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Restricted colored permutations and Chebyshev polynomials

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Abstract

Several authors have examined connections between restricted permutations and Chebyshev polynomials of the second kind. In this paper we prove analogues of these results for colored permutations. First we define a distinguished set of length two and length three patterns, which contains only 312 when just one color is used. Then we give a recursive procedure for computing the generating function for the colored permutations which avoid this distinguished set and any set of additional patterns, which we use to find a new set of signed permutations counted by the Catalan numbers and a new set of signed permutations counted by the large Schröder numbers. We go on to use this result to compute the generating functions for colored permutations which avoid our distinguished set and any layered permutation with three or fewer layers. We express these generating functions in terms of Chebyshev polynomials of the second kind and we show that they are special cases of generating functions for involutions which avoid 3412 and a layered permutation.

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1. Introduction and notation

Let c denote a nonnegative integer and let CS_n denote the set of permutations of $\{1, 2, \ldots, n\}$, written in one-line notation, in which each element has an associated *color* from among the integers $0, 1, \ldots, c$. We refer to the elements of CS_n as *colored permutations*, and we write the colors of their entries as exponents, as in $2^3 3^1 1^0$ and $2^0 1^0$. For each $\pi \in CS_n$ and each i, $1 \leqslant i \leqslant n$, we write $\pi(i)$ to denote the ith entry of π. When $c = 0$ we identify CS_n with the set S_n of ordinary permutations, and we omit the color. When $c = 1$ we identify CS_n with the set B_n of *signed permutations*, and we sometimes omit the color 0 and replace the color 1 with an overbar.

Suppose π and σ are colored permutations. We say a subsequence of π has *type* σ whenever it has all of the same pairwise comparisons as σ and each entry of the subsequence of π has the same color as the corresponding entry of σ. For example, the subsequence $2^1 8^0 6^2 9^1$ of the colored permutation $2^1 1^0 4^0 5^2 3^1 8^0 7^0 6^2 9^1$ has type $1^1 3^0 2^2 4^1$. We say π *avoids* σ whenever π has no subsequence of type σ. For example, the colored permutation $2^1 1^0 4^0 5^2 3^1 8^0 7^0 6^2 9^1$ avoids $3^1 1^1 2^0$ and $1^1 3^2 2^2$, but it has $2^1 8^0 6^2$ as a subsequence so it does not avoid $1^1 3^0 2^2$. In this setting (and especially when $c = 0$) σ is sometimes called a *pattern* or a *forbidden subsequence* and π is sometimes called a *restricted permutation* or a *pattern-avoiding permutation*. In this paper we will be interested in colored permutations which avoid several

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patterns, so for any set R of colored permutations we write $CS_n(R)$ to denote the set of colored permutations of length n which avoid every pattern in R and we write $CS(R)$ to denote the set of all colored permutations which avoid every pattern in R. When $R=\{\pi_1,\dots,\pi_r\}$ we often write $CS_n(R)=CS_n(\pi_1,\dots,\pi_r)$ and $CS(R)=CS(\pi_1,\dots,\pi_r)$. When we wish to discuss ordinary permutations or signed permutations, respectively, we replace CS with S or B in the above notation.

As several authors have shown, generating functions for $S_n(132,\pi)$ for various π can be computed recursively, and can often be expressed nicely in terms of Chebyshev polynomials of the second kind. For example, Mansour and Vainshtein have given [11, Theorem 2.1] the following recursive formula for $f_\pi(x)=\sum_{n=0}^{\infty}|S_n(132,\pi)|x^n$, which makes it possible to compute $f_\pi(x)$ for any π which avoids 132:

$$f_\pi(x)=1+x\sum_{j=0}^{r}\left(f_{\pi^j}(x)-f_{\pi^{j-1}}(x)\right)f_{\sigma^j}(x). \tag{1}$$

Here π^{j-1}, π^j, and σ^j are the types of certain subsequences of π. Moreover, several authors [2,7,10] have shown that for all $k\geqslant 1$,

$$f_{k(k-1)\dots 21}(x)=\frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}. \tag{2}$$

Here $U_n(x)$ is the nth Chebyshev polynomial of the second kind, which may be defined by $U_n(\cos x)=\sin((n+1)t)/\sin t$. For additional results along these lines, see [3,5,12,14].

