where $D_{n-1} \times \{1, 2\}$ is direct product of the chain of length $n - 1$, denoted by D_{n-1}, and the anti-chain of rank 2, with the order relation $(k, i) \prec (k + 1, j)$ for all $i, j \in \{1, 2\}$. Also $\hat{0}$ and $\hat{1}$ are the unique minimal and the unique maximal elements of this poset, respectively. Clearly, $T_n = \Sigma^*(T_{n-1})$.

A larger class of posets to consider is the class of *triangular posets.*

Definition 2.9. A finite poset P with $\hat{0}$ and $\hat{1}$ is called a *(finite) triangular poset* if it satisfies the following two conditions.
(i) Every interval $[x, y]$ is graded; hence P has a rank function ρ.
(ii) Every two intervals $[x, y]$ and $[u, v]$ such that $\rho(x) = \rho(u) = m$ and $\rho(y) = \rho(v) = n$ have the same number $B(m, n)$ of maximal chains.

All posets considered in this paper are finite. By binomial, Sheffer and triangular posets, we mean finite binomial, finite Sheffer and finite triangular posets.

3. Finite Eulerian binomial posets

For undefined poset terminology and further information about binomial posets, see [12]. In this section for an Eulerian binomial poset P of rank n we describe its structure as follows.

(i) If $n = 3$, then $P = \boxplus_{i=1 \ldots k} P_{q_i}$ for some $q_1, \ldots, q_r$ such that $q_i \geq 2$, where P_{q_i} is the face lattice of q_i-gon.
(ii) If n is an even integer, then $P = B_n$ or T_n.
(iii) If n is an odd integer and $n \geq 5$, then there is an integer $k \geq 1$, such that $P = \boxplus^k(B_n)$ or $P = \boxplus^k(T_n)$ (see Definition 2.6).

First we provide some examples of finite binomial posets.

Example 3.1. The boolean lattice B_n of rank n is an Eulerian binomial poset with factorial function $B(k) = k!$ and atom function $A(k) = k$, $k \leq n$. Every interval of length k of this poset is isomorphic to B_k.

Example 3.2. Let D_n be the chain containing $n + 1$ elements. This poset has factorial function

$B(k) = 1$ and atom function $A(k) = 1$, for each $k \leq n$.

Example 3.3. The butterfly poset T_n of rank n is an Eulerian binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$ and atom function $A(k) = 2$, for $2 \leq k \leq n$, and $A(1) = 1$.

Example 3.4. Let $\mathbb{F}_q$ be the q-element field where q is a prime power and let $V_n = V_n(q)$ be an n-dimensional vector space over $\mathbb{F}_q$. Let $L_n = L_n(q)$ denote the poset of all subspaces of V_n, ordered by inclusion. L_n is a graded lattice of rank n. It is easy to see that every interval of size $1 \leq k \leq n$ is isomorphic to L_k. Hence $L_n(q)$ is a binomial poset. This poset is not Eulerian for $q \geq 3$.

It is not hard to see that in any n-interval of an Eulerian binomial poset P with factorial function $B(k)$ for $1 \leq k \leq n$, the Euler-Poincaré relation is stated as follows:

$$\sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k)B(n-k)} = 0. \tag{4}$$

The following Lemma can be found in [4].

Lemma 3.5. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*

Lemma 3.6. *Let P be a Eulerian binomial poset of rank 3. Then the factorial function $B(n)$ for $1 \leq n \leq 3$ and the poset P satisfy the following conditions:*

- (i) *$B(2) = 2$ and $B(3) = 2q$, where q is a positive integer such that $q \geq 2$.*
- (ii) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$, such that $P = \boxplus_{i=1\ldots k} P_{q_i}$, where P_{q_i} is the face lattice of q_i-gon.*

Proof. The proof is omitted. It is a consequence of Theorem 4.3. □

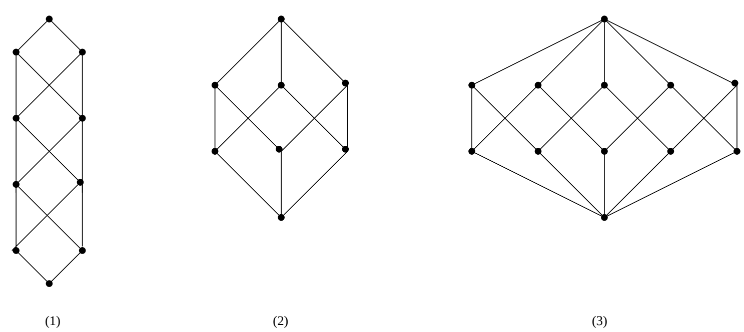


Fig. 1. (1): T_5, (2): B_3 and (3): P_5, the face lattice of a 5-gon

R. Ehrenborg and M. Readdy proved the following two propositions in [4].

Proposition 3.7. Let P be a binomial poset of rank n with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the butterfly poset T_n.

Proposition 3.8. Let P be an Eulerian binomial poset of rank n with factorial function $B(k) = k!$ for $1 \leq k \leq n$. Then the poset P is isomorphic to the boolean lattice B_n of rank n.

It is easy to obtain the following lemma for Eulerian binomial posets by applying the proof of Lemma 2.12 in [4].

Lemma 3.9. *Let $P^{'}$ and P be two Eulerian binomial posets of rank $2m+2$, $m \geq 2$, having atom functions $A^{'}(n)$ and $A(n)$, respectively, which agree for $n \leq 2m$. Then the following equality holds:*

$$\frac{1}{A(2m+1)}\left(1-\frac{1}{A(2m+2)}\right)=\frac{1}{A'(2m+1)}\left(1-\frac{1}{A'(2m+2)}\right). \tag{5}$$

Lemma 3.10. *Every Eulerian binomial poset P of rank 4 is either isomorphic to T_4 or B_4.*

Proof. Applying Lemma 3.6 gives $B(3) = 2k$, where $k \geq 2$. Eq.(3) implies that the number of elements of rank one is the same as the number of elements of rank three in P. We denote this number by n. Hence

$$n = \frac{B(4)}{B(3)B(1)} = \frac{B(4)}{B(3)}. \tag{6}$$

We can also enumerate the number r of elements of rank 2 as follows:

$$r = \frac{B(4)}{B(2)B(2)}. \tag{7}$$

The Euler-Poincaré relation on intervals of length four is $2 + r = 2n$. By enumerating the number of maximal chains, we conclude $B(4) = rB(2)B(2) = nB(3)$ and since always $B(2) = 2$, we have $2r = kn$. The Euler-Poincaré relation implies that $\frac{kn}{2}+2 = 2n$, and so $k < 4$. We have the following cases.

(*i*) $k = 1$. $\frac{n}{2} + 2 = 2n$, so $n = \frac{4}{3}$. This case is not possible.

(*ii*) $k = 2$. $n + 2 = 2n$, so $n = 2$ and $r = 2$. We conclude that $B(k) = 2^{k-1}$, for $1 \leq k \leq 4$. By Proposition 3.7, $P = T_4$.

(*iii*) $k = 3$. $\frac{3n}{2} + 2 = 2n$, so $n = 4$ and $r = 6$. Thus $B(k) = k!$, for $1 \leq k \leq 4$. By Proposition 3.8, $P = B_4$.

□

In the following theorem we obtain the structure of Eulerian binomial posets of even rank.

Theorem 3.11. *Every Eulerian binomial poset P' of even rank $n = 2m \geq 4$ is either isomorphic to T_n or B_n (the butterfly poset of rank n or boolean lattice of rank n).*

Proof. We proceed by induction on m. The claim is true for $2m = 4$, by Lemma 3.10. Assume that the theorem holds for Eulerian binomial posets of rank $2m \geq 4$. We wish to show that it also holds for Eulerian binomial posets of rank $2m + 2$.

Let P' be a Eulerian binomial poset of rank $2m + 2$. The factorial and atom function of this poset are denoted by $B'(n)$ and $A'(n)$, respectively. By Lemma 3.10, every interval of size 4 is either isomorphic to B_4 or T_4. So the factorial function $B'(3)$ of intervals of rank 3, can only take the values $4, 6$ and we have the following two cases:

- $B'(3) = 6$. We wish to show that P' is isomorphic to B_{2m+2} by induction on m. By Lemma 3.10, the claim is true for $2m = 4$. By the induction hypothesis, the claim holds for $n = 2m$, and we wish to prove it for $n = 2m + 2$. Let $P = B_{2m+2}$, so P has the atom function $A(n) = n$ for $1 \leq n \leq 2m + 2$. By the induction hypothesis, $A^{'}(j) = A(j) = j$ for $j \leq 2m$. Now Lemma 3.9 implies that

$$\frac{1}{A(2m+1)}\left(1-\frac{1}{A(2m+2)}\right)=\frac{1}{A'(2m+1)}\left(1-\frac{1}{A'(2m+2)}\right). \tag{8}$$

Since $2m = A^{'}(2m) \leq A^{'}(2m + 2) < \infty$, we obtain the following equation:

$$2m+1-\frac{2}{2m} < A^{'}(2m+1) < 2m+2. \tag{9}$$

Thus $A^{'}(2m+1) = 2m+1$. Eq.(8) implies that $A^{'}(2m+2) = 2m+2$. By Proposition 3.8, the poset P' is isomorphic to B_{2m+2}, as desired.

- $\acute{B}(3) = 4$. We claim that the poset $P^{'}$ of rank $n = 2m+2$ is isomorphic to T_n. By the induction hypothesis, our claim holds for even $n \leq 2m$, and we would like to prove it for $n = 2m+2$. Consider the poset T_{2m+2}. This poset has the atom function $A(n) = 2$ for $1 \leq n \leq 2m+2$. By the induction hypothesis the intervals of length $2m$ in $P^{'}$ are isomorphic to T_{2m}, so $A^{'}(j) = 2$ for $1 \leq j \leq 2m$.

