



Permutation patterns and statistics

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ABSTRACT

Let \mathfrak{S}_n denote the symmetric group of all permutations of $\{1, 2, \dots, n\}$ and let $\mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n$. If $\Pi \subseteq \mathfrak{S}$ is a set of permutations, then we let $\text{Av}_n(\Pi)$ be the set of permutations in \mathfrak{S}_n which avoid every permutation of Π in the sense of pattern avoidance. One of the celebrated notions in pattern theory is that of Wilf-equivalence, where Π and Π' are Wilf equivalent if $\#\text{Av}_n(\Pi) = \#\text{Av}_n(\Pi')$ for all $n \geq 0$. In a recent paper, Sagan and Savage proposed studying a q -analogue of this concept defined as follows. Suppose $\text{st} : \mathfrak{S} \rightarrow \{0, 1, 2, \dots\}$ is a permutation statistic and consider the corresponding generating function $F_n^{\text{st}}(\Pi; q) = \sum_{\sigma \in \text{Av}_n(\Pi)} q^{\text{st}\sigma}$. Call Π, Π' *st-Wilf equivalent* if $F_n^{\text{st}}(\Pi; q) = F_n^{\text{st}}(\Pi'; q)$ for all $n \geq 0$. We present the first in-depth study of this concept for the *inv* and *maj* statistics. In particular, we determine all *inv*- and *maj*-Wilf equivalences for any $\Pi \subseteq \mathfrak{S}_3$. This leads us to consider various q -analogues of the Catalan numbers, Fibonacci numbers, triangular numbers, and powers of two. Our proof techniques use lattice paths, integer partitions, and Foata's second fundamental bijection. We also answer a question about Mahonian pairs raised in the Sagan–Savage article.

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1. Introduction

We begin with some well-known definitions. Let \mathfrak{S}_n denote the set of permutations $\pi = a_1 a_2 \dots a_n$ of $[n] = \{1, 2, \dots, n\}$, and let $\mathfrak{S} = \bigcup_{n \geq 0} \mathfrak{S}_n$. We denote by $\pi(k)$ the entry a_k . And if $\pi \in \mathfrak{S}_n$ then we say that π has *length* n . Two sequences of distinct integers, $a_1 a_2 \dots a_n$ and $b_1 b_2 \dots b_n$, are said to be *order isomorphic* whenever they satisfy

$$a_i < a_j \quad \text{if and only if} \quad b_i < b_j$$

for all $1 \leq i < j \leq n$. We say that $\sigma \in \mathfrak{S}_n$ *contains a copy of* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is a subsequence of σ order isomorphic to π . For example, $\sigma = 436152$ contains the pattern $\pi = 132$ because of the copy 365. On the other hand, σ *avoids* π if it does not contain π and we let

$$\text{Av}_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \pi\}.$$

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More generally, if $\Pi \subseteq \mathfrak{S}$ then we define $Av_n(\Pi) = \bigcap_{\pi \in \Pi} Av_n(\pi)$ so that these permutations avoid every element of Π . Just as with symmetric groups, it will be useful to let

$$Av(\Pi) = \bigcup_{n \geq 0} Av_n(\Pi).$$

We say that Π, Π' are Wilf equivalent and write $\Pi \equiv \Pi'$ if, for all $n \geq 0$,

$$\#Av_n(\Pi) = \#Av_n(\Pi')$$

where the number sign denotes cardinality. Given a specific set Π , we will drop the curly brackets enclosing the set when writing $Av_n(\Pi)$ and in similar notations below. A famous result in this area states that $\pi \equiv \pi'$ for any $\pi, \pi' \in \mathfrak{S}_3$.

In a recent paper [17], Sagan and Savage proposed studying a q -analogue of Wilf equivalence defined as follows. A permutation statistic is a function

$$st : \mathfrak{S} \rightarrow \mathbb{N},$$

where \mathbb{N} is the nonnegative integers. Associated with any statistic st and any $\Pi \subseteq \mathfrak{S}$ we have the generating function

$$F_n^{st}(\Pi; q) = \sum_{\sigma \in Av_n(\Pi)} q^{st \sigma}.$$

Call Π, Π' st -Wilf equivalent, written $\Pi \stackrel{st}{\equiv} \Pi'$, if

$$F_n^{st}(\Pi; q) = F_n^{st}(\Pi'; q)$$

for all $n \geq 0$. We denote the st -Wilf equivalence class of Π by $[\Pi]_{st}$. Note that st -Wilf equivalence implies Wilf equivalence since setting $q = 1$ gives $F_n^{st}(\Pi; 1) = \#Av_n(\Pi)$. In particular, if $\pi \stackrel{st}{\equiv} \pi'$ then π and π' must be in the same symmetric group since this is true of Wilf equivalence.

Our goal is to study this concept and related ideas for two famous permutation statistics: the inversion number and the major index. The set of inversions of $\sigma = a_1 a_2 \dots a_n$ is

$$Inv \sigma = \{(i, j) \mid i < j \text{ and } a_i > a_j\}.$$

So $Inv \sigma$ records pairs of indices where the corresponding elements in σ are out of order. The inversion number of σ is just

$$inv \sigma = \#Inv \sigma.$$

By way of illustration, if $\sigma = 41523$

$$Inv \sigma = \{(1, 2), (1, 4), (1, 5), (3, 4), (3, 5)\}$$

since $a_1 = 4 > a_2 = 1$ and so forth, giving $inv \sigma = 5$.