Although some results concerning pattern avoidance in colored permutations are known (see [9], for instance), the topic has not received as much attention as has pattern avoidance in ordinary permutations. In this paper we prove analogues of (1) and (2) and several similar results for pattern-avoiding colored permutations. In particular, for each nonnegative integer c, let P_c denote the set consisting of all patterns of the form 2^a1^b where $0\leqslant a\leqslant c$ and $1\leqslant b\leqslant c$, together with all patterns of the form $3^a1^02^0$ where $0\leqslant a\leqslant c$. Observe that $P_0=\{312\}$, which is the complement of 132. We prove that if $F_\pi(x)=\sum_{n=0}^{\infty}|CS_n(P_c,\pi)|x^n$ then

$$F_\pi(x)=1+cxF_\beta(x)+x\sum_{i=1}^{k}(F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_i}}(x)-F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}}(x))F_{\alpha_i\oplus\cdots\oplus\alpha_k}(x), \tag{3}$$

where the various subscripts of F on the right are the type of certain subsequences of π, which are defined (along with the operator $\oplus$) in the next section. The recurrence in (3), which is an analogue of (1), allows one to compute $F_\pi(x)$ for any colored permutation π. For instance, using (3) we prove that for all $k\geqslant 1$,

$$F_{k(k-1)\dots 21}(x)=\frac{U_{k-1}\left(\frac{1-cx}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1-cx}{2\sqrt{x}}\right)}.$$

Building on this result, which is an analogue of (2), we also show that

$$F_{[k,l]}(x)=F_{[k+l]}(x)$$

and

$$F_{[l_1,l_2,l_3]}(x)=\frac{V_{l_1+l_2+l_3}V_{l_1+l_2+l_3-1}+V_{l_1+l_2}V_{l_1+l_3}V_{l_2+l_3}}{\sqrt{x}V_{l_1+l_2-1}V_{l_1+l_3-1}V_{l_2+l_3-1}},$$

where $[l_1,\dots,l_m]$ is the layered permutation given by

$$l_1,l_1-1,\dots,1,l_2+l_1,l_2+l_1-1,\dots,l_1+1,\dots,\sum_{i=1}^{m}l_i,\sum_{i=1}^{m}l_i-1,\dots,\sum_{i=1}^{m-1}l_i+1$$

and we abbreviate $V_n = U_n((1-cx)/2\sqrt{x})$. We have not found quite as nice a form for the generating function $F_{[l_1,\ldots,l_m]}(x)$ when $m \geqslant 4$, but we conjecture that $F_{[l_1,\ldots,l_m]}(x)$ is symmetric in $l_1, \ldots, l_m$ for all $m \geqslant 1$ and all $l_1, \ldots, l_m \geqslant 1$. We have verified this conjecture for $m = 4$ and $l_i \leqslant 10$, for $m = 5$ and $l_i \leqslant 6$, for $m = 6$ and $l_i \leqslant 4$, and for $m = 7$ and $l_i \leqslant 3$ using a Maple program.

To state the last of our main results, recall that $|S_n(312)| = C_n$ for all $n \geqslant 0$, where C_n is the nth Catalan number, which may be defined by setting $C_0 = 1$ and

$$C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k} \quad (n \geqslant 1).$$

We can generalize the Catalan numbers by defining, for each $c \geqslant 0$, the c-Schröder numbers $r_n(c)$ by setting $r_0(c) = 1$ and

$$r_n(c) = c r_{n-1}(c) + \sum_{k=1}^{n} r_{k-1}(c) r_{n-k}(c) \quad (n \geqslant 1). \tag{4}$$

Observe that for all $n \geqslant 0$ we have $r_n(0) = C_n$ and $r_n(1) = r_n$, the nth large Schröder number. Using (4), we routinely find that if $R_c(x) = \sum_{n=0}^{\infty} r_n(c) x^n$ then

$$R_c(x) = 1 + cx R_c(x) + x R_c^2(x) \tag{5}$$

and

$$R_c(x) = \frac{1 - cx - \sqrt{c^2x^2 - (2c+4)x + 1}}{2x}. \tag{6}$$

Using a simpler version of the analysis we employ to prove (3), we show that for all $c \geqslant 0$ we have $|CS_n(P_c)| = r_n(c)$. When $c = 0$ this reduces to the fact that $|S_n(312)| = C_n$, and when we set $c = 1$ we find that the signed permutations which avoid $2\bar{1}$, $\overline{21}$, 312, and $\bar{3}12$ are counted by the large Schröder numbers. For more information concerning pattern-avoiding permutations counted by the Schröder numbers, see [4,8,15].

2. A recurrence relation

For each $c \geqslant 0$, let P_c denote the set consisting of all patterns of the form $2^a 1^b$ where $0 \leqslant a \leqslant c$ and $1 \leqslant b \leqslant c$, together with all patterns of the form $3^a 1^0 2^0$ where $0 \leqslant a \leqslant c$. For example, $P_0 = \{312\}$ and $P_1 = \{2\bar{1}, \overline{21}, \bar{3}12, 312\}$. For any set T of colored permutations we write $F_T(x)$ to denote the generating function given by

$$F_T(x) = \sum_{n=0}^{\infty} |CS_n(P_c, T)| x^n.$$

Observe that every permutation contains the empty subsequence, so $F_\varepsilon(x) = 0$, where ε is the empty permutation. In addition, note that if $\pi \in CS_n$ avoids 1^0 then π contains no entries of color 0. If π also avoids P_c then π can have no decreases of any color combination, but π may have the form $1^{a_1} \cdots n^{a_n}$ for any colors $a_1, \ldots, a_n$ with $1 \leqslant a_i \leqslant c$. Therefore $|CS_n(P_c, 1^0)| = c^n$ and $F_{1^0}(x) = 1/(1-cx)$. In this section we prove a recurrence relation which allows one to compute $F_T(x)$ for any T, given these two initial values.