  Clearly $2 = A^{'}(2m) \leq A^{'}(2m+2) < \infty$. Eq.(8) implies that $2 \leq A^{'}(2m+1) < 4$. The case $A^{'}(2m+1) = 3$ is forbidden by similar idea that appeared in the proof of Theorem 2.16 in [4]: Assume that $A^{'}(2m+1) = 3$. Let $[x,y]$ be a $(2m+1)$-interval in $P^{'}$. For $1 \leq k \leq 2m$ there are $B'(2m+1)/(B'(k) \cdot B'(2m+1-k)) = 3 \cdot 2^{2m-1}/(2^{k-1} \cdot 2^{2m-k}) = 3$ elements of rank k in this interval. Let c be a coatom. The interval $[x,c]$ has two atoms, say a_1 and a_2. Moreover, the interval $[x,c]$ has two elements of rank 2, say b_1 and b_2. Moreover we know that each b_j covers each a_i. Let a_3 and b_3 be the third atom, respectively the third rank 2 element, in the interval $[x,y]$. We know that b_3 covers two atoms in $[x,y]$. One of them must be a_1 or a_2, say a_1. But then a_1 is covered by the three elements b_1, b_2 and b_3. But this contradicts the fact that each atom is covered by exactly two elements. Hence this rules out the case $A^{'}(2m+1) = 3$.

  Hence $A^{'}(2m+1) = A^{'}(2m+2) = 2$. Lemma 3.7 implies that $P^{'}$ is isomorphic to T_{2m+2}. □

Theorem 3.12. *Let P be an Eulerian binomial poset of odd rank $n = 2m+1 \geq 5$. Then P satisfies one of the following conditions:*

(*i*) *There is a positive integer k such that P is the k-summation of the boolean lattice of rank n. In other words, $P = \boxplus^k(B_n)$.*

(*ii*) *There is a positive integer k such that P is the k-summation of the butterfly poset of rank n. In other words, $P = \boxplus^k(T_n)$.*

Proof. Lemma 3.10 implies that every interval of length 4 is isomorphic either to B_4 or T_4. Thus the factorial function $B(3)$ can only take the values 4 or 6. Therefore we have the following two cases.

(i) $B(3) = 6$. In this case we claim that there is a positive integer k such that $P = \boxplus^k(B_n)$. When we remove the $\hat{1}$ and $\hat{0}$ from P, the remaining poset is a disjoint union of connected components. Consider one of them and add minimal element $\hat{0}$ and maximal element $\hat{1}$ to it. Denote the resulting poset by Q. It is not hard to see that Q is an Eulerian binomial poset, and also the posets P and Q have the same factorial functions and atom functions up to rank $2m$. Hence $B_Q(k) = B_P(k)$ and $A_Q(k) = A_P(k)$, for $1 \leq k \leq 2m$. Eq.(3) implies that in the poset Q the number of atoms and number of coatoms are the same. Denote this number by t. Let $x_1, \ldots, x_t$ and $a_1, \ldots, a_t$ be an ordering of the atoms and coatoms of Q, respectively. Also, let $c_1, \ldots, c_l$ be the set of elements of rank $2m-1$ in Q. For each element y of rank at least 2 in Q, let $S(y)$ be the set of atoms of Q that are below y. Set $A_i := S(a_i)$ for each element a_i of rank $2m$, $1 \leq i \leq t$, and also set $C_i := S(c_i)$ for each element c_i of rank $2m-1$, $1 \leq i \leq l$. By considering factorial functions, Theorem 3.11 implies that the intervals $[\hat{0}, a_i]$ and $[x_j, \hat{1}]$ are isomorphic to B_{2m}, where $1 \leq i \leq t$ and $1 \leq j \leq t$. We conclude that any interval $[\hat{0}, c_k]$ of rank $2m-1$ is isomorphic to B_{2m-1}. As a consequence, $|A_i| = |S(a_i)| = 2m$, $1 \leq i \leq t$ and also $|C_k| = |S(c_k)| = 2m-1$, $1 \leq k \leq l$.

In the case that there are i_1 and j_1 such that $A_{i_1} \cap A_{j_1} \neq \phi$, where $1 \leq i_1, j_1 \leq t$, we claim that $2m-1 \leq |A_{i_1} \cap A_{j_1}| \leq 2m$. Consider an atom $x_k \in A_{i_1} \cap A_{j_1}$, $1 \leq k \leq t$. Theorem 3.11

implies that $[x_k, \hat{1}] = B_{2m}$. Thus, there is an element c_h of rank $2m-2$ in this interval which is covered by a_{i_1} and a_{j_1}, $1 \leq h \leq l$. Notice that c_h is an element of rank $2m-1$ in Q. Therefore, $|C_h| = 2m-1 \leq |A_{i_1} \cap A_{j_1}| \leq |A_{i_1}| = |S(a_{i_1})| = 2m$.

We claim that for all distinct pairs i and j, $1 \leq i, j \leq t$, we have $A_i \cap A_j \neq \emptyset$. Associate the graph G_Q to the poset Q as follows: $A_1, \ldots, A_t$ are vertices of this graph, and we connect vertices A_i and A_j if and only if $A_i \cap A_j \neq \phi$. Since $Q - \{\hat{0}, \hat{1}\}$ is connected, we conclude that G_Q is a connected graph. If $\{A_{i_1}, A_{j_1}\}$ and $\{A_{j_1}, A_{k_1}\}$ are different edges of G_Q, we wish to show that $\{A_{i_1}, A_{k_1}\}$ is also an edge of G_Q. $|A_{i_1} \cap A_{j_1}| \geq 2m-1$ as well as $|A_{j_1} \cap A_{k_1}| \geq 2m-1$. On other hand, since $|A_{i_1}| = |A_{j_1}| = |A_{k_1}| = 2m$, we conclude that $A_{i_1} \cap A_{k_1} \neq \phi$. Therefore $\{A_{i_1}, A_{k_1}\}$ is also an edge of G_Q. As a consequence, the connected graph G_Q is a complete graph. Thus for all different i and j $A_i \cap A_j \neq \phi$ and also $2m-1 \leq |A_i \cap A_j| \leq 2m$, where $1 \leq i, j \leq t$.

Now we show that $|A_i \cap A_j| = 2m-1$ for different i, j. Suppose this claim doesn't hold. Then there are different $i^{'}, j^{'}$ such that $|A_{i'} \cap A_{j'}| = 2m$. We claim that there are two elements of rank $2m-1$ in Q such that they both are covered by coatoms $a_{i'}$ and $a_{j'}$. To prove this claim, consider an atom $x_f \in A_{i'} \cap A_{j'}$, so $[x_f, \hat{1}] = B_{2m}$. Hence, there is a unique element c_h of rank $2m-2$ in this interval which is covered by both $a_{i'}$ and $a_{j'}$. By induction on m, Lemma 3.6, and the property that $|C_h| \leq |A_{i'} \cap A_{j'}| = 2m$ we conclude that $[\hat{0}, c_h]$ is isomorphic to B_{2m-1} and so $|C_h| = 2m-1$. Therefore there is an atom $x_d \in A_{i'} \cap A_{j'} \setminus C_h$. Since the interval $[x_d, \hat{1}]$ is isomorphic to B_{2m}, there is an element $c_k \neq c_h$ of rank $2m-1$ which is covered by coatoms $a_{i'}$ and $a_{j'}$.

Since $|C_h| = |S(c_h)| = |C_k| = |S(c_k)| = 2m-1$ and C_k, C_h are both subsets of $A_i \cap A_j$, we conclude that there should be an atom $x_s \in C_k \cap C_h$. Therefore the interval $[x_s, \hat{1}]$ has two elements c_k and c_h of rank $2m-2$ such that they both are covered by two elements a_i and a_j of rank $2m-1$ in the interval $[x_s, \hat{1}]$. We know $[x_s, \hat{1}] = B_{2m}$ and there are no two elements of rank $2m-2$ covered by two elements of rank $2m-1$ in B_{2m}. This contradicts our assumption, and so $|A_i \cap A_j| = 2m-1$ for all different i, j, as desired.

In summary:

(a) $|A_i| = 2m$ for $1 \leq i \leq t$,

(b) $|A_i \cap A_j| = 2m-1$ for all $1 \leq i < j \leq t$,

(c) $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$.

As a consequence, we have $t > 2m$.

Next, we are going to show that $t = 2m+1$. Without loss of generality, consider the three different sets $A_1 = S(a_1), A_2 = S(a_2)$ and $A_3 = S(a_3)$ which are associated with the three coatoms a_1, a_2 and a_3. We know that $|A_1| = |A_2| = |A_3| = 2m$ and $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$. Without loss of generality, let us that assume $A_1 = \{x_1, x_2, \ldots, x_{2m-1}, y_1\}$ and $A_2 = \{x_1, x_2, \ldots, x_{2m-1}, y_2\}$ where $y_i \neq x_1, \ldots, x_{2m-1}$, $i = 1, 2$. We have two different cases: either A_3 contains at least one of y_1 and y_2, or A_3 contains neither of them. First we study the second case, $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_3\}$ where $y_3 \neq y_1, y_2, x_1, \ldots, x_{2m-1}$. Considering the $t-3$ other coatoms a_k, $4 \leq k \leq t$, there are different atoms y_k, $4 \leq k \leq t$, such that $y_k \neq y_1, y_2, y_3, x_1, \ldots, x_{2m-1}$ and $A_k = S(a_k) = \{x_1, x_2, \ldots, x_{2m-1}, y_k\}$. This implies that the number of atoms is $|\bigcup_{i=1}^{t} A_i| = t + 2m - 1$, which is a contradiction. Hence only the first case can happen and A_3 should contain one of y_1 or y_2. In this case $|A_2 \cap A_3| = |A_1 \cap A_3| = 2m-1$ implies that $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_1, y_2\} \setminus \{x_j\} \subset A_1 \cup A_2$ for some x_j. Since A_3 was chosen arbitrarily, it follows that for each A_k we have $A_k \subset A_1 \cup A_2$. Hence

$$\bigcup_{i=1}^{t} A_k = \{x_1, \ldots, x_{2m-1}, y_1, y_2\}. \tag{10}$$

Thus the number of coatoms in the poset Q is $t = 2m + 1$. By Theorem 3.11, $B_Q(k) = k!$, $1 \leq k \leq 2m$, therefore $B_Q(2m + 1) = (2m + 1)!$. By Proposition 3.8, Q is isomorphic to B_{2m+1} and so P is a union of copies of B_{2m+1} by identifying their minimal elements and their maximal elements. In other words, $P = \boxplus^k(B_{2m+1})$. It can be seen that P is binomial and Eulerian and the proof follows.