To define the major index, we must first define descents. Permutation $\sigma = a_1 a_2 \dots a_n$ has descent set

$$Des \sigma = \{i \mid a_i > a_{i+1}\}$$

and descent number

$$des \sigma = \#Des \sigma.$$

Descents keep track of the first index in inversion pairs consisting of adjacent elements. The major index of σ is given by

$$maj \sigma = \sum_{i \in Des \sigma} i.$$

Continuing our example from the previous paragraph, we see that

$$Des \sigma = \{1, 3\}$$

and so $maj \sigma = 1 + 3 = 4$.

The statistics inv and maj are intimately connected and, in fact, are equidistributed over \mathfrak{S}_n in the sense that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{inv \sigma} = \sum_{\sigma \in \mathfrak{S}_n} q^{maj \sigma}.$$

A statistic whose distribution over \mathfrak{S}_n equals this one is called Mahonian. Since we will use them so often, we will adopt the following abbreviations for the generating functions for these statistics over avoidance sets

$$I_n(\Pi; q) = F_n^{inv}(\Pi; q) \text{ and } M_n(\Pi; q) = F_n^{maj}(\Pi; q).$$

This article is devoted to a study of $I_n(\Pi)$ and $M_n(\Pi)$ and, by the end, we will have completely classified all these polynomials for all $\Pi \subseteq \mathfrak{S}_3$ just as Simion and Schmidt [18] did with the corresponding cardinalities when $q = 1$. Along the way we will meet a number of interesting q -analogues for known combinatorial quantities such as Catalan numbers, Fibonacci numbers, triangular numbers, and powers of two. We will establish connections with lattice paths, integer partitions, and Foata's second fundamental bijection. We will also answer a question of Sagan and Savage [17] concerning certain objects called Mahonian pairs. The rest of this paper is structured as follows. In the following section we will consider inv - and maj -Wilf equivalence in the case $\#\Pi = 1$. Section 3 will talk about q -analogues of the Catalan numbers arising in this context. In Sections 4 and 5 we will look at $I_n(\Pi)$ and $M_n(\Pi)$, respectively, when $\Pi \subset \mathfrak{S}_3$ has cardinality 2. The next two sections investigate the analogous problem for $\#\Pi = 3$. The final section contains concluding remarks. Various conjectures are scattered throughout the paper.

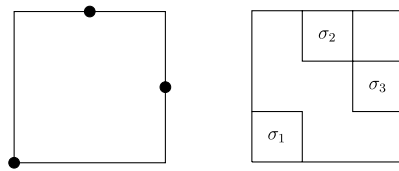


Fig. 1. The diagram of 132 (left) and 132[σ₁, σ₂, σ₃] (right).

2. Equivalence for single permutations

It will be useful to have a geometric way to describe permutations. We can diagram $\sigma = a_1a_2 \dots a_n \in \mathfrak{S}_n$ using the points $(1, a_1), (2, a_2), \dots, (n, a_n)$ in the first quadrant of the Cartesian plane. For example, the diagram of $\sigma = 132$ is in Fig. 1 where we have enclosed the region $1 \leq x, y \leq 3$ in a square.

Inflations of permutations will play an important role in our work. Given permutations $\pi = a_1a_2 \dots a_k \in \mathfrak{S}_k$ and $\sigma_1, \sigma_2, \dots, \sigma_k \in \mathfrak{S}$, the *inflation* of π by the σ_i is the permutation $\pi[\sigma_1, \sigma_2, \dots, \sigma_k]$ whose diagram is obtained from the diagram of π by replacing the point (i, a_i) by the diagram of a permutation order isomorphic to σ_i for $1 \leq i \leq k$. The σ_i are called the *components* of the inflation. Fig. 1 displays the a schematic diagram of $132[\sigma_1, \sigma_2, \sigma_3]$. As a specific example, we have $132[21, 1, 213] = 216435$. Note that we permit $\sigma_i = \epsilon$, the empty permutation, which has the effect of removing the corresponding point of π . By way of illustration, $132[\epsilon, 1, 213] = 4213$.

Since the diagram of σ lies in a square, we can let the dihedral group of the square, D_4 , act on permutations. To set notation, let

$$D_4 = \{R_0, R_{90}, R_{180}, R_{270}, r_{-1}, r_0, r_1, r_\infty\},$$

where R_θ is rotation counterclockwise through an angle of θ degrees and r_m is reflection in a line of slope m . A couple of these rigid motions have easy descriptions in terms of the one line notation for permutations. If $\sigma = a_1a_2 \dots a_n$ then its *reversal* is

$$\sigma^r = a_n \dots a_2a_1 = r_\infty(\sigma),$$

and its *complement* is

$$\sigma^c = (n + 1 - a_1), (n + 1 - a_2), \dots, (n + 1 - a_n) = r_0(\sigma).$$

We will apply these operations to sets of permutations by doing so element-wise.

We will now investigate what these symmetries do to the inversion polynomial. Clearly each $f \in D_4$ either preserves inversions and non-inversions, or interchanges the two. Since the total number of pairs (i, j) with $1 \leq i < j \leq n$ is $\binom{n}{2}$, we immediately have the first part of following lemma. The second part is obtained from the first by standard generating function manipulations and the fact that σ avoids π if and only if $f(\sigma)$ avoids $f(\pi)$.