To state our recurrence relation, we first need some notation concerning a few simple ways colored permutations can be put together and taken apart. In particular, suppose $\pi \in CS_m$ and $\sigma \in CS_n$. We write $\pi \oplus \sigma$ to denote the colored permutation in CS_{m+n} given by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) + m & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

and we refer to $\pi \oplus \sigma$ as the *direct sum* of π and σ. We call a colored permutation π *direct sum indecomposable* whenever there do not exist nonempty colored permutations π_1 and π_2 such that $\pi = \pi_1 \oplus \pi_2$, and we observe that every colored permutation π has a unique decomposition $\pi = \alpha_1 \oplus \cdots \oplus \alpha_k$ in which $\alpha_1, \ldots, \alpha_k$ are direct sum indecomposable. Along the same lines, we write $\pi \ominus \sigma$ to denote the colored permutation in CS_{m+n} given by

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + n & \text{if } 1 \leqslant i \leqslant m, \\ \sigma(i-m) & \text{if } m+1 \leqslant i \leqslant m+n, \end{cases}$$

and we refer to $\pi \ominus \sigma$ as the *skew sum* of π and σ. We will find it useful to combine the direct and skew sums by writing $\pi * \sigma$ to denote the colored permutation in CS_{m+n+1} given by

$$\pi * \sigma = (\pi \ominus 1^0) \oplus \sigma.$$

Finally, if π is a colored permutation such that $\pi = \pi_1 \ominus 1^0$ for some colored permutation π_1 then we write $\overline{\pi} = \pi_1$. If π does not have this form then we set $\overline{\pi} = \pi$.

Example 2.1. Set $c = 2$. If $\pi = 4^0 2^1 1^0 3^2$ and $\sigma = 2^2 4^0 1^1 3^1$ then

$$\pi \oplus \sigma = 4^0 2^1 1^0 3^2 6^2 8^0 5^1 7^1,$$

$$\pi \ominus \sigma = 8^0 6^1 5^0 7^2 2^2 4^0 1^1 3^1,$$

and

$$\pi * \sigma = 5^0 3^1 2^0 4^2 1^0 7^2 9^0 6^1 8^1.$$

To prove our recurrence we will need the following result concerning the structure of those colored permutations which avoid P_c.

Lemma 2.2. *Fix $c \geqslant 0$ and let σ denote a colored permutation in which* 1 *has color b.*

(i) *Suppose $b > 0$. Then $\sigma \in CS(P_c)$ if and only if $\sigma = 1^b \oplus \sigma_1$ for some $\sigma_1 \in CS(P_c)$.*
(ii) *Suppose $b = 0$. Then $\sigma \in CS(P_c)$ if and only if $\sigma = \sigma_1 * \sigma_2$ for some $\sigma_1, \sigma_2 \in CS(P_c)$.*

Proof. (i) First observe that if $\sigma \in CS(P_c)$ does not begin with 1^b then $\sigma(1)1^b$ is a subsequence of type $2^a 1^b$, where a is the color of $\sigma(1)$. This is a forbidden subsequence, so every element of $CS(P_c)$ in which 1 has color $b > 0$ begins with 1^b, and thus has the form $1^b \oplus \sigma_1$ for some $\sigma_1 \in CS(P_c)$. Since no element of P_c begins with 1^b, the fact that $\sigma_1 \in CS(P_c)$ implies $1^b \oplus \sigma_1 \in CS(P_c)$, and (i) follows.

(ii) Suppose $\sigma \in CS(P_c)$ and there are elements x, y of σ such that x is to the left of 1, y is to the right of 1, and $x > y$. If the color of y is not 0 then xy is a forbidden subsequence of type $2^a 1^b$, where a is the color of x and b is the color of y. If the color of y is 0 then $x1y$ is a forbidden subsequence of type $3^a 1^0 2^0$, where a is the color of x. Therefore every element of σ to the left of 1 is less than every element of σ to the right of 1 and it follows that $\sigma = \sigma_1 * \sigma_2$ for $\sigma_1, \sigma_2 \in CS(P_c)$. It is routine to verify that if $\sigma_1, \sigma_2 \in CS(P_c)$ then $\sigma_1 * \sigma_2 \in CS(P_c)$, and (ii) follows. $\square$

Lemma 2.2 allows us to find the cardinality of $CS_n(P_c)$.

Proposition 2.3. *For all $n \geqslant 0$ and all $c \geqslant 0$,*

$$|CS_n(P_c)| = r_n(c). \tag{7}$$

Proof. The set $CS(P_c)$ can be partitioned into three sets: the set A_1 containing only the empty permutation, the set A_2 of those colored permutations in which the color of 1 is positive, and the set A_3 of those colored permutations in which the color of 1 is 0.