(*ii*) $B(3) = 4$. With the same argument as part (*i*), we construct the binomial poset Q by adding $\hat{1}$ and $\hat{0}$ to one of the connected components of $P - \{\hat{0}, \hat{1}\}$. We claim that Q is isomorphic to T_{2m+1}. Similar to part (*i*), let $a_1, \ldots, a_t$ and $x_1, \ldots, x_t$ denote coatoms and atoms of Q. Set $A_i = S(a_i)$. By Theorem 3.11, $|A_i| = 2$. It is easy to see that $\bigcup_{i=1}^t A_i = \{x_1, \ldots, x_t\}$. Define G_Q to be the graph with vertices $x_1, \ldots, x_t$ and edges $A_1, \ldots, A_t$. Since $Q \setminus \{\hat{0}, \hat{1}\}$ is a connected component, G_Q is a connected graph. Since $[x_i, \hat{1}] \cong T_{2m}$, the degree of each vertex of G_Q is 2 and G_Q is the cycle of length t. Therefore if $t > 2$, $|A_i \cap A_j| = 1$ or 0, $1 \leq i < j \leq t$.

We claim $t = 2$. Suppose this claim does not hold, so $t > 2$. Consider an element c of rank 3 in Q, Lemma 3.6 and Theorem 3.11 imply that both intervals $[\hat{0}, c]$ and $[c, \hat{1}]$ are the butterfly posets. Hence there are two coatoms above c, say a_k and a_l, and similarly there are two atoms below c, say x_h and x_s. That is, $A_k = A_l = \{x_h, x_k\}$. As which is not possible when $t > 2$. As a consequence, $t = 2$ and all A_i's have 2 elements and $|\bigcup_1^t A_i| = |\{x_1, \ldots, x_t\}| = 2 = t$.

Similar to part (*i*), $B_Q(k) = 2^{k-1}$ for $1 \leq k \leq 2m + 1$. By Proposition 3.7, we conclude that Q is isomorphic to T_{2m+1}. Therefore, there is an integer $k > 0$ such that $P = \boxplus^k(T_n)$.

□

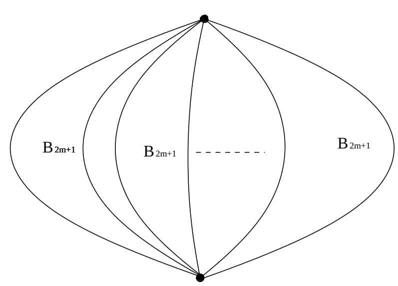


Fig. 2. A poset that is obtained by identifying all mimimal elements and all maximal elements of copies of B_{2m+1}

4. Finite Eulerian Sheffer posets

For basic definitions regarding Sheffer posets, see Section 2. In this section, we give an almost complete classification of the factorial functions and the structure of Eulerian Sheffer posets.

First, we provide some examples of Eulerian Sheffer posets. We study Eulerian Sheffer posets of small ranks $n = 3, 4$ in Lemma 4.3 and 4.4. By Lemma 4.3 and 4.4, the only possible values of $B(3)$ are 4 and 6. In Section 4.1, Lemma 4.5 and Theorems 4.6, 4.12, 4.13, 4.14 deal with Eulerian Sheffer posets with $B(3) = 6$. Finally in Section 4.2, Theorems 4.15, 4.16, 4.17 deal with Eulerian Sheffer posets with $B(3) = 4$.

The results of this Section are summarized below.

Let P be a Eulerian Sheffer poset of rank n. Then P satisfies one of following conditions.

(i) $n = 3$. $P = \boxplus_{i=1\ldots k} P_{q_i}$ for some $q_1, \ldots, q_r$ such that $q_i \geq 2$.

(ii) $n = 4$. The complete classification of factorial functions of the poset P follows from Lemma 4.4.

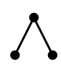
(iii) n is odd and $n \geq 4$. Then one of the following is true:
   (a) $B(3) = D(3) = 6$. Then $P = \boxplus^\alpha(B_n)$ for some α.
   (b) $B(3) = 6, D(3) = 8$. This case is open.
   (c) $n = 5$, $B(3) = 6, D(3) = 10$. This case remains open.
   (d) $B(3) = 6, D(3) = 4$. Then $P = \boxplus^\alpha(\Sigma^*(B_{n-1}))$ for some α.
   (e) $B(3) = 4$. The classification follows from Theorems 3.11 and 3.13 in [4].

(iv) n is even and $n \geq 6$. Then one of the following is true:
   (a) $B(3) = D(3) = 6$. Then, $P = B_n$.
   (b) $B(3) = 6, D(3) = 8$. The poset P has the same factorial function as the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$ and $B(k) = k!$.
   (c) $B(3) = 6, D(3) = 4$. Then $P = \Sigma^*(\boxplus^\alpha(B_{n-1}))$ for some α.
   (d) $B(k) = 2^{k-1}$, for $1 \leq k \leq 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$ for some $\alpha > 1$. In this case P is isomorphic to $\Sigma^* \boxplus^\alpha(T_{2m+1})$.
   (e) $B(k) = 2^{k-1}$, $1 \leq k \leq 2m+1$. The classification follows from Theorems 3.11 and 3.13 in [4]

It is clear that every binomial poset is also a Sheffer poset. Here are some other examples of Sheffer posets.

Example 4.1. Let P be a binomial poset of rank n with the factorial functions $B(k)$. By adjoining a new minimal element $\widehat{-1}$ to P, we obtain a Sheffer poset of rank $n+1$ with binomial factorial functions $B(k)$ for $1 \leq k \leq n$ and Sheffer factorial functions, $D(k) = B(k-1)$ for $1 \leq k \leq n+1$.

Example 4.2. Let T be the following three element poset:



Let T^n be the Cartesian product of n copies of the poset T. The poset $C_n = T^n \cup \{\hat{0}\}$ is the face lattice of an n-dimensional cube, also known as the *cubical lattice*. The cubical lattice is a Sheffer poset with $B(k) = k!$ for $1 \leq k \leq n$ and $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n+1$.

Let P be an Eulerian Sheffer poset of rank n. The Euler-Poincaré relation for every m-Sheffer interval, $2 \leq m \leq n$, becomes

$$1 + \sum_{k=1}^{m} (-1)^k \cdot \frac{D(m)}{D(k)B(m-k)} = 0. \tag{11}$$

It is clear that B_2 is the only Eulerian Sheffer poset of rank 2.

In the next lemma, we characterize the structure of Eulerian Sheffer posets of rank 3. The characterization of the factorial function is an immediate consequence.

Lemma 4.3. *Let P be a Eulerian Sheffer poset of rank* 3.
  (*i*) *The poset P has the factorial functions $D(2) = 2$ and $D(3) = 2q$, where q is a positive integer such that $q \geq 2$.*
  (*ii*) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$ such that $P = \boxplus_{i=1\ldots k} P_{q_i}$, where P_{q_i} is the face lattice of a q_i-gon.*

Proof. Consider an Eulerian Sheffer poset P of rank 3. Now $P - \{\hat{0}, \hat{1}\}$ consists of elements of rank 1 and rank 2 of P. By the Euler-Poincaré relation, it is easy to see that $B(2) = 2$ and every interval of length 2 is isomorphic to B_2. So in $P - \{\hat{0}, \hat{1}\}$, every element of rank 2 is connected to two elements of rank 1 and vice versa. Therefore, the Hasse diagram of $P - \{\hat{0}, \hat{1}\}$ is just the disjoint union of the cycles of even lengths $2q_1, \ldots, 2q_r$ where $q_i \geq 2$. We conclude that P is obtained by identifying all minimal elements of the posets $P_{q_1}, \ldots, P_{q_r}$ and identifying all of their maximal elements. Hence $P = \boxplus_{i=1\ldots k} P_{q_i}$ and $D(3) = 2(q_1 + \cdots + q_r)$. Thus every Eulerian Sheffer poset of rank 3 has the factorial functions $D(3) = 2q$ where $q \geq 2$ and $B(2) = D(2) = 2$. □

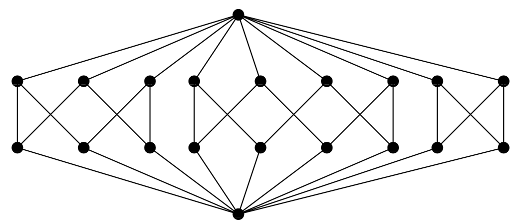


Fig. 3. $P = \boxplus_{k=2\ldots 4} P_k$.

Lemma 4.4 deals with Eulerian Sheffer posets of rank 4.

Lemma 4.4. *Let poset P be an Eulerian Sheffer poset of rank 4. Then one of the following conditions hold.*

(i) $B(3) = 2r$, $D(3) = 4$, $D(4) = 4r$, *where* $r \geq 2$.
(ii) $B(3) = 10$, $D(3) = 3!$, $D(4) = 120$.
(iii) $B(3) = 8$, $D(3) = 3!$, $D(4) = 2^3.3!$.
(iv) $B(3) = 3!$, $D(3) = 3!$, $D(4) = 4!$.
(v) $B(3) = 4$, $D(3) = 3!$, $D(4) = 2 \cdot 3!$.
(vi) $B(3) = 3!$, $D(3) = 10$, $D(4) = 120$.
(vii) $B(3) = 3!$, $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.
(viii) $B(3) = 3!$, $D(3) = 4$, $D(4) = 2 \cdot 3!$.
(ix) $B(3) = 4$, $D(3) = 2r$, $D(4) = 4r$ *where* $r \geq 2$.

Proof. Let P be an Eulerian Sheffer poset of rank 4. Note that for every Eulerian Sheffer poset $B(1) = D(1) = 1$ as well as $B(2) = D(2) = 2$. The variables m, r, n denote the number of elements of rank 1, 2 and 3 of P, respectively. By the Euler-Poincaré relation $2 + r = m + n$. The number of maximal chains in P is given by $4r = B(3)m = D(3)n$. Lemma 4.3 implies that there are positive integers k_1, k_2 such that $D(3) = 2k_2$ and $B(3) = 2k_1$. Thus $r + 2 = (\frac{2}{k_1} + \frac{2}{k_2})r$. We conclude that $\frac{2}{k_1} + \frac{2}{k_2} > 1$; therefore the case $k_1, k_2 > 3$ cannot happen. Next we study the remaining cases as follows.