Lemma 2.1. For any $\sigma \in \mathfrak{S}$ we have

$$\text{inv} f(\sigma) = \begin{cases} \text{inv} \sigma & \text{if } f \in \{R_0, R_{180}, r_{-1}, r_1\}, \\ \binom{n}{2} - \text{inv} \sigma & \text{if } f \in \{R_{90}, R_{270}, r_0, r_\infty\}. \end{cases}$$

It follows that

$$I_n(f(\pi); q) = \begin{cases} I_n(\pi; q) & \text{if } f \in \{R_0, R_{180}, r_{-1}, r_1\}, \\ q^{\binom{n}{2}} I_n(\pi; q^{-1}) & \text{if } f \in \{R_{90}, R_{270}, r_0, r_\infty\}. \end{cases}$$

for all $n \geq 0$. □

We call the $f \in \{R_0, R_{180}, r_{-1}, r_1\}$ *inv preserving* and the $f \in \{R_{90}, R_{270}, r_0, r_\infty\}$ *inv reversing*. Translating the previous result into the language of inv-Wilf equivalence gives the following.

Corollary 2.2. For every permutation π and $f \in \{R_0, R_{180}, r_{-1}, r_1\}$ we have

$$f(\pi) \stackrel{\text{inv}}{\equiv} \pi. \tag{1}$$

Furthermore, for any $f \in D_4$, we have that $\pi \stackrel{\text{inv}}{\equiv} \pi'$ implies $f(\pi) \stackrel{\text{inv}}{\equiv} f(\pi')$. □

Because of Eq. (1), we call equivalences arising from these symmetries *trivial*. Restricting our attention to \mathfrak{S}_3 , the above corollary gives

$$132 \stackrel{\text{inv}}{\equiv} 213 \quad \text{and} \quad 231 \stackrel{\text{inv}}{\equiv} 312.$$

To see that these are the only inv-Wilf equivalences, note that if $\pi \in \mathfrak{S}_k$ then $\text{Av}_k(\pi) = \mathfrak{S}_k - \{\pi\}$. It follows that

$$I_k(\pi; q) = I_k - q^{\text{inv}\pi},$$

where $I_k = \sum_{\sigma \in \mathfrak{S}_k} q^{\text{inv}\sigma}$. So $\pi \stackrel{\text{inv}}{\equiv} \pi'$ forces $\text{inv}\pi = \text{inv}\pi'$. Since in \mathfrak{S}_3 the latter equation only holds for the pairs 132, 213 and 231, 312, we have proved the following result.

Theorem 2.3. *We have*

$$\begin{aligned} [123]_{\text{inv}} &= \{123\}, \\ [321]_{\text{inv}} &= \{321\}, \\ [132]_{\text{inv}} &= \{132, 213\}, \\ [231]_{\text{inv}} &= \{231, 312\}, \end{aligned}$$

and these are all the inv-Wilf equivalence classes involving single elements of \mathfrak{S}_3 . \square

In fact, we believe that all inv-Wilf equivalences are trivial. The following conjecture has been verified by computer for single permutations up through \mathfrak{S}_5 and for all subsets of permutations of \mathfrak{S}_3 . Note that if $\Pi = \{\pi_1, \dots, \pi_l\}$ then we let $f(\Pi) = \{f(\pi_1), \dots, f(\pi_l)\}$. In a similar vein, we will let other operators distribute over sets, e.g., if $S = \{a_1, \dots, a_l\}$ is a set of integers then $n + S = \{n + a_1, \dots, n + a_l\}$.

Conjecture 2.4. *For $\Pi, \Pi' \subseteq \mathfrak{S}$ we have $\Pi \stackrel{\text{inv}}{\equiv} \Pi'$ if and only if $f(\Pi) = \Pi'$ for some $f \in \{R_0, R_{180}, r_{-1}, r_1\}$.*

We now turn our attention to the maj polynomial. Here the situation is more complicated because there are no maj-preserving symmetries in D_4 . In fact, the only symmetry which permits one to deduce something about maj-Wilf equivalence is complementation. The next result is proved in much the same way as Lemma 2.1 and so its demonstration is omitted.

Lemma 2.5. *For any $\sigma \in \mathfrak{S}$ we have*

$$\text{maj}\sigma^c = \binom{n}{2} - \text{maj}\sigma.$$

It follows that

$$M_n(\pi^c; q) = q^{\binom{n}{2}} M_n(\pi, q^{-1})$$

for all $n \geq 0$. So $\pi \stackrel{\text{maj}}{\equiv} \pi'$ if and only if $\pi^c \stackrel{\text{maj}}{\equiv} (\pi')^c$. \square

We now come to our first non-trivial equivalences. In the proof as well as later we will say that a map f on permutations is Des preserving if $\text{Des}\sigma = \text{Des}f(\sigma)$ for all σ in the domain of f . Note that such a map must preserve both the des and maj statistics in the same sense.

Theorem 2.6. *We have*

$$\begin{aligned} [123]_{\text{maj}} &= \{123\}, \\ [321]_{\text{maj}} &= \{321\}, \\ [132]_{\text{maj}} &= \{132, 231\}, \\ [213]_{\text{maj}} &= \{213, 312\}, \end{aligned}$$

and these are all the maj-Wilf equivalence classes involving single elements of \mathfrak{S}_3 . \square

Proof. Using arguments like those before Theorem 2.3, we see that $132 \stackrel{\text{maj}}{\equiv} 231$ and $213 \stackrel{\text{maj}}{\equiv} 312$ are the only possible maj-Wilf equivalences in \mathfrak{S}_3 . Also, the previous lemma shows that it suffices to prove one of these two equivalences, and we will choose the former.