Using Lemma 2.2, we find that the generating functions for these sets are 1, $cxF_\emptyset(x)$, and $xF_\emptyset^2(x)$, respectively. Add these generating functions to obtain

$$F_\emptyset(x) = 1 + cxF_\emptyset(x) + xF_\emptyset^2(x).$$

Compare this with (5) to conclude that $F_\emptyset(x) = R_c(x)$, and the result follows. $\square$

Observe that when we set $c=0$ in (7) we recover the well-known result that $|S_n(312)|=C_n$ for $n\geqslant 0$. When we set $c=1$ in (7) we obtain the following new result:

$$|B_n(2\overline{1},\overline{21},312,\overline{3}12)|=r_n \quad (n\geqslant 0). \tag{8}$$

Now that we have found $|CS_n(P_c)|$, we turn our attention to the promised recurrence for $F_T(x)$. We begin with the case in which T contains just one element.

Theorem 2.4. *Fix $c\geqslant 0$ and suppose $\pi=\alpha_1\oplus\cdots\oplus\alpha_k$ is a colored permutation, where $\alpha_1,\ldots,\alpha_k$ are direct sum indecomposable. Then*

$$F_\pi(x)=1+cxF_\beta(x)+x\sum_{i=1}^{k}(F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_i}}(x)-F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}}(x))F_{\alpha_i\oplus\cdots\oplus\alpha_k}(x). \tag{9}$$

Here $\beta=\alpha_2\oplus\cdots\oplus\alpha_k$ if $\alpha_1=1^a$ and $a>0$, and $\beta=\pi$ otherwise.

Proof. The set $CS(P_c,\pi)$ can be partitioned into three sets: the set A_1 containing only the empty permutation, the set A_2 of those colored permutations in which the color of 1 is positive, and the set A_3 of those colored permutations in which the color of 1 is 0.

The generating function for A_1 is 1.

In view of Lemma 2.2(i), the generating function for A_2 is $cxF_\beta(x)$, where $\beta=\alpha_2\oplus\cdots\oplus\alpha_k$ if $\alpha_1=1^a$ and $a>0$, and $\beta=\pi$ otherwise.

To obtain the generating function for A_3, we first observe that in view of Lemma 2.2(ii), all elements of A_3 have the form $\sigma_1*\sigma_2$ for unique $\sigma_1,\sigma_2\in CS(P_c)$. Since each α_i is direct sum indecomposable, if $\sigma_1*\sigma_2$ contains a subsequence of type α_i then that subsequence is entirely contained in either $\sigma_1\ominus 1^0$ or σ_2. As a result, A_3 can be partitioned into sets $B_1,\ldots,B_k$, where B_i is the set of those colored permutations in A_3 in which σ_1 contains $\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}$ but avoids $\overline{\alpha_1\oplus\cdots\oplus\alpha_i}$. Now observe that if $\sigma_1*\sigma_2\in B_i$ then σ_2 avoids $\alpha_i\oplus\cdots\oplus\alpha_k$, since otherwise $\sigma_1*\sigma_2$ would contain $(\alpha_1\oplus\cdots\oplus\alpha_{i-1})\oplus(\alpha_i\oplus\cdots\oplus\alpha_k)=\pi$. Conversely, note that if σ_1 contains $\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}$ but avoids $\overline{\alpha_1\oplus\cdots\oplus\alpha_i}$ and σ_2 avoids $\alpha_i\oplus\cdots\oplus\alpha_k$ then $\sigma_1*\sigma_2$ avoids π. It follows that the generating function for A_3 is $\sum_{i=1}^k G_i(x)F_{\alpha_i\oplus\cdots\oplus\alpha_k}(x)$, where $G_i(x)$ is the generating function for those permutations in $CS(P_c)$ which contain $\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}$ but avoid $\overline{\alpha_1\oplus\cdots\oplus\alpha_i}$. Since $CS_n(P_c,\alpha_1\oplus\cdots\oplus\alpha_{i-1})\subseteq CS_n(P_c,\alpha_1\oplus\cdots\oplus\alpha_i)$, we have $G_i(x)=x(F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_i}}(x)-F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}}(x))$, and we find that the generating function for A_3 is $x\sum_{i=1}^k(F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_i}}(x)-F_{\overline{\alpha_1\oplus\cdots\oplus\alpha_{i-1}}}(x))F_{\alpha_i\oplus\cdots\oplus\alpha_k}(x)$.

Add the generating functions for A_1, A_2, and A_3 to obtain (9). □

Observe that when we set $c=0$ in Theorem 2.4 we recover [11, Theorem 2.1].

In order to state our recurrence relation for $F_T(x)$ when T has more than one element, we first need some additional notation.