(1) $k_2 = 1$. Then $n = 2r$ and $2r \leq r + 2$. Therefore $r = 1, 2$, and we have the following cases:
   (a) $r = 1$. Then $m = 1$ and $n = 2$, so the Sheffer interval of length 2 in P does not satisfy the Euler-Poincaré relation. This case is not possible.
   (b) $r = 2$. Then $n = 4$ and $m = 0$, which is not possible.

(2) $k_2 = 2$. Then $2r = 2n$, so $n = r$, $m = 2$ and $k_1 = r$. The fact that every interval of rank 2 is isomorphic to B_2 implies that $r \geq 2$. Thus $B(1) = 1$, $B(2) = 2$ and $B(3) = 2r$, as well as $D(1) = 1$, $D(2) = 2$, $D(3) = 4$, and $D(4) = 4r$. The poset $T = \Sigma^*(P_r)$, where P_r is the face lattice of r-polygon, is an Eulerian Sheffer poset with the described factorial functions.

(3) $k_2 = 3$. The equation $r + 2 = m + n = (\frac{2}{3} + \frac{2}{k_1})r$ implies that $k_1 < 6$, so we need to consider the following cases.
   (a) $k_1 = 5$. Then $r + 2 = \frac{2}{5}r + \frac{2}{3}r$, so $\frac{1}{15}r = 2$, $r = 30$, $n = 20$ and $m = 12$. Thus P has the following factorial functions $B(3) = 10$, $D(3) = 3!$ and $D(4) = 120$. The face lattice of icosahedron is an Eulerian Sheffer poset with the same factorial functions.
   (b) $k_1 = 4$. Similarly, P has the same factorial functions as the dual of the cubical lattice of rank 4, $B(3) = 8$, $D(3) = 3!$ and $D(4) = 2^3 \cdot 3!$.
   (c) $k_1 = 3$. Similarly, P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. And P is isomorphic to B_4.

(d) $k_1 = 2$. Similarly, P has the factorial functions $B(3) = 4$, $D(3) = 3!$ and $D(4) = 2 \cdot 3!$. The suspension of poset B_3, $\Sigma(B_3)$, is an Eulerian Sheffer poset with the same factorial functions.

(e) $k_1 = 1$. Then $r + 2 = 2r + \frac{2}{3}r$, which is not possible.

(4) $k_1 = 3$. Then $r + 2 = (\frac{2}{k_1} + \frac{2}{k_2})r$ implies that $k_2 < 6$, so we have the following cases.

(a) $k_2 = 5$. Then $r + 2 = \frac{2}{5}r + \frac{2}{3}r$, so $\frac{1}{15}r = 2$. Therefore $r = 30$, $m = 20$ and $n = 12$ and so P has the same factorial functions face lattice of a dodecahedron, $B(3) = 3!$, $D(3) = 10$ and $D(4) = 120$.

(b) $k_2 = 4$. Similarly, P has the same factorial functions as the cubical lattice of rank 4, $B(3) = 3!$ and $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.

(c) $k_2 = 3$. P has the factorial functions $B(3) = 3!$, $D(3) = 3!$ and $D(4) = 4!$. So, $P = B_4$.

(d) $k_2 = 2$. It is easy to see that P has the factorial functions as $\Sigma^*(B_3)$, $B(3) = 3!$, $D(3) = 2 \cdot 2!$ and $D(4) = 2 \cdot 3!$.

(e) $k_2 = 1$. Then $r + 2 = 2r + \frac{2}{3}r$, which is not possible.

(5) $k_1 = 2$. Then $2r = 2m$, so $m = r$ and $n = 2$. Therefore , $B(3) = 4$, $D(3) = 2r$ and $D(4) = 4r$ where $r \geq 2$. $T = \Sigma(P_r)$, the suspension of poset P_r, is an Eulerian Sheffer poset with the described factorial functions.

(7) $k_1 = 1$. Then $n = 2r$ where $2r \leq r + 2$, so $r = 1, 2$.

(a) $r = 1$. Then $n = 1$ and $m = 2$. The Sheffer interval of length 2 in this poset does not satisfy the Euler-Poincaré relation, so this case is not possible.

(b) $r = 2$. Then $m = 4$ and $n = 0$, this case is not possible.

□

4.1. **Characterization of the factorial functions and structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$.**

In this section we consider Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 3!$. Lemma 4.5 shows that for any such poset of rank $n \geq 5$, $D(3)$ can only take the values $4, 6, 8$. In Subsections 4.1.1, 4.1.2, 4.1.3, we consider the three different cases $D(3) = 4, 6, 8$, respectively.

Lemma 4.5. *Let P be a Eulerian Sheffer poset of rank $n \geq 6$ with $B(3) = 3!$. Then $D(3)$ can take only the values $4, 6, 8$.*

Proof. By Lemma 4.4, the Sheffer factorial function of poset P for Sheffer 3-intervals can take the following values $D(3) = 4, 6, 8, 10$. We claim that the case $D(3) = 10$ is not possible. Suppose there is an Eulerian Sheffer poset P of rank of at least 6 with the factorial functions $D(3) = 10$ and $B(3) = 3!$. By Lemma 4.4, P has the following factorial functions $D(1) = 1$, $D(2) = 2$, $D(3) = 10$, $D(4) = 120$, $B(1) = 1$, $B(2) = 2!$ and $B(3) = 3!$. Set $C(6) = A$, $C(5) = B$, where $C(5)$ and $C(6)$ are coatom functions of P. By Theorems 3.11 and 3.12, we conclude there is $\alpha > 0$ such that $B(4) = 4!$ and $B(5) = \alpha.5!$. The Euler-Poincaré relation implies that

$$1 + \sum_{k=1}^{6} (-1)^k \cdot \frac{D(6)}{D(k)B(6-k)} = 0,$$

therefore, by substituting the values in above equation, we have:

$$2 = \frac{AB}{\alpha} - AB + A, \alpha(A-2) = (\alpha - 1)AB. \tag{12}$$

we have the two following cases:

(i) $\alpha = 1$. Eq.(12) implies that $A = 2$. However, $A \geq A(5) = 5$ where $A(5)$ is an atom function of B_5. This case is not possible.

(ii) $\alpha > 1$. By Eq.(12),

$$\left(\frac{\alpha}{\alpha-1}\right)\left(\frac{A}{A-2}\right) = B.$$

$A \geq A(5) = 5$ implies that $B < 4$. On the other hand, since $B \geq A(4) \geq 4$. This case is also not possible.

We conclude that there is no Eulerian Sheffer poset of rank at least 6 with $D(3) = 10$ and $B(3) = 3!$, as desired. □

4.1.1. **Characterization of the factorial function of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$.**

In this subsection, we study the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$. Theorem 4.6 characterizes the factorial functions of such posets of even rank. However, the question of characterizing factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 3!$ and $D(3) = 8$ still remains open.

Theorem 4.6. *Let P be an Eulerian Sheffer poset of even rank $n = 2m + 2 \geq 6$ with $B(3) = 3!$ and $D(3) = 8$. Then P has the same factorial functions as the cubical lattice of rank n, C_n. That is, $D(k) = 2^{k-1}(k-1)!$, $1 \leq k \leq n$ and $B(k) = k!$, $1 \leq k \leq n-1$.*

In order to prove Theorem 4.6, we prove the following three Lemmas 4.7, 4.9 and 4.10:

Lemma 4.7. *Let Q be an Eulerian Sheffer poset of odd rank $2m+1$, $m \geq 2$, with $B(3) = 3!$. Then Q cannot have the following sequence of coatom functions: $C(n) = 2(n-1)$ for $2 \leq n \leq 2m$ and $C(2m+1) = 4m+1$.*

Proof. We proceed by contradiction. Assume Q is such a poset. Theorem 3.11 implies that P has the binomial factorial functions $B(k) = k!$ for $1 \leq k \leq 2m$. By Eq.(2) we enumerate the number elements of ranks 1, $2m-1$, $2m$ in this Sheffer poset. Let $\{a_1, \ldots, a_{4m+1}\}$, $\{e_1, \ldots, e_{(4m+1)(2m-1)}\}$ and $\{x_1, \ldots, x_t\}$ denote the sets of elements of rank $2m$, $2m-1$ and 1 in Q, respectively, where $t = \frac{4m+1}{2m} \cdot 2^{2m-1}$. For each element y of rank at least 2, let $S(y)$ be the set of atoms in $[\hat{0}, y]$. Set $A_j = S(a_j)$ for each element a_j of rank $2m$ and also $E_j = S(e_j)$ for each element e_j of rank $2m-1$. Eq.(2) implies that $|S(y)| = 2^{r-1}$ for any element y of rank $2 \leq r \leq 2m$.

We claim that for all different $1 \leq i, j \leq 4m+1$, $A_i \cap A_j \neq \phi$. Suppose this claim does not hold, then there exist two different s, l such that $|A_s \cap A_l| = 0$ where $1 \leq s, l \leq 4m+1$. Since $|A_s| + |A_l| < t$, there is a set $A_k = S(a_k)$ such that $A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l) \neq \phi$, $1 \leq k \leq 4m+1$. Generally speaking, $A_i \cap A_j$ is the set of atoms which are below $a_i \wedge a_j$. Thus,

$$|A_i \cap A_j| = |S(a_i \wedge a_j)| = 2^{rank(a_i \wedge a_j)-1}. \tag{13}$$

Let us recall the following facts:

(i) $A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l) \neq \phi$

(ii) $|A_i \cap A_j| = |S(a_i \wedge a_j)| = 2^{rank(a_i \wedge a_j)-1}$ for all different i, j, $1 \leq i, j \leq 4m+1$.