We first note that $\sigma \in \text{Av}_n(132)$ if and only if $\sigma = 231[\sigma_1, 1, \sigma_2]$ for some $\sigma_1, \sigma_2 \in \text{Av}(132)$ as shown in the diagram on the left in Fig. 2. This is well known, but since we will be using such characterizations frequently in the sequel, we will present a proof this time and leave future demonstrations to the reader.

Suppose first that $\sigma \in \text{Av}_n(132)$ and write $\sigma = \sigma_L n \sigma_R$ where σ_L and σ_R are the subsequences of σ to the left and right of n , respectively. To avoid having a copy of 132 where n acts as the 3, we must have every element of σ_L larger than every element of σ_R . It follows that $\sigma = 231[\sigma_1, 1, \sigma_2]$ where σ_1 is order isomorphic to σ_L and similarly with σ_2 and σ_R . Clearly σ_1 and σ_2 must avoid 132 since σ does, and this completes the forward direction of the characterization.

Now suppose σ is an inflation of the given form and that, towards a contradiction, it contains a copy of 132. Consider the element a of σ playing the role of the 2. If a comes from σ_1 then the whole copy is in σ_1 , contradicting the fact that $\sigma_1 \in \text{Av}(132)$. It is impossible for n to be the 2. So the only other choice is for a to come from σ_2 . But since n as well as every element of σ_1 is larger than every element of σ_2 , the element playing the role of the 1 in the 132 copy must also come from

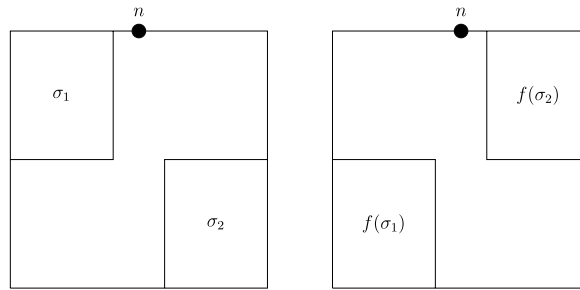


Fig. 2. An element of $Av_n(132)$ (left) and its image under f (right).

σ_2 , placing the whole copy in σ_2 . This contradicts $\sigma_2 \in Av(132)$ and finishes the characterization. Note that exactly the same line of reasoning shows that $\sigma \in Av_n(231)$ if and only if $\sigma = 132[\sigma_1, 1, \sigma_2]$ where $\sigma_1, \sigma_2 \in Av(231)$.

We will now prove $132 \stackrel{\text{maj}}{\cong} 231$ by inductively constructing a Des-preserving bijection $f : Av_n(132) \rightarrow Av_n(231)$ for all $n \geq 0$. For $n = 0$ we define $f(\epsilon) = \epsilon$. Suppose that f has been defined for permutations with fewer than n elements and consider $\sigma \in Av_n(132)$. If $\sigma = 231[\sigma_1, 1, \sigma_2]$ then define the image of σ under f to be

$$f(\sigma) = 132[f(\sigma_1), 1, f(\sigma_2)].$$

This is illustrated by the right-hand diagram in Fig. 2. By induction and the characterization above, we have that $f(\sigma) \in Av_n(231)$. So f does indeed map the domain into the desired range. It is also clear inductively that f preserves the descent set and so preserves maj. It is an easy matter to construct an inverse for f , thus the map is bijective and we are done. \square

The two maj-Wilf equivalences proved in the previous theorem seem to be part of a larger picture. To state this result, we use the notation

$$\iota_n = 12 \dots n$$

and

$$\delta_n = n \dots 21$$

for the monotone increasing and decreasing permutations. We will suppress the subscript if we do not wish to specify the number of elements in the permutation.

Conjecture 2.7. For all $m, n \geq 0$ we have

$$132[\iota_m, 1, \delta_n] \stackrel{\text{maj}}{\cong} 231[\iota_m, 1, \delta_n].$$

Equivalently

$$213[\delta_m, 1, \iota_n] \stackrel{\text{maj}}{\cong} 312[\delta_m, 1, \iota_n].$$

Also, for $n \geq 5$ these are the only maj-Wilf equivalences of the form $\pi \stackrel{\text{maj}}{\cong} \pi'$ with $\pi \neq \pi'$.

The maj-Wilf equivalences in this conjecture have been checked by computer up to \mathfrak{S}_7 . We have also checked that there are no other such equivalences in \mathfrak{S}_5 . In \mathfrak{S}_4 , however, there appear to be some other sporadic ones. Aside from the two sets of equivalences below, all other maj-Wilf equivalence classes in \mathfrak{S}_4 are singletons.

Conjecture 2.8. We have

$$[1423]_{\text{maj}} = \{1423, 2314, 2413\}.$$

Equivalently

$$[3142]_{\text{maj}} = \{3142, 3241, 4132\}.$$

3. q-Catalan numbers

The Catalan numbers can be defined as

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

It is well known that $\#Av_n(\pi) = C_n$ for any $\pi \in \mathfrak{S}_3$. So the polynomials $I_n(\pi; q)$ and $M_n(\pi; q)$ for $\pi \in \mathfrak{S}_3$ are q -analogues of these numbers in that they evaluate to C_n when setting $q = 1$. In view of Lemmas 2.1 and 2.5 as well as Theorems 2.3 and 2.6, only four of these polynomials are really different up to a change of variable. So we will concentrate on the cases when $\pi = 312$ and 321 .