Definition 2.5. Fix $c\geqslant 0$, let $T=\{\pi_1,\ldots,\pi_m\}$ denote a set of colored permutations and fix direct sum indecomposable permutations α^i_j, $1\leqslant i\leqslant m$, $1\leqslant j\leqslant k_i$, such that $\pi_i=\alpha^i_1\oplus\cdots\oplus\alpha^i_{k_i}$. For all $i_1,\ldots,i_m$ such that $0\leqslant i_j\leqslant k_j$, let $T^{\text{right}}_{i_1,\ldots,i_m}=\{\alpha^1_{i_1}\oplus\cdots\oplus\alpha^1_{k_1},\ldots,\alpha^m_{i_m}\oplus\cdots\oplus\alpha^m_{k_m}\}$. For any subset $Y\subseteq\{1,\ldots,m\}$, set

$$T_Y=\bigcup_{j\in Y}\{\overline{\alpha^j_1\oplus\cdots\oplus\alpha^j_{i_j-1}}\}\bigcup_{j\notin Y,1\leqslant j\leqslant m}\{\overline{\alpha^j_1\oplus\cdots\oplus\alpha^j_{i_j}}\}.$$

The general recurrence relation is an application of the inclusion–exclusion principle.

Theorem 2.6. *With reference to Definition* 2.5,

$$F_T(x)=1+cxF_{\beta(T)}(x)+x\sum_{i_1,\ldots,i_m=1}^{k_1,\ldots,k_m}\left(\sum_{Y\subseteq\{1,2,\ldots,m\}}(-1)^{|Y|}F_{T_Y}(x)\right)F_{T^{\text{right}}_{i_1,\ldots,i_m}}(x). \tag{10}$$

Here $\beta(\pi_i)=\alpha_2^i\oplus\cdots\oplus\alpha_{k_i}^i$ *if* $\alpha_1^i=1^a$ *and* $a>0$, *and* $\beta(\pi_i)=\pi_i$ *otherwise. Moreover,* $\beta(T)$ *is the set of permutations obtained by applying* β *to every element of* T.

We omit the proof of Theorem 2.6 for the sake of brevity.

Theorems 2.4 and 2.6 allow us to enumerate many sets of pattern-avoiding colored permutations. We conclude this section with some of the enumerations which follow from these results. The first of these is a new occurrence of the Catalan numbers.

Corollary 2.7. *For all* $n\geqslant 0$ *we have*

$$|B_n(\overline{21},2\overline{1},\overline{2}1,312)|=C_{n+1}.$$

Proof. First observe that $B_n(P_1,\overline{2}1)=B_n(\overline{21},2\overline{1},\overline{2}1,312)$ for all $n\geqslant 0$, so we compute $F_{\overline{2}1}(x)$. To do this, set $c=1$ and $\pi=\overline{2}1$ in (9) to obtain

$$F_{\overline{2}1}(x)=1+xF_{\overline{2}1}(x)+xF_{\overline{1}}(x)F_{\overline{2}1}(x). \tag{11}$$

Observe that $B_n(\overline{21},2\overline{1},312,\overline{3}12,\overline{1})=S_n(312)$, so $F_{\overline{1}}(x)=(1-\sqrt{1-4x})/2x$. Use this to eliminate $F_{\overline{1}}(x)$ in (11) and solve the resulting equation for $F_{\overline{2}1}(x)$ to obtain

$$F_{\overline{2}1}(x)=\frac{\dfrac{1-\sqrt{1-4x}}{2x}-1}{x}.$$

Since $\sum_{n=0}^{\infty}C_nx^n=\frac{1-\sqrt{1-4x}}{2x}$ and $C_0=1$, the result follows. $\square$

Using the same techniques one can generalize Corollary 2.7 by showing that the number of colored permutations of $\{1,2,\ldots,n\}$ with colors $0,1,\ldots,c$ which avoid P_c and 2^11^0 is $r_{n+1}(c-1)/c$. As an aside, it follows that $r_n(c)$ is divisible by $c+1$ for all $c\geqslant 0$ and all $n\geqslant 1$. By setting $c=2$ we find a new set of colored permutations counted by the little Schröder numbers.

Next we give a new proof of an old occurrence of the Fibonacci numbers.

Corollary 2.8 (*Mansour and West [13, Eq. (3.5)]*). *For all* $n\geqslant 0$ *we have*

$$|B_n(2\overline{1},\overline{21},12)|=F_{2n+1}.$$

Here F_n *is the nth Fibonacci number, defined by* $F_0=0$, $F_1=1$, *and* $F_n=F_{n-1}+F_{n-2}$ *for all* $n\geqslant 2$.

Proof. This is similar to the proof of Corollary 2.7, with $c=1$ and $\pi=12$. $\square$

We conclude this section with an enumeration involving powers of 2.

Corollary 2.9. *For all* $n\geqslant 0$, *the number of colored permutations of* $\{1,2,\ldots,n\}$ *with colors* $0,1,2$ *which avoid* 2^01^0, 2^01^1, 2^11^1, 2^21^1, 2^01^2, 2^11^2, 2^21^2, $3^11^02^0$, *and* $3^21^02^0$ *is* $(2^{2n+1}+1)/3$.