The above equations yield $|A_l \cap A_k|, |A_s \cap A_k| \leq 2^{2m-2}$. Furthermore, since $|\{x_1, \ldots, x_t\}| = t = \frac{4m+1}{2m} \cdot 2^{2m-1}$, $|A_l| = |A_s| = |A_k| = 2^{2m-1}$ and $|A_l \cap A_s| = 0$, we conclude that

$$|A_k \cap (\{x_1, \ldots, x_t\} - A_s \cup A_l)| \leq |\{x_1, \ldots, x_t\} - A_s \cup A_l| = \frac{2^{2m-1}}{2m}. \tag{14}$$

Since $|A_l \cap A_k|, |A_s \cap A_k| \leq 2^{2m-2}$, Eq.(14) implies that $|A_l \cap A_k|, |A_s \cap A_k| \neq 0$. Consider an arbitrary atom $x_1 \in A_l \cap A_k$. Clearly a_l, a_k are elements of rank $2m-1$ in the interval $[x_1, \hat{1}]$. By

Proposition 3.8, $[x_1, \hat{1}] = B_{2m}$. So $a_l \wedge a_k$ is covered by a_k and a_l, therefore $|A_l \cap A_k| = 2^{2m-2}$ and similarly, $|A_s \cap A_k| = 2^{2m-2}$.

We have seen $|A_s \cap A_k| = 2^{2m-2}$, $|A_l \cap A_k| = 2^{2m-2}$ and $|A_k| = 2^{2m-1}$. Moreover, since we assumed $A_s \cap A_l = \phi$, we conclude that $A_k = A_s \cup A_l$. On the other hand $A_k \cap (\{x_1, \dots, x_t\} - A_s \cup A_l) \neq \phi$, which is not possible when $A_k = A_s \cup A_l$. This contradicts our assumption. Therefore $|A_i \cap A_j| \neq 0$ for $1 \leq i, j \leq 4m+1$. So for every distinct pair a_i and a_j, there is an atom $x_h \in A_i \cap A_j$. As above $[x_h, \hat{1}] = B_{2m}$, so there is at least one element of rank $2m-2$ in this interval, e_k, $1 \leq k \leq (4m+1)(2m-1)$, and it is covered by both a_i and a_j. In addition, for every element e_l of rank $2m-1$ in Q $[e_l, \hat{1}]$ is isomorphic to B_2. As a consequence, for every e_l there is exactly one pair a_i, a_j such that e_l is covered by them. Hence, the number of the disjoint pairs of elements of rank $2m$ in poset Q is at most the number of elements of rank $2m-1$. That is, $(4m+1)(2m-1) \geq (4m+1)(2m)$ which is not possible. This contradicts the assumption. So there is no poset Q with the described factorial and coatom functions, as desired. □

Lemma 4.7, implies the following.

Corollary 4.8. *Let P be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with $B(k) = k!$, for $1 \leq k \leq 2m$. P cannot have the following sequence of coatom functions: $C(n) = 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) = 4m+1$ and $C(2m+2) = 4(2m+1)$.*

Lemma 4.9. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$ with the binomial factorial functions $B(k) = k!$, for $1 \leq k \leq 2m+1$. Then Q cannot have the following sequence of coatom functions: $C(n) = 2(n-1)$ for $2 \leq n \leq 2m$, $C(2m+1) = 4m-2$ and $C(2m+2) = 2m+1$.*

Proof. We proceed by contradiction, assume that Q is such a poset of rank $2m+2$ as described above. By Eq.(2), we can enumerate the number elements of rank k in this Sheffer poset of rank $n = 2m+2$ for $1 \leq k \leq n$. Let $\{a_1, \dots, a_{2m+1}\}$, $\{e_1, \dots, e_{(2m)^2-1}\}$ and $\{x_1, \dots, x_t\}$, where $t = \frac{4m-2}{2m} \cdot 2^{2m-1}$, be the sets of elements of rank $2m+1$, $2m$ and 1 in poset Q, respectively.

With the same argument as Lemma 4.7, for any element y of at least 2 we define $S(y)$ to be the set of atoms in interval $[\hat{0}, y]$. Set $A_j = S(a_j)$ for $1 \leq j \leq 2m+1$, and also set $E_j = S(e_j)$, $1 \leq j \leq (2m)^2 - 1$. By Eq.(2), $|E_j| = |S(e_j)| = 2^{2m-1}$ for $1 \leq j \leq (2m)^2 - 1$ and also $|A_i| = |S(a_i)| = \frac{4m-2}{2m} \cdot 2^{2m-1} = t$, $1 \leq i \leq 2m+1$.

For each element e_i of rank $2m$, $[e_i, \hat{1}] = B_2$. Hence, each element e_i of rank $2m$ covered by exactly two coatoms such as a_r, a_s where $1 \leq r, s \leq 2m+1$ in Q. By Eq.(2), the number of elements of rank $2m$ is $(2m)^2 - 1$ and also the number of pairs of elements of rank $2m+1$ is $m(2m+1)$. We deduce, there are at least two different coatoms such as a_k, a_l that both cover two different elements e_i, e_j for some particular i, j . We know the following facts:

(i) $|A_k| = |A_l| = \frac{4m-2}{2m} \cdot 2^{2m-1} = |\{x_1, \dots, x_t\}| = t$
(ii) $|E_i| = |E_j| = 2^{2m-1}$
(iii) $E_i, E_j \subseteq A_k = A_l = \{x_1, \dots, x_t\}$.

By the above facts $|E_i| + |E_j| > |A_k|, |A_l|$. Hence, there is at least one atom $x_r \in E_i, E_j, A_k, A_l$ such that e_i, e_j are elements of rank $2m-1$ in the intervals $[x_r, a_l]$ and $[x_r, a_k]$. By Proposition 3.8, $[x_r, a_k] = [x_r, a_l] = B_{2m}$, so there is an element c of rank $2m-2$ in this interval $[x_r, a_l]$ which is covered by e_i and e_j. Therefore the interval $[c, \hat{1}]$ has two elements e_i, e_j of rank 1 and they both are covered by two elements a_k, a_l of rank 2. By Proposition 3.8, $[c, \hat{1}] = B_3$. Since B_3, dose not have two elements of rank 1 which are both covered by two elements of rank 2, it lead us to contradiction. There is no poset with described conditions, as desired.

□

Lemma 4.10. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with binomial factorial function $B(k) = k!$ for $1 \leq k \leq 2m$. Then the poset Q cannot have the following sequence of coatom functions: $C(n) = 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) = 4m-1$ and $C(2m+2) = \frac{4}{3}(2m+1)$.*

Proof. We proceed by contradiction. So, suppose Q is such a poset of rank $2m+2$ with the described factorial functions. We enumerate the number of elements of rank k in Q as follows,

$$\frac{D(2m+2)}{B(k)D(2m+2-k)} = \frac{C(2m+2)\cdots C(2m+2-k+1)}{k!}. \tag{15}$$

Thus, $\{a_1, \ldots, a_{\frac{4}{3}(2m+1)}\}$, $\{e_1, \ldots, e_{\frac{4}{6}(2m+1)(4m-1)}\}$ are the sets of elements of rank $2m+1$ and $2m$ in Q, respectively. For every element e_i of rank $2m$, $[e_i, \hat{1}]$ is isomorphic to B_2. So, each element of rank $2m$ covered by exactly two different elements of rank $2m+1$.

There are exactly $\frac{4}{6}(2m+1)(4m-1)$ elements of rank $2m$ in Q, and we also know that there are $(\frac{4}{6}(2m+1))(\frac{4}{3}(2m+1)-1)$ different pairs of coatoms $\{a_i, a_j\}$ in Q, $1 \leq i < j \leq \frac{4}{3}(2m+1)$. We conclude there are at least two different coatoms a_k, a_l such that they both cover two different elements e_i, e_j of rank $2m$. The interval $T = [\hat{0}, a_k]$ has binomial factorial functions $B_T(k) = k!$ for $1 \leq k \leq 2m$ and coatom functions $C_T(n) = 2(n-1)$ for $2 \leq n \leq 2m$ and $C_T(2m+1) = 4m-1$. Let $\{y_1, \ldots, y_t\}$ be the set of atoms in poset T where $t = \frac{(4m-1)}{2m}.2^{2m-1}$. Thus $A_k = \{y_1, \ldots, y_t\}$. Set $E_j = S(e_j), E_i = S(e_i)$, so $E_j, E_i \subset A_k$. By Eq.(2), $|E_i| = |E_j| = 2^{2m-1}$, therefore $|E_i| + |E_j| > |A_k|$. We conclude that there is at least one atom $y_1 \in T$ which is below e_i, e_j and a_k.

Proposition 3.8, implies that $[y_1, a_k] = B_{2m}$. By the boolean lattice properties, there is an element c of rank $2m-2$ in $[y_1, a_k]$ such that c is covered by e_i, e_j. By Proposition 3.8, $[c, \hat{1}] = B_3$. Consider the interval $[c, \hat{1}]$, a_k and a_l are two elements of rank 2 in this interval and they both cover two elements e_i and e_j of rank 1. It contradicts the fact that $[c, \hat{1}] = B_3$. We conclude that $[c, \hat{1}] \neq B_3$. It lead us to contradiction, there is no poset Q with describe conditions, as desired. □

The following lemma can be obtained by applying the proof of Lemma 4.8 in [4].