The Catalan numbers can also be defined as the unique solution to the initial condition $C_0 = 1$ and recursion

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

for $n \geq 1$. Carlitz and Riordan [3] defined a q -analogue $C_n(q)$ of C_n . For our purposes, it will be easier to work with the polynomials

$$\tilde{C}_n(q) = q^{\binom{n}{2}} C_n(q^{-1}) \tag{2}$$

obtained by reversing the order of the coefficients of $C_n(q)$. These can be defined by $\tilde{C}_0(q) = 1$ and, for $n \geq 1$,

$$\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^k \tilde{C}_k(q) \tilde{C}_{n-k-1}(q). \tag{3}$$

The Carlitz–Riordan polynomials have been studied by numerous authors. They have a close connection with the space of diagonal harmonics and Haglund’s text [13] gives more information and references.

It turns out that some of our polynomials are related to those of Carlitz and Riordan.

Theorem 3.1. *We have $I_0(312; q) = 1$ and*

$$I_n(312; q) = \sum_{k=0}^{n-1} q^k I_k(312; q) I_{n-k-1}(312; q). \tag{4}$$

for $n \geq 1$. It follows that, for $n \geq 0$,

$$\begin{aligned} C_n(q) &= I_n(132; q) = I_n(213; q), \\ \tilde{C}_n(q) &= I_n(231; q) = I_n(312; q). \end{aligned}$$

Proof. To prove the recursion, suppose that $\sigma \in Av_n(312)$ so that $\sigma = 213[\sigma_1, 1, \sigma_2]$. If $\sigma(k+1) = 1$ for some $0 \leq k < n$ then $\sigma_1 \in Av_k(312)$, $\sigma_2 \in Av_{n-k-1}(312)$, and

$$\text{inv } \sigma = k + \text{inv } \sigma_1 + \text{inv } \sigma_2.$$

Rewriting the previous equation in terms of generating functions gives the desired result.

The statements about $\tilde{C}_n(q)$ now follow by comparing (3) and (4) and applying Theorem 2.3. Those about $C_n(q)$ are then obtained from (2) and Lemma 2.1. \square

Although a similar recursion seems to hold for $I_n(321)$, we have been unable to prove it and so state it as a conjecture.

Conjecture 3.2. *We have, for all $n \geq 1$,*

$$I_n(321; q) = I_{n-1}(321; q) + \sum_{k=0}^{n-2} q^{k+1} I_k(321; q) I_{n-k-1}(321; q).$$

We have not been able to discover a recursion for the polynomials $M_n(312; q)$ and it would be interesting to do so. However, one can get a result by using a bivariate generating function. For $\Pi \subseteq \mathfrak{S}$ we let

$$M(\Pi; q, t) = \sum_{\sigma \in Av_n(\Pi)} q^{\text{maj } \sigma} t^{\text{des } \sigma}.$$

Theorem 3.3. *We have, for $n \geq 1$,*

$$M_n(312; q, t) = M_{n-1}(312; q, qt) + \sum_{k=1}^{n-1} q^k t M_k(312; q, t) M_{n-k-1}(312; q, q^{k+1}t).$$

Proof. Keeping the notation of the previous proof, we have

$$\text{des } \sigma = \begin{cases} \text{des } \sigma_2 & \text{if } k = 0, \\ 1 + \text{des } \sigma_1 + \text{des } \sigma_2 & \text{if } k \geq 1, \end{cases}$$

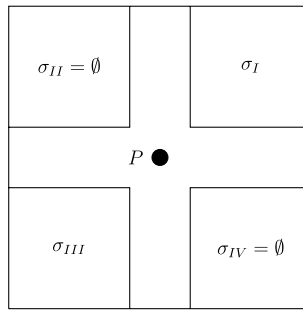


Fig. 3. An element of $Av_{2^k-1}(321)$ fixed by R_{180} .

and, for all $k \geq 0$,

$$\text{maj } \sigma = k + \text{maj } \sigma_1 + \text{maj } \sigma_2 + (k + 1)\text{des } \sigma_2.$$

Turning these statements into an equation for the generating functions completes the proof. \square

Computer experimentation has yielded no nice recursion for $M_n(321; q, t)$ so we leave this as an open problem

Question 3.4. Do the $M_n(321; q, t)$ satisfy a (q, t) -analogue of the Catalan recursion?

We will now show how one can use one of our q -analogues to give a new proof of an interesting result about Catalan numbers. It is well known that

$$C_n \text{ is odd if and only if } n = 2^k - 1 \text{ for some } k \geq 0. \tag{5}$$

In fact, one can characterize the 2-adic valuation (highest power of 2) dividing C_n . For a statement and combinatorial proof of this stronger result, see the paper of Deutsch and Sagan [5]. We will prove a refinement of the “if” direction of (5). To do so it will be convenient to have the notation that, for any polynomial $f(q)$,

$$\langle q^i \rangle f(q) = \text{the coefficient of } q^i \text{ in } f(q).$$

Theorem 3.5. For all $k \geq 0$ we have

$$\langle q^i \rangle I_{2^k-1}(321; q) = \begin{cases} 1 & \text{if } i = 0, \\ \text{an even number} & \text{if } i \geq 1. \end{cases}$$

Proof. The constant coefficient of $I_n(321; q)$ is 1 because ι_n is the only permutation in \mathfrak{S}_n with no inversions. And if $n = 2^0 - 1 = 0$ then all other coefficients are zero, which is even. So we will proceed by induction on k .

Assume the result is true for indices less than k and consider

$$S_i = \{\sigma \in Av_{2^k-1}(321) \mid \text{inv } \sigma = i\}.$$

For $i \geq 1$, we will partition this set into blocks (subsets) each of even size and that will prove the theorem. Since $R_{180}(321) = 321$ and R_{180} preserves inv by Lemma 2.1, the involution R_{180} acts on S_i .