Proof. This is similar to the proof of Corollary 2.7, with $c=2$ and $\pi=2^01^0$. $\square$

The sequence which appears in Corollary 2.9 is sequence A007583 in the Encyclopedia of Integer Sequences. This sequence is known to have several other combinatorial interpretations; for instance, its nth term is the number of walks of length $2n+1$ between adjacent vertices in the cycle graph C_6.

3. Colored permutations and Chebyshev polynomials

In this section we use (9) to find $F_\pi(x)$ for certain π. In each case we express $F_\pi(x)$ in terms of Chebyshev polynomials of the second kind, generalizing the results of Chow and West [2], Krattenthaler [7], and Mansour and Vainshtein [11] for permutations which avoid 132. However, we obtain our results in a new way, by relating our generating functions with generating functions for involutions which avoid 3412.

We begin by recalling the Chebyshev polynomials of the second kind.

Definition 3.1. For all $n \geqslant -1$, we write $U_n(x)$ to denote the *nth Chebyshev polynomial of the second kind*, which is defined by $U_n(\cos t) = \sin((n+1)t)/\sin t$. Recall that these polynomials satisfy $U_{-1}(x) = 0$, $U_0(x) = 1$, and

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

Throughout this section we will focus on $F_\pi(x)$ when π is a layered permutation; we recall layered permutations next.

Definition 3.2. For all $n \geqslant 0$ and all $c \geqslant 0$, set $[n] = n^0(n-1)^0 \ldots 2^0 1^0$. For any sequence $l_1, \ldots, l_m$ of positive integers we write $[l_1, \ldots, l_m] = [l_1] \oplus \cdots \oplus [l_m]$. We call a colored permutation *layered* whenever it has the form $[l_1, \ldots, l_m]$ for some sequence $l_1, \ldots, l_m$.

Observe that $\overline{[1]} = \emptyset$, $\overline{[n]} = [n-1]$ for $n \geqslant 2$, and $\overline{[l_1, \ldots, l_m]} = [l_1, \ldots, l_m]$ for $m \geqslant 2$.

As we will see, the generating function $F_{[l_1,\ldots,l_m]}(x)$ can be neatly expressed in terms of Chebyshev polynomials of the second kind for any layered permutation $[l_1, \ldots, l_m]$. To obtain these expressions, we exploit a new connection between $F_{[l_1,\ldots,l_m]}(x)$ and certain generating functions for involutions which avoid 3412. To describe this connection, we first recall some results concerning these latter generating functions.

Recall that an involution π is a permutation such that $\pi(\pi(i)) = i$ for all i, let I_n denote the set of involutions of length n, and let $I_n(\sigma_1, \ldots, \sigma_k)$ denote the set of involutions in I_n which avoid $\sigma_1, \ldots, \sigma_k$. For any permutation π let $G_\pi(x)$ be given by

$$G_\pi(x) = \sum_{n=0}^{\infty} |I_n(3412, \pi)| x^n. \tag{12}$$

Egge has shown [3, Corollary 5.6] that $G_\pi(x)$ satisfies a recurrence relation which is similar to (9). When $\pi = [l_1, \ldots, l_m]$ this recurrence may be written as

$$\begin{aligned} G_{[l_1,\ldots,l_m]}(x) = 1 &+ xG_\beta(x) + x^2 G_{[l_1-2]}(x) G_{[l_1,\ldots,l_m]}(x) \\ &+ x^2 G_{[l_1,l_2]}(x) G_{[l_2,\ldots,l_m]}(x) - x^2 G_{[l_1-2]}(x) G_{[l_2,\ldots,l_m]}(x) \\ &+ x^2 \sum_{i=3}^{m} (G_{[l_1,\ldots,l_i]}(x) - G_{[l_1,\ldots,l_{i-1}]}(x)) G_{[l_i,\ldots,l_m]}(x). \end{aligned} \tag{13}$$

where $\beta = [l_2, \ldots, l_m]$ if $l_1 = 1$ and $\beta = [l_1, \ldots, l_m]$ otherwise. As we show next, this recurrence enables us to express $F_{[l_1,\ldots,l_m]}(x)$ in terms of $G_{[2l_1,\ldots,2l_m]}(x)$.

Theorem 3.3. *Fix $c \geqslant 0$. For all $m > 0$ and all $l_1, \ldots, l_m > 0$ we have*

$$F_{[l_1,\ldots,l_m]}(x) = \frac{1}{1+\sqrt{x}-cx} G_{[2l_1,\ldots,2l_m]}\left(\frac{\sqrt{x}}{1+\sqrt{x}-cx}\right). \tag{14}$$

Proof. We argue by induction on m.