Lemma 4.11. *Let P and $P^{'}$ be two Eulerian Sheffer posets of rank $2m+2$, $m \geq 2$, such that their binomial factorial functions and coatom functions agree up to rank $n \leq 2m$. That is $B(n) = B^{'}(n)$ and $C(n) = C^{'}(n)$, where $m \geq 2$. Then the following equation holds,*

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \tag{16}$$

Proof of Theorem 4.6. Let $C(k)$ and $C^{'}(k) = 2(k-1)$ be the coatom functions of the Eulerian Sheffer poset P and C_n, the cubical lattice of rank n, for $2 \leq k \leq n = 2m+2$. We only need to show that $C(n) = C^{'}(n) = 2(n-1)$ for $2 \leq n \leq 2m+2$ We prove this claim by induction on m. By Lemma 4.4, $C(4) = C^{'}(4) = 6$ and the claim is hold for $m = 1$. By induction hypothesis, $C(n) = C^{'}(n) = 2(n-1)$ for $2 \leq n \leq 2m$. Set $B = C(2m+1)$ and $A = C(2m+2)$. Theorem 3.12 implies that $B(k) = k!$ for $1 \leq k \leq 2m$ and there is a positive integer α such that $B(2m+1) = \alpha(2m+1)!$. We know that $D(k) = 2^{k-1}.(k-1)!$ for $1 \leq k \leq 2m$, so $D(2m+1) = B2^{2m-1}(2m-1)!$ and $D(2m+2) = AB2^{2m-1}(2m-1)!$. Since P is an Eulerian Sheffer poset, the Euler-Poincaré relation implies that,

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{17}$$

By substituting the values of the factorial functions we have,

$$2 - A + \frac{AB}{2}\left[\frac{1}{2m} - \frac{1}{2m(2m+1)} + \frac{2^{2m}}{2m(2m+1)} - \frac{2^{2m}}{2\alpha m(2m+1)}\right] = 0. \tag{18}$$

Thus,

$$A\left(1-B\left(\frac{2\alpha m+(\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right)\right)=2. \tag{19}$$

We can see that if $\alpha \geq 2$, the left side of Eq.(19) become negative, therefore $\alpha = 1$ and posets P and C_{2m+2} have the same binomial factorial functions. Since $2m+1 = A(2m+1) \leq C(2m+2) < \infty$, Lemma 4.11 implies that $4m-2 \leq C(2m+1) = B \leq 4m+1$. Since $\alpha = 1$, Eq.(19) implies that $2 - A + \frac{AB}{4m+2} = 0$. Therefore A and B should satisfy one of the following cases:

(1) $B = 4m-2$ and $A = 2m+1$.
(2) $B = 4m-1$ and $A = \frac{4}{3}(2m+1)$.
(3) $B = 4m$ and $A = 4m+2$.
(4) $B = 4m+1$ and $A = 4(2m+1)$.

As we have discussed in Corollary 4.8 as well as Lemma's 4.9 and 4.10, the cases (1), (2), (4) are not possible. The case (3) happens in the cubical lattice of rank $2m+2$, C_{2m+2}. Thus, P has same factorial functions as C_{2m+2}, as desired. □

Classification of the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6$ and $D(3) = 8$ is still remaining open. Let α be a positive integer and set $Q_\alpha = \boxplus^\alpha(C_{2m+1})$. It can be seen that Q_α is an Eulerian Sheffer poset and it has the following factorial functions $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n-1$, $D(n) = \alpha \cdot 2^{n-1}(n-1)!$ and $B(k) = k!$ for $1 \leq k \leq n-1$. We ask the following question:

Question: Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer α, such that P has the same factorial function as poset $Q_\alpha = \boxplus^\alpha(C_{2m+1})$, where C_{2m+1} is a cubical lattice of rank $2m+1$.

4.1.2. **Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$, and $D(3) = 3! = 6$.**

In this section, we prove the following:

Theorem 4.12. *Let P be an Eulerian Sheffer poset of rank $n \geq 3$ with $B(3) = D(3) = 3! = 6$ for 3-intervals. P satisfy one of the following cases:*

(i) n is an odd. There is an integer $k \geq 1$ such that $P = \boxplus^k(B_n)$.
(ii) n is an even. $P = B_n$.

Proof. We proceed by induction on n. Theorem 4.3 and Lemma 4.4 imply that this theorem holds for $n = 3, 4$. Assume that Theorem 4.12 holds for $n \leq m$, we wish to show that it also holds for $n = m+1 \geq 5$. This problem divides into the following cases:

(i) $n = m+1$ is odd. Consider poset Q which is obtained by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$. So Q is an Eulerian Sheffer poset with $B(3) = D(3) = 3! = 6$. By induction hypothesis, every intervals of rank $k \leq m$ isomorphic to B_k. So, the Sheffer and binomial factorial functions of Q the boolean lattice of rank $m+1$ are the same up to rank $m = n-1$. Therefore, Q and also P are binomial posets. Theorem 3.12 implies there is a positive integer k such that $P = \boxplus^k(B_n)$, as desired.

(ii) $n = m+1$ is even. We proceed by induction on n, the rank of P. Let $C(k)$ and $C^{'}(k) = k$ be the coatom functions of posets P and B_n, respectively, where $k \leq n$. By induction hypothesis $C(k) = C^{'}(k)$ for $k \leq n-2$. So, Lemma 4.11 implies that

$$\frac{1}{C(n-1)}\left(1-\frac{1}{C(n)}\right)=\frac{1}{C^{'}(n-1)}\left(1-\frac{1}{C^{'}(n)}\right). \tag{20}$$

By induction hypothesis, there is a positive integer α such that $C(n-1) = \alpha(n-1)$. Moreover, we know that $C^{'}(n-1) = n-1$ and $C^{'}(n) = n$. Eq.(20) implies that $\alpha = 1$ and $C(n) = n$, so poset P has the same factorial function as B_n and $P = B_n$, as desired.

□

4.1.3. **Characterization of the structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 4$**

Let P be an Eulerian Sheffer poset of rank $n \geq 5$, with $B(3) = 3!$ and $D(3) = 4$. In this section we show that in case $n = 2m+2$, $P = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$ for some $\alpha \geq 1$ and in case $n = 2m+1$, $P = \boxplus^{\alpha}(\Sigma^*(B_{2m}))$, for some $\alpha \geq 1$.

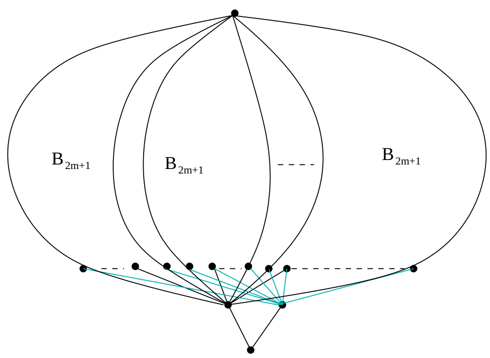


Fig. 4. $\Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$

Theorem 4.13. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 \geq 4$ with $B(3) = 3!$ and $D(3) = 4$. Then $P = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$, where $\alpha = \frac{B(2m+1)}{(2m+1)!}$ is positive integer, as a consequence P has the following binomial and Sheffer factorial functions:*

$B(k) = k!$ for $1 \leq k \leq 2m$, and $B(2m+1) = \alpha(2m+1)!$,

$D(1) = 1$, $D(k) = 2(k-1)!$ for $2 \leq k \leq 2m+1$, and $D(2m+2) = 2\alpha(2m+1)!$.

Proof. By Theorem 3.12, we know that there is a positive integer α such that P has the binomial factorial function $B(2m+1) = \alpha(2m+1)!$ and $B(k) = k!$, $1 \leq k < n = 2m+1$. We proceed by induction on m. The case $m = 1$ implies that $\alpha = 1$, $B(3) = 3!$ and $D(3) = 4$. By applying Lemma 4.4, it can be seen that the poset P has the same factorial functions as $\Sigma^* B_3$; therefore, poset P has two atoms and its binomial 3-intervals are isomorphic to B_3. We conclude that $P = \Sigma^* B_3$ and so Theorem 4.13 holds for $m = 1$. In case $m > 1$, by Theorem 3.12, $Q = \boxplus^{\alpha}(B_{2m+1})$ is the only Eulerian binomial poset of rank $2m+1$ with the binomial factorial functions $B(k) = k!$ for $1 \leq k \leq 2m$ and $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer. Set $P^{'} = \Sigma^* Q = \Sigma^* \boxplus^{\alpha}(B_{2m+1})$. It can be seen that $P^{'}$ is an Eulerian Sheffer poset of rank $2m+2$ with coatom functions $C^{'}(2m+2) = \alpha(2m+1)$ and $C^{'}(k) = (k-1)$ for $3 \leq k \leq 2m+1$ as well as $C^{'}(2) = 2$.

By induction hypothesis, the theorem holds for $m-1$ and $n = 2m$. We wish to show it also holds for m and $n = 2m+2$. Let $C(k)$, $2 \leq k \leq 2m+2$, be the coatom function of P of rank $2m+2$ which satisfies Theorem conditions. By induction hypothesis, $C(k) = 2(k-1)$ for $2 \leq k \leq 2m$. Lemma 4.11, implies the following Eq.(21)

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \tag{21}$$

By substituting the values of $C^{'}(2m+2)$ and $C^{'}(2m+1)$, we have

$$\frac{1}{C(2m+1)}\left(1-\frac{1}{C(2m+2)}\right)=\frac{1}{2m}\left(1-\frac{1}{\alpha(2m+1)}\right). \tag{22}$$

The poset P has the binomial factorial functions $B(2m+1) = \alpha(2m+1)!$, where α is a positive integer, and $B(k) = k!$ for $1 \le k < 2m+1$. We conclude that $A(2m+1) = \alpha(2m+1)$ and $A(2m) = 2m$. So $C(2m+2) \ge A(2m+1) = \alpha(2m+1)$ as well as $C(2m+1) \ge A(2m) = 2m$. Eq.(22) implies that $C(2m+1) = 2m$ and also $C(2m+2) = \alpha(2m+1)$. By induction hypothesis, $D(k) = 2(k-1)!$ for $2 \le k \le 2m$. Since $C(2m+1) = 2m$ as well as $C(2m+2) = \alpha(2m+1)$, we conclude that P has the same factorial functions as poset $P^{'} = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$.

Applying Eq.(2), P has $\frac{D(2m+2)}{B(2m+1)} = 2$ elements of rank 1, let us call them $\hat{0}_1$ and $\hat{0}_2$. Using Eq.(2), the number elements of rank $1 \le k \le 2m+1$ in posets $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{23}$$

The intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ both have the factorial functions, $B(k) = k!$ for $1 \le k \le 2m$ and $B(2m+1) = \alpha(2m+1)!$. It can be seen that the intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ satisfy the Euler-Poincaré relation and these intervals are Eulerian and binomial. Applying Theorem 3.12 implies that both intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ are isomorphic to the poset $\boxplus^{\alpha}(B_{2m+1})$. Since, P has the same factorial functions as poset $P^{'} = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$, Eq.(2) yields that the number of elements of rank $k+1$ in P is the same as the number of elements of rank k in intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ for $1 \le k \le 2m+1$, that is

$$\frac{\alpha(2m+1)!}{k!(2m+1-k)!}. \tag{24}$$

In summary, we have

(1) $[\hat{0}_1, \hat{1}] = [\hat{0}_2, \hat{1}] = Q = \boxplus^{\alpha}(B_{2m+1})$.
(2) The number of elements of rank $k+1$ in P is the same as the number of elements of rank k in intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$, $1 \le k \le 2m+1$.
(3) P has only two atoms $\hat{0}_1, \hat{0}_2$.