Suppose $\sigma \in S_i$ is such that $R_{180}(\sigma) \neq \sigma$. Then let the block containing σ be $B = \{\sigma, R_{180}(\sigma)\}$. By construction, all these blocks have size two which is even. Now put all of the remaining σ in one block

$$B' = \{\sigma \in S_i \mid R_{180}(\sigma) = \sigma\}.$$

For $R_{180}(\sigma) = \sigma$ to hold, one of the points of the diagram of σ must be the center of rotation $P = (2^{k-1}, 2^{k-1})$ which splits the diagram up into four parts as in Fig. 3. Note that since $\sigma_{IV} = R_{180}(\sigma_{II})$ both parts must be empty or nonempty. If they are both nonempty then this will result in a copy of 321, so they are both empty. Also $\sigma_{III} = R_{180}(\sigma_I)$ and so once σ_I is chosen, σ_{III} is determined. Thus we can characterize the $\sigma \in B'$ as those of the form $\sigma = 123[R_{180}(\tau), 1, \tau]$ where $\tau \in Av_{2^{k-1}-1}(321)$. Furthermore $i = \text{inv } \sigma = 2\text{inv } \tau$. It follows that

$$\#B' = \langle q^{i/2} \rangle I_{2^{k-1}-1}(321; q)$$

and, by induction, this is an even number. This finishes the proof. \square

We conjecture that a similar statement can be made for one of the maj polynomial sequences.

Conjecture 3.6. For all $k \geq 0$ we have

$$\langle q^i \rangle M_{2^k-1}(321; q) = \begin{cases} 1 & \text{if } i = 0, \\ \text{an even number} & \text{if } i \geq 1. \end{cases}$$

4. The inversion polynomial for doubletons

By the Erdős–Szekeres theorem on monotonic subsequences [10], any permutation of length 5 or greater will contain an increasing or decreasing subsequence of length 3. Due to this fact, we have that $I_n(\Pi; q) = 0$ (similarly $M_n(\Pi; q) = 0$) for $n \geq 5$ and $\Pi \supseteq \{123, 321\}$. For such Π different techniques will be used and so we will not consider the set $\{123, 321\}$ here. Such Π will be discussed in Section 8.1.

For $\Pi \subseteq \mathfrak{S}$, we define

$$\text{inv } \Pi = \{\{\text{inv } \pi \mid \pi \in \Pi\}\}$$

to be the multiset of $\text{inv } \pi$ for each Π . We define $\text{maj } \Pi$ similarly. For example, the set $\Pi = \{123, 132, 213, 231, 321\}$ has $\text{inv } \Pi = \{\{0, 1, 1, 2, 3\}\}$ and $\text{maj } \Pi = \{\{0, 1, 2, 2, 3\}\}$. Note that if $\Pi, \Pi' \subseteq \mathfrak{S}_n$ for a fixed n and $\Pi \stackrel{\text{inv}}{\equiv} \Pi'$, then $\text{inv } \Pi = \text{inv } \Pi'$ and the same result holds true for $\text{maj } \Pi$. The argument is similar to the one given before Theorem 2.3 in the case $\#\Pi = \#\Pi' = 1$.

In the next result, the inv -Wilf equivalences are obtained from Eq. (1). On the other hand, it is easy to check that $\text{inv } \Pi \neq \text{inv } \Pi'$ for Π and Π' listed in different classes and so, by the discussion above, there can be no other such equivalences among these doubletons.

Theorem 4.1. *We have*

$$\begin{aligned} [123, 132]_{\text{inv}} &= \{\{123, 132\}, \{123, 213\}\}, \\ [312, 321]_{\text{inv}} &= \{\{231, 321\}, \{312, 321\}\}, \\ [123, 231]_{\text{inv}} &= \{\{123, 231\}, \{123, 312\}\}, \\ [213, 321]_{\text{inv}} &= \{\{132, 321\}, \{213, 321\}\}, \\ [132, 231]_{\text{inv}} &= \{\{132, 231\}, \{132, 312\}, \{213, 231\}, \{213, 312\}\}. \end{aligned}$$

All other inv -Wilf equivalence classes for $\Pi \subset \mathfrak{S}_3$ with $|\Pi| = 2$ and $\Pi \neq \{123, 321\}$ consist of a single pair. \square

The equivalence classes above have been paired, with the class of Π appearing in a line adjacent with the class of its complement. By Lemma 2.1, $I_n(\Pi; q)$ for one class determines the polynomial for its pair. So we only compute the inversion polynomial for one doubleton from each pair consisting of a class and its complement; the reader can easily do the computations for the complementary class.

Before giving the next result, we first must make some remarks about integer partitions. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be an integer partition, that is, a weakly decreasing sequence of positive integers called parts. We use the notation

$$|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_k$$

and

$$l(\lambda) = \text{the number of parts of } \lambda.$$

If we further suppose that $n > \lambda_1 > \lambda_2 > \dots > \lambda_k$, then λ is a partition of $|\lambda|$ into distinct parts, with no part larger than $n - 1$. We will use the notation $P_d(n - 1)$ for this set. It is well known that the generating function for these partitions is expressible as a simple product

$$\sum_{\lambda \in P_d(n-1)} q^{|\lambda|} t^{l(\lambda)} = (1 + qt)(1 + q^2t) \cdots (1 + q^{n-1}t). \tag{6}$$

We also define the following permutations which will be useful here as well as in Section 5

$$\check{\iota}_n = n1 \cdots (n - 1) = 21[1, \iota_{n-1}], \tag{7}$$

and

$$\hat{\iota}_n = 2 \cdots n1 = 21[\iota_{n-1}, 1]. \tag{8}$$

Note that the polynomials in the following theorem give two different q -analogues of the number 2^{n-1} .