First suppose $m = 1$. In this case we argue by induction on l_1. Observe that when $l_1 = 0$ both sides of (14) are equal to 0, since [0] is the empty permutation, which is contained in every permutation. To handle the case $l_1 = 1$, first observe that if $\pi \in I_n(3412, [2])$ then $\pi = 12 \ldots n$, since $[2] = 21$. Therefore $G_{[2]}(x) = 1/(1-x)$, and we see that when $l_1 = 1$ both sides of (14) are equal to $1/(1-cx)$. Since the result holds when $l_1 = 0$ and when $l_1 = 1$, suppose $l_1 \geqslant 2$. Set $m = 1$, replace l_1 with $2l_1$ in (13), and rearrange the resulting equation to obtain

$$(1 - x - x^2 G_{[2l_1-2]}(x)) G_{[2l_1]}(x) = 1.$$

Now replace x with $\sqrt{x}/(1+\sqrt{x}-cx)$ and use induction to find

$$(1-cx-xF_{[l_1-1]}(x))\frac{1}{1+\sqrt{x}-cx}G_{[2l_1]}\left(\frac{\sqrt{x}}{1+\sqrt{x}-cx}\right)=1. \tag{15}$$

Now set $\pi=[l_1]$ in (9) and rearrange the resulting equation to obtain

$$(1-cx-xF_{[l_1-1]}(x))F_{[l_1]}(x)=1. \tag{16}$$

Compare (15) with (16) to find (14) holds when $m=1$.

Now suppose $m\geqslant 2$. Replace $[l_1,\ldots,l_m]$ with $[2l_1,\ldots,2l_m]$ and x with $\sqrt{x}/(1+\sqrt{x}-cx)$ in (13), use induction and rearrange the resulting equation to obtain

$$\begin{aligned}(1-cx-xF_{[l_1-1]}(x)-xF_{[l_m]}(x))&\frac{1}{1+\sqrt{x}-cx}G_{[2l_1,\ldots,2l_m]}\left(\frac{\sqrt{x}}{1+\sqrt{x}-cx}\right)\\ &=1+xF_{[l_1,l_2]}(x)F_{[l_2,\ldots,l_m]}(x)-xF_{[l_1-1]}(x)F_{[l_2,\ldots,l_m]}(x)\\ &\quad+x\sum_{i=3}^{m-1}(F_{[l_1,\ldots,l_i]}(x)-F_{[l_1,\ldots,l_{i-1}]}(x))F_{[l_i,\ldots,l_m]}(x)-xF_{[l_1,\ldots,l_{m-1}]}(x)F_{[l_m]}(x).\end{aligned} \tag{17}$$

Now set $\pi=[l_1,\ldots,l_m]$ in (9) and rearrange the resulting equation to obtain

$$\begin{aligned}(1-cx-xF_{[l_1-1]}(x)-xF_{[l_m]}(x))&F_{[l_1,\ldots,l_m]}(x)\\ &=1+xF_{[l_1,l_2]}(x)F_{[l_2,\ldots,l_m]}(x)-xF_{[l_1-1]}(x)F_{[l_2,\ldots,l_m]}(x)\\ &\quad+x\sum_{i=3}^{m-1}(F_{[l_1,\ldots,l_i]}(x)-F_{[l_1,\ldots,l_{i-1}]}(x))F_{[l_i,\ldots,l_m]}(x)-xF_{[l_1,\ldots,l_{m-1}]}(x)F_{[l_m]}(x).\end{aligned} \tag{18}$$

Compare (17) with (18) to complete the proof. □

Theorem 3.3 allows us to use results from [3] to obtain $F_\pi(x)$ for various π. In these results we abbreviate

$$V_k(x)=U_k\left(\frac{1-cx}{2\sqrt{x}}\right).$$

Corollary 3.4. *Fix $c\geqslant 0$. Then for all $k\geqslant 0$ we have*

$$F_{[k]}(x)=\frac{V_{k-1}(x)}{\sqrt{x}V_k(x)}.$$

Proof. Combine (14) with [3, Eq. (37)]. □

Observe that when we set $c=0$ in Corollary 3.4 we recover [7, Eq. (3.4)] and [2, Theorem 3.6, second case].

Corollary 3.5. *Fix $c\geqslant 0$. Then for all $l_1,l_2\geqslant 1$ we have*

$$F_{[l_1,l_2]}(x)=F_{[l_1+l_2]}(x).$$

Proof. Combine (14) with [3, Eq. (42)]. □

Observe that when we set $c=0$ in Corollary 3.5 we recover [11, Theorem 2.4].

Corollary 3.6. *Fix $c\geqslant 0$. Then for all $l_1,l_2,l_3\geqslant 1$ we have*

$$F_{[l_1,l_2,l_3]}(x)=\frac{V_{l_1+l_2+l_3}V_{l_1+l_2+l_3-1}+V_{l_1+l_2-1}V_{l_1+l_3-1}V_{l_2+l_3-1}}{\sqrt{x}V_{l_1+l_2}V_{l_1+l_3}V_{l_2+l_3}}. \tag{19}$$

Proof. Combine (14) with [3, Eq. (44)]. □

Observe that when we set $c=0$ in Corollary 3.6 we recover [11, Theorem 2.5].