Statements (1), (2), (3) imply that $P = P^{'} = \Sigma^*(\boxplus^{\alpha}(B_{2m+1}))$, as desired.

□

Theorem 4.14. *Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \ge 5$ with $B(3) = 6$ and $D(3) = 4$. Then $P = \boxplus^{\alpha}(\Sigma^*(B_{2m}))$.*

Proof. We obtain the poset Q by adding $\hat{0}$ and $\hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$. It is easy to see that Q is an Eulerian Sheffer poset and also P and Q have the same factorial functions and coatom functions up to rank $2m$. That is $B_Q(k) = B_P(k)$ and $D_Q(k) = D_P(k)$ for $1 \le k \le 2m$. By $B(k)$, $D(k)$, $C(k)$, $A(k)$ we denote the factorial functions and the coatom functions and atom functions of Q.

By Theorem 3.11, Q has the binomial factorial functions $B(k) = k!$ for $1 \le k \le 2m$. We have $C(2m+1) \ge A(2m) = 2m$. Since every interval of rank 2 in the Q is isomorphic to B_2, Q has at least two coatoms. For every coatom such as a_i in Q, Theorem 4.13 imply that $[\hat{0}, a_i] = \Sigma^*(\boxplus^{\alpha}(B_{2m-1}))$, by considering the factorial functions we conclude that $\alpha = 1$ as well as $[\hat{0}, a_i] = \Sigma^*(B_{2m-1})$. Since Q is obtained by adding $\hat{0}, \hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, we conclude that there are at least two particular coatoms a_1, a_2 such that there is an element $c \in [\hat{0}, a_1]$, $[\hat{0}, a_2]$ where $c \neq \hat{0}$. By considering the interval $[c, \hat{1}]$ factorial functions, Theorems 3.11 and 3.12 imply that there is a positive integer k such that $[c, \hat{1}] = B_k$. Therefore, there is an element b of rank $k-2$ in $[c, \hat{1}]$ such that $b = a_1 \wedge a_2$, b is also an element of rank $2m-2$ in Q. The interval $[\hat{0}, b]$ is subinterval of $[\hat{0}, a_1]$, so $[\hat{0}, b] = \Sigma^*(B_{2m-2})$. We conclude $[\hat{0}, b]$ only has two atoms say x_1, x_2.

Since $[\hat{0}, a_1] = [\hat{0}, a_2] = \Sigma^*(B_{2m-1})$, so the intervals $[\hat{0}, a_1]$ and $[\hat{0}, a_2]$ only have two atoms x_1 and x_2.

Define a graph G_Q as follows; vertices of G_Q are coatoms of poset Q and two vertices (coatoms) a_i and a_j adjacent in G_Q if and only if there is an element $d \neq \hat{0}$ such that $d \in [\hat{0}, a_i], [\hat{0}, a_j]$. Since Q is obtained by adding $\hat{0}, \hat{1}$ to a connected component of $P - \{\hat{0}, \hat{1}\}$, G_Q is a connected graph. Thus, every coatom of rank $2m$ in Q is just above two atoms x_1, x_2 in Q. Hence the number of elements of rank 1 in poset Q is 2, and by Eq.(2),

$$\frac{C(2m+1)D(2m)}{B(2m)} = 2. \tag{25}$$

Thus, $C(2m+1) = 2m$ and also Q has the same factorial function as $\Sigma^*(B_{2m})$. By the same argument as Theorem 4.13, we conclude that $Q = \Sigma^*(B_{2m})$. So $P = \boxplus^\alpha(\Sigma^*(B_{2m}))$ for some positive integer α, as desired. □

4.2. Characterization of the structure and factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$

In this section, we characterize Eulerian Sheffer posets of rank $n \geq 5$ with $B(3) = 4$. Let P be an Eulerian Sheffer poset of rank $n \geq 5$ with $B(3) = 4$. It can be seen that the poset P satisfies one of the following cases:

(i) P has the following binomial factorial functions $B(k) = 2^{k-1}$, where $1 \leq k \leq n-1$;

(ii) n is even and there is a positive integer $\alpha > 1$ such that poset P has the binomial factorial functions $B(k) = 2^{k-1}$ for $1 \leq k \leq n-2$ and $B(n-1) = \alpha.2^{n-2}$.

As a consequence of Theorems 3.11 and 3.12 in [4], we characterize posets in the case (i).

Theorem 4.17 deals with the case (ii). It shows that if the Eulerian Sheffer posets P of rank $n = 2m+2$ has the binomial factorial functions $B(k) = 2^{k-1}$ for $1 \leq k \leq 2m$ and $B(2m+1) = \alpha.2^{2m}$, where $\alpha > 1$ is an integer, then $P = \Sigma^*\boxplus^\alpha(T_{2m+1})$. See Figure 5.

Given two ranked posets P and Q, define the rank product $P * Q$ by

$$P * Q = \{(x, z) \in P \times Q \ : \ \rho_P(x) = \rho_Q(z)\}.$$

Define the order relation by $(x, y) \leq_{P*Q} (z, w)$ if $x \leq_P z$ and $y \leq_Q w$. The rank product is also known as the Segre product; see [2].

Theorem 4.15. *[Consequence of Theorem 3.11 [4]] Let P be an Eulerian Sheffer poset of rank $n \geq 4$ with the binomial factorial functions $B(k) = 2^{k-1}$ for $1 \leq k \leq n-1$. Then its coatom function $C(k)$ and P satisfy the following conditions:*

(i) *$C(3) \geq 2$, and a length 3 Sheffer interval is isomorphic to a poset of the form $P_{q_1,\ldots,q_r}$, as described before.*

(ii) *$C(2k) = 2$, for $\lfloor \frac{n}{2} \rfloor \geq k \geq 2$ and the two coatoms in a length $2k$ Sheffer interval cover exactly the same element of rank $2k-2$.*

(iii) *$C(2k+1) = h$ is an even positive integer for $\lfloor \frac{n-1}{2} \rfloor \geq k \geq 2$. Moreover, the set of h coatoms in a Sheffer interval of length $2k+1$ partitions into $\frac{h}{2}$ pairs, $\{c_1, d_1\}, \{c_2, d_2\}, \ldots, \{c_{\frac{h}{2}}, d_{\frac{h}{2}}\}$, such that c_i and d_i cover the same two elements of rank $2k-1$*

Theorem 4.16. *[Consequence of Theorem 3.12 [4]] Let P be an Eulerian Sheffer poset of rank $n > 4$ with the binomial factorial functions $B(k) = 2^{k-1}$, $1 \leq k \leq n-1$ and the coatom functions $C(k)$, $1 \leq k \leq n$. Then a Sheffer k-interval $[\hat{0}, y]$ of P factors in the rank product as $[\hat{0}, y] \cong$*

$(T_{k-2} \cup \{\hat{0}, \hat{-1}\}) * Q$, *where* $T_{k-2} \cup \{\hat{0}, \hat{-1}\}$ *denotes the butterfly interval of rank* $k-2$ *with two new minimal elements attached in order, and* Q *denotes a poset of rank* k *such that*

(i) *each element of rank* 2 *through* $k-1$ *in* Q *is covered by exactly one element,*
(ii) *each element of rank* 1 *in* Q *is covered by exactly two elements,*
(iii) *each element of even rank* 4 *through* $2\lfloor \frac{k}{2} \rfloor$ *in* Q *covers exactly one element,*
(iv) *each element of odd rank* r *from* 5 *through* $2\lfloor \frac{k}{2} \rfloor + 1$ *in* Q *covers exactly* $\frac{C(r)}{2}$ *elements, and*
(v) *each* 3*-interval* $[\hat{0}, x]$ *in* Q *is isomorphic to a poset of the form* $P_{q_1,\ldots,q_r}$, *where* $q_1 + \cdots + q_r = C(3)$.

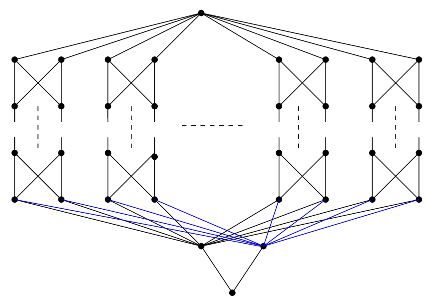


Fig. 5. $P = \Sigma^*(\boxplus^\alpha(T_{2m+1}))$

In the following theorem we study the only remaining case (*ii*)

Theorem 4.17. *Let* P *be an Eulerian Sheffer poset of even rank* $n = 2m+2 > 4$ *with the binomial factorial functions* $B(k) = 2^{k-1}$ *for* $1 \le k \le 2m$, *and* $B(2m+1) = \alpha \cdot 2^{2m}$, *where* $\alpha > 1$ *is a positive integer. Then* $P = \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.