Proposition 4.2. *We have*

$$\begin{aligned} I_n(231, 321; q) &= (1 + q)^{n-1} \\ I_n(132, 231; q) &= (1 + q)(1 + q^2) \cdots (1 + q^{n-1}) \end{aligned}$$

for all $n \geq 1$.

Proof. For the first equation, it is easy to see that σ is determined by its left-right maxima $m_1 < m_2 < \dots < m_j = n$. In fact, $\sigma = \iota_j[\check{\iota}_{n_1}, \check{\iota}_{n_2}, \dots, \check{\iota}_{n_j}]$ where $n_i = m_i - m_{i-1}$ for $1 \leq i < k$. (By convention $m_0 = 0$.) Note that $\text{inv } \sigma = n - j$ since

each element which is not a left-right maximum participates in exactly one inversion. And there are $\binom{n-1}{j-1}$ ways to chose the m_i since $m_j = n$ is always a left-right maximum. So we have

$$I_n(231, 321; q) = \sum_{j=1}^n \binom{n-1}{j-1} q^{n-j} = \sum_{j=0}^{n-1} \binom{n-1}{j} q^j = (1+q)^{n-1}.$$

The second equation comes from noticing that $\sigma \in Av_n(132, 231)$ if and only if $\sigma = \sigma_L 1 \sigma_R$ where σ_L is a decreasing sequence and σ_R is an increasing sequence. We now define a bijection $f : Av_n(132, 231) \rightarrow P_d(n-1)$. From the discussion before the theorem, it suffices to show that for $\lambda = f(\sigma)$, we have that $\text{inv } \sigma = |\lambda|$. Note that for any entry i in σ_L , all entries $1, \dots, i-1$ occur after i and that σ_R contributes nothing to $\text{inv } \sigma$. So $\text{inv } \sigma = (\sigma(1) - 1) + \dots + (\sigma(k) - 1)$. Defining $f(\sigma) = (\sigma(1) - 1, \dots, \sigma(k) - 1)$, it is easy to check that f has the desired properties. \square

For the remaining two equivalence classes, $[132, 321]_{\text{inv}}$ and $[132, 213]_{\text{inv}}$, we have not been able to obtain closed forms for their inversion polynomials. Therefore, only summation formulae will be proved here.

Proposition 4.3. For all $n \geq 1$, we have the following formulae

$$I_n(132, 321; q) = 1 + \sum_{k=1}^{n-1} \frac{q^{k(n-k+1)} - q^k}{q^k - 1},$$

$$I_n(132, 213; q) = \sum_{k=1}^n q^{k(n-k)} I_{n-k}(132, 213; q),$$

where the latter equation is a recurrence for one of the singletons not listed in Theorem 4.1.

Proof. By considering the placement of n , one sees that $\sigma \in Av_n(132, 321)$ if and only if one has that $\sigma = 12[\sigma', 1]$ with $\sigma' \in Av_{n-1}(132, 321)$ or $\sigma = 21[\iota_k, \iota_{n-k}]$ for some $k, 0 < k < n$. These two cases result in $\text{inv } \sigma = \text{inv } \sigma'$ or $\text{inv } \sigma = k(n-k)$, respectively. It follows that, for $n \geq 1$, $I_n(132, 321; q) = I_{n-1}(132, 321; q) + \sum_{k=1}^{n-1} q^{k(n-k)}$. Iterating this recursion gives the desired sum.

For the second equality, note that $\sigma \in Av_n(132, 213)$ if and only if $\sigma = 21[\iota_k, \sigma']$ where $\sigma' \in Av_{n-k}(132, 213)$ for some $k \in [n]$. Therefore, $\text{inv } \sigma = \text{inv } \sigma' + k(n-k)$ and the second recurrence follows. \square

5. The major index polynomial for doubletons

As discussed in the previous section, we will consider the set $\{123, 321\}$ and its supersets at the end of the paper. For the remaining 14 pairs, there is only one maj-Wilf equivalence class which does not consist of a single pair.

Theorem 5.1. We have

$$[132, 213]_{\text{maj}} = \{\{132, 213\}, \{132, 312\}, \{213, 231\}, \{231, 312\}\}.$$

All other maj-Wilf equivalence classes for $\Pi \subset \mathfrak{S}_3$ with $|\Pi| = 2$ and $\Pi \neq \{123, 321\}$ contain a single pair.

Proof. The non-equivalences follow from computer-generated example polynomials. The equivalence of the sets $\{132, 213\}, \{132, 312\}$, and $\{213, 231\}$ will follow from the next proposition, where we will explicitly compute the corresponding polynomials and see that they are the same. By complementing the equivalence $\{132, 213\} \stackrel{\text{maj}}{\equiv} \{132, 312\}$ we obtain $\{231, 312\} \stackrel{\text{maj}}{\equiv} \{132, 312\}$, completing the proof. \square

We will now explicitly calculate $M_n(\Pi; q, t)$ for the various Π under consideration. For every pair Π, Π^c , we will only compute one of the polynomials since the other can be obtained using the fact that $M_n(\Pi^c; q) = q^{\binom{n}{2}} M_n(\Pi; q^{-1})$ by Lemma 2.5.