We have now found $F_{[l_1,\ldots,l_m]}(x)$ for all $m \leqslant 3$. Although this generating function appears to be more complicated for larger values of m, our results suggest the following conjecture.

Conjecture 3.7. Fix $c \geqslant 0$. Then for all $m \geqslant 1$ and all $l_1, \ldots, l_m \geqslant 1$, the generating function $F_{[l_1,\ldots,l_m]}(x)$ is symmetric in $l_1, \ldots, l_m$.

We have verified Conjecture 3.7 for $m=4$ and $l_i \leqslant 10$, for $m=5$ and $l_i \leqslant 8$, for $m=6$ and $l_i \leqslant 4$, and for $m=7$ and $l_i \leqslant 3$ using a Maple program. In view of Theorem 3.3, Conjecture 3.7 is a special case of the following.

Conjecture 3.8 (*Egge [3, Conjecture 6.9]*). For all $m \geqslant 1$ and all $l_1, \ldots, l_m \geqslant 1$, the generating function $G_{[l_1,\ldots,l_m]}(x)$ is symmetric in $[l_1, \ldots, l_m]$.

Conjectures 3.7 and 3.8 have resisted the efforts of the author and several others, and seem to require a new approach. In the hope of fostering such a new approach we close this section by restating these conjectures combinatorially, emphasizing their similarities with the main results of [1,6]. To state these reformulations, recall that two sets R_1 and R_2 of forbidden patterns are called *Wilf-equivalent* (resp. *involution Wilf-equivalent*) whenever $|CS_n(R_1)|=|CS_n(R_2)|$ (resp. $|I_n(R_1)|=|I_n(R_2)|$) for all $n \geqslant 0$. With this terminology, Conjectures 3.7 and 3.8 are equivalent, respectively, to the following.

Conjecture 3.9. Fix $c \geqslant 0$. Then the sets

$$P_c \cup \{[l_1, \ldots, l_m]\}$$

and

$$P_c \cup \{[l_1, \ldots, l_{i-1}, l_{i+1}, l_i, l_{i+2}, \ldots, l_m]\}$$

are Wilf-equivalent for all $m \geqslant 1$, all $l_1, \ldots, l_m \geqslant 1$, and all i with $1 \leqslant i \leqslant m-1$.

Conjecture 3.10. The sets $\{3412, [l_1, \ldots, l_m]\}$ and $\{3412, [l_1, \ldots, l_{i-1}, l_{i+1}, l_i, l_{i+2}, \ldots, l_m]\}$ are involution Wilf-equivalent for all $m \geqslant 1$ and all $l_1, \ldots l_m \geqslant 1$.

References

[1] E. Babson, J. West, The permutations $123p_4 \cdots p_m$ and $321p_4 \cdots p_m$ are Wilf-equivalent, Graphs Combin. 16 (4) (2000) 373–380.
[2] T. Chow, J. West, Forbidden subsequences and Chebyshev polynomials, Discrete Math. 204 (1–3) (1999) 119–128.
[3] E.S. Egge, Restricted 3412-avoiding involutions, continued fractions, and Chebyshev polynomials, Adv. Appl. Math. 33 (2004) 451–475.
[4] E.S. Egge, Restricted signed permutations counted by the Schröder numbers, Discrete Math. 306 (2006) 552–563.
[5] E.S. Egge, T. Mansour, Permutations which avoid 1243 and 2143, continued fractions, and Chebyshev polynomials, Electron. J. Combin. 9 (2) (2003) #R7.
[6] A. Jaggard, Prefix exchanging and pattern avoidance by involutions, Electron. J. Combin. 9 (2) (2003) #R16.
[7] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, Adv. Appl. Math. 27 (2/3) (2001) 510–530.
[8] D. Kremer, Permutations with forbidden subsequences and a generalized Schröder number, Discrete Math. 218 (2000) 121–130.
[9] T. Mansour, Pattern avoidance in coloured permutations, Sém. Lothar. Combin. 46 (2001) Article B46g.
[10] T. Mansour, A. Vainshtein, Restricted permutations, continued fractions, and Chebyshev polynomials, Electron. J. Combin. 7 (1) (2000) #R17.
[11] T. Mansour, A. Vainshtein, Restricted 132-avoiding permutations, Adv. Appl. Math. 26 (3) (2001) 258–269.
[12] T. Mansour, A. Vainshtein, Restricted permutations and Chebyshev polynomials, Sém. Lothar. Combin. 47 (2002) Article B47c.
[13] T. Mansour, J. West, Avoiding 2-letter signed patterns, Sém. Lothar. Combin. 49 (2002) Article B49a.
[14] A. Reifegerste, On the diagram of Schröder permutations, Electron. J. Combin. 9 (2) (2003) #R8.
[15] J. West, Generating trees and the Catalan and Schröder numbers, Discrete Math. 146 (1995) 247–262.