Proof. Let $D(k)$, $1 \le k \le 2m+2$, and also $B(k)$, $1 \le k \le 2m+1$, be the Sheffer and binomial factorial functions of poset P, respectively. The Euler-Poincaré relation for interval of size $2m+2$ states as follows,

$$1 + \sum_{k=1}^{2m+2} (-1)^k \cdot \frac{D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{26}$$

The above Euler-Poincaré relation for the interval of even rank $2m+2$ can also be stated as follows,

$$\frac{2}{D(2m+2)} + \sum_{k=1}^{2m+1} \frac{(-1)^k}{D(k)B(2m+2-k)} = 0. \tag{27}$$

By expanding the left side of Eq.(27), we have:

$$\frac{(-1)}{\alpha \cdot 2^{2m}} + \sum_{k=2}^{2m+2} \frac{(-1)^k}{D(k) \cdot 2^{2m+2-k-1}} = 0. \tag{28}$$

Here, Eq.(27) for Sheffer $2m$-intervals can be stated as follows,

$$\sum_{k=1}^{2m} \frac{(-1)^k}{D(k) \cdot 2^{2m-1-k}} = 0. \tag{29}$$

Thus,

$$\frac{1}{2^{2m}} = \sum_{k=2}^{2m} \frac{(-1)^k}{D(k) \cdot 2^{2m+1-k}}. \tag{30}$$

It follows from Eq.(28) and Eq.(30) that

$$\frac{-1}{\alpha \cdot 2^{2m}} + \frac{1}{2^{2m}} + \frac{-1}{D(2m+1)} + \frac{2}{D(2m+2)} = 0. \tag{31}$$

Let k be the number of atoms in a Sheffer interval of size $2m+1$ and $c = C(2m+2)$, so $D(2m+1) = k \cdot 2^{2m-1}$ and $D(2m+2) = ck \cdot 2^{2m-1}$. Therefore

$$\frac{1}{2^{2m}} - \frac{1}{\alpha \cdot 2^{2m}} = \frac{1}{k \cdot 2^{2m-1}} - \frac{1}{\frac{ck}{2} \cdot 2^{2m-1}}. \tag{32}$$

Thus,

$$\frac{1}{2} - \frac{1}{2\alpha} = \frac{1}{k} - \frac{1}{(\frac{c}{2})k}. \tag{33}$$

Comparing coatom and atom functions of Sheffer and binomial intervals, we have $k \geq 2$ as well as $c \geq 2\alpha$. By Eq.(33), we conclude that $k = 2$ and $c = 2\alpha$. So $D(2m+1) = 2^{2m}$ and $D(2m+2) = \alpha \cdot 2^{2m+1}$. Since $B(2m+1) = \alpha \cdot 2^{2m}$ and $B(2m) = 2^{2m-1}$, the number of atoms in poset P is $\frac{D(2m+2)}{B(2m+1)} = 2$. Since the only Eulerian Sheffer interval of rank 2 is B_2, every Sheffer j-interval has two atoms for $1 \leq j \leq 2m+1$.

$D(k) = 2B(k-1) = 2^{k-1}$ for $2 \leq k \leq 2m+1$ as well as $D(2m+2) = \alpha \cdot 2^{2m+1}$. Let $\hat{0}_1$, $\hat{0}_2$ be atoms of P. By Theorem 3.12, both intervals $[\hat{0}_1, \hat{1}]$ and $[\hat{0}_2, \hat{1}]$ are isomorphic to the poset $Q = \boxplus^{\alpha}(T_{2m+1})$. It follows from Eq.(2) that the number of elements of rank $k-1$ in the intervals $Q = [\hat{0}_1, \hat{1}] = [\hat{0}_2, \hat{1}]$ is the same as the number of elements of rank k in poset P and it can be computed as follows,

$$\frac{D(2m+2)}{D(k)B(2m+2-k)} = \frac{B(2m+1)}{B(k)B(2m+1-k)}. \tag{34}$$

We know that $\hat{0}_1$,$\hat{0}_2$ are the only atoms in P, so by the above fact we conclude that $P = \Sigma^* Q = \Sigma^*(\boxplus^{\alpha}(T_{2m+1}))$, as desired.

□

5. Finite Eulerian triangular posets

As we discussed before, the larger class of posets to consider are triangular posets. For definitions regarding triangular posets, see Section 2. A non-Eulerian example of triangular poset is the the face lattice of the 4-dimensional regular polytope known as the 24-cell. In the following theorem, we characterize the Eulerian triangular posets of rank $n \geq 4$ such that $B(k, k+3) = 6$ for $1 \leq k \leq n-3$.

Theorem 5.1. *Let P be an Eulerian triangular poset of rank $n \geq 4$ such that for every $0 \leq k \leq n-3$, $B(k, k+3) = 6$. Then P can be characterized as follows:*
(i) n is odd, there is an integer $\alpha \geq 1$ such that $P = \boxplus^{\alpha}(B_n)$.
(ii) n is even, then $P = B_n$.

Proof. We proceed by induction on the rank of poset n.

- $n = 4$. A triangular poset of rank 4 is also a Sheffer poset. Since $B(1, 4) = 6$, by Lemma 4.4 we conclude that $P = B_4$.
- $n = 2m+1$. By induction hypothesis, every interval of rank $k \leq 2m$ in P is isomorphic to B_k. Hence P is a Sheffer poset and Theorem 4.12 implies that $P = \boxplus^{\alpha}(B_n)$, where $\alpha \geq 1$ is a positive integer.
- $n = 2m+2$. Let r and t be the number of elements of rank 1 and $2m+1$ in P. By induction hypothesis, there are positive integers k_t and k_r such that $B(1, 2m+2) = k_t(2m+1)!$ and $B(0, 2m+1) = k_r(2m+1)!$. Therefore, $B(0, 2m+2) = tk_r(2m+1)! = rk_t(2m+1)!$ and also

$B(n, n+k) = k!$, where $1 \leq k \leq 2m+1-n$ and $n \geq 1$. The Euler-Poincaré relation for interval of size $2m+2$ state as follows,

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k B(0, 2m+2)}{B(0,k)B(k, 2m+2)} = 0. \tag{35}$$

By substituting the values in Eq.(35), we have

$$1 + tk_r \left(\sum_{k=2}^{2m} \frac{(-1)^k (2m+1)!}{k!(2m+2-k)!} \right) + \frac{-tk_r(2m+1)!}{k_t(2m+1)!} + \frac{-tk_r(2m+1)!}{k_r(2m+1)!} + 1 = 0. \tag{36}$$

Eq.(36) lead us to

$$2 - t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r}\right) + tk_r \left(\sum_{k=2}^{2m} \left(\frac{(-1)^k (2m+1)!}{k!(2m+2-k)!} \right) \right) = 0, \tag{37}$$

so,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right). \tag{38}$$

Without loss of generality, let us assume that $k_r \geq k_t \geq 1$. Therefore,

$$2 = t\left(\frac{k_r}{k_t} + \frac{k_r}{k_r} + \frac{-(k_r)(4m+2)}{2m+2}\right) \leq t\left(k_r + 1 - \left(\frac{4m+2}{2m+2}\right) k_r\right) \leq t\left(1 - \frac{2m}{2m+2} k_r\right). \tag{39}$$

The right-hand side of the above equation is positive only if $k_r = 1$. So $k_r = 1$ and since $k_r \geq k_t \geq 1$, we conclude that $k_t = 1$. Therefore, $2 = t\frac{2}{2m+2}$ and so $t = 2m+2$. Similarly, we conclude that $r = 2m+2$. Thus, P has the same factorial function as B_{2m+2} and by Proposition 3.8, this poset is isomorphic to B_{2m+2}, as desired.

□

6. Conclusions and remarks

An interesting research problem is to classify the factorial functions of Eulerian triangular posets. It is also interesting to classify Eulerian trianguler posets with specific factorial functions on their smaller intervals. In Theorem 5.1, we characterize the Eulerian triangular posets of rank $n \geq 4$ such that $B(k, k+3) = 6$, for $1 \leq k \leq n-3$.

Readers can find the following result of Stanley. A graded poset P is a boolean lattice if every 3-interval is a boolean lattice and for every $[x, y]$ of rank of at least 4 the open interval (x, y) is connected (See [7], Lemma 8). Using Stanley's result, it might be possible to obtain different proofs for Theorems 3.11, 3.12, 4.12 and 5.1.

This research is motivated by the above result of Stanley. We characterize Eulerian binomial and Sheffer posets by considering the factorial functions of 3-intervals. The project of studying Eulerian Sheffer posets is almost complete. Only the following cases remain to be studied:

- Finite Eulerian Sheffer posets of odd rank with $B(3) = 6, D(3) = 8$. In this case we ask the following question: Let P be a Eulerian Sheffer poset of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer k such that P has the same factorial function as the poset $Q_k = \boxplus^k(C_{2m+1})$?
- Finite Eulerian Sheffer posets of rank 5 with $B(3) = 6, D(3) = 10$. We conjecture that there is no poset with these conditions.

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References

[1] J. Backelin, Binomial posets with non-isomorphic intervals, arXiv math.CO/0508397, 22 August, 2005.
[2] A. Bjorner, V. Welker,Segre and Rees products of posets, with ring-theoretic applications, J. Pure Appl. Algebra, 198 (2005), 43-55.
[3] P. Doubilet, G.-C.Rota, R. Stanley, On the foundations of combinatorial theory (VI). The idea of generating functions, in Sixth Berkeley Symp. on Math. Stat. and Prob., vol. 2: Probability Theory, Univ. of California, Berkeley, 1972, pp. 267-318.
[4] R. Ehrenborg and M. Readdy, Classification of the factorial functions of Eulerian binomial and Sheffer posets, J. Combin. Theory Ser.A 114 (2007) 339-359.
[5] R. Ehrenborg and M. Readdy, Sheffer posets and r-signed permutations, Ann. Sci. Math. Québec 19 (1995) 173196.
[6] J. Farley and S. Schmidt, Posets that locally resemble distributive lattices, J. Combin. Theory Ser. A 92 (2000) 119-137.
[7] D. J. Grabiner, Posets in which every interval is a product of chains, and natural local actions of the symmetric group, Discrete Math. 199 (1999) 77-84.
[8] G. Hetyei, Matrices of formal power series associated to binomial posets. J. Algebraic Combin. 22 (2005), no. 1, 65–104.
[9] V. Reiner, Upper binomial posets and signed permutation statistics, Eouropean J. Combin. 14(1993) 581-588.
[10] R. Simion, R. Stanley, Flag-symmetry of the poset of shuffles and a local action of the symmetric group, Discrete Mathematics 204 (1999) 369-396.
[11] R. Stanley, Binomial posets, Möbius inversion, and permutation enumeration, J.Combin.Theory Ser.A 20 (1976) 336-356.
[12] R. Stanley, Enumerative Combinatorics, vol.I, Wadsworth and Brooks/Cole, Pacific Grove, 1986.
[13] R. Stanley, Flag-symmetric and locally rank-symmetric partially ordered sets, Electron. J. Combin. 3 (1996).