Proposition 5.2. For $n \geq 0$ and $\Pi = \{132, 213\}, \{132, 312\}$, or $\{213, 231\}$ we have

$$M_n(\Pi; q, t) = (1 + qt)(1 + q^2t) \cdots (1 + q^{n-1}t).$$

Proof. When $\Pi = \{132, 213\}$, in view of (6), it suffices to establish a bijection $f : Av_n(132, 213) \rightarrow P_d(n-1)$ such that if $\sigma \in Av_n(132, 213)$ and $\lambda = f(\sigma)$, then $\text{des } \sigma = l(\lambda)$ and $\text{maj } \sigma = |\lambda|$. Recursively applying the characterization of $Av_n(132, 213)$ given in the proof of Proposition 4.3, we see that σ avoids 132, 213 if and only if $\sigma = \delta_k[\iota_{m_1}, \iota_{m_2}, \dots, \iota_{m_k}]$. Thus σ is uniquely determined by the elements of its descent set, which we shall denote $\text{Des } \sigma = \{\lambda_1 > \lambda_2 > \dots > \lambda_{k-1}\}$. But by letting $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k-1})$, we have our bijection which clearly satisfies $l(\lambda) = \text{des } \sigma$ and $|\lambda| = \text{maj } \sigma$.

For $\Pi = \{132, 312\}$, it now suffices to show that there is a Des-preserving bijection $f : Av_n(132, 312) \rightarrow Av_n(132, 213)$. Since avoiding 132 and 312 forces the permutation to end in 1 or n , we have $\sigma \in Av_n(132, 312)$ if and only if

$$\sigma = 21[\sigma', 1] \quad \text{or} \quad 12[\sigma', 1],$$

where $\sigma' \in Av_{n-1}(132, 312)$. By induction, we can assume $f(\sigma') = \tau' \in Av_{n-1}(132, 213)$ where the image permutation has the form $\tau' = \delta_k[l_{m_1}, l_{m_2}, \dots, l_{m_k}]$. So define

$$f(\sigma) = \delta_{k+1}[l_{m_1}, l_{m_2}, \dots, l_{m_k}, 1] \quad \text{or} \quad \delta_k[l_{m_1}, l_{m_2}, \dots, l_{m_k+1}],$$

where the two cases for $f(\sigma)$ correspond respectively to the two cases for σ listed above. It is easy to see that this process can be reversed to obtain an inverse, which proves that f is a bijection. Also $\text{Des } \sigma' = \text{Des } \tau'$ by induction and the final ascent or descent has been preserved by construction. So f is Des preserving as desired.

For $\Pi = \{213, 231\}$, we again wish to construct a Des-preserving bijection, this time from $Av_n(213, 231)$ to $Av_n(132, 213)$. But $\sigma \in Av_n(213, 231)$ if and only if $\sigma(1) = 1$ or $\sigma(1) = n$ and $\sigma' = \sigma(2) \dots \sigma(n) \in Av_{n-1}(213, 231)$. It should now be clear how the proof of the previous paragraph can be adapted to provide the desired bijection. \square

The next result is easy to show using the previous characterizations of $Av_n(132, 231)$ and $Av_n(132, 321)$ in the demonstrations of Propositions 4.2 and 4.3. So we omit the proof. In the result for the second pair, we use the standard q -analogue of a natural number n :

$$[n]_q = 1 + q + \dots + q^{n-1}.$$

Proposition 5.3. For $n \geq 0$, we have

$$M_n(132, 231; q, t) = \sum_{k=0}^{n-1} \binom{n-1}{k} q^{\binom{k+1}{2}} t^k$$

and

$$M_n(132, 321; q, t) = 1 + qt \frac{n - [n]_q}{1 - q}. \quad \square$$

Proposition 5.4. For $n \geq 0$, we have

$$M_n(213, 321; q, t) = 1 + t \sum_{k=1}^{n-1} kq^k.$$

Proof. The permutations in $Av_n(213, 321)$ are those of the form $\sigma = 132[l_{i_1}, l_{i_2}, l_{i_3}]$. As usual, let $k = \sigma^{-1}(n)$ and $\sigma = \sigma_L n \sigma_R$. If $k < n$, then the number of ways of partitioning σ_L is equal to k . If $k = n$, then $\sigma = \iota_n$ which contributes the 1 present in our expression. Translating these facts into a generating function completes the proof. \square

It is known that $\#Av_n(213, 321) = 1 + \binom{n}{2}$ from [18], and thus our polynomial can be interpreted as a q -analogue of the expression

$$\binom{n}{2} = \sum_{k=1}^{n-1} k.$$

The final two sets of permutations do not have known closed forms for their major index polynomials. However, their generating functions

$$M(\Pi; x) = \sum_{n \geq 0} M_n(\Pi; q, t) x^n$$

are equal to generating functions over binary words. It will be informative to look at these words and their intimate connection with integer partitions.

Given a 0-1 word w with m zeros and n ones, we associate with it a *lattice path*, $P(w)$, from $(0, 0)$ to (n, m) with each zero representing a unit step to the north, and each one a unit step to the east. In Fig. 4 we have the path 001101001. The unit boxes northwest of the path and contained in the rectangle with diagonal corners $(0, 0)$ and (n, m) form the *Ferrers diagram* of the path. This diagram can be interpreted as a partition $\lambda(w)$, with each row of boxes being a part. For example, the partition in Fig. 4 is $(3, 3, 2)$. It is well known [17] that

$$\text{inv } w = |\lambda(w)|. \tag{9}$$

For a given Ferrers diagram of a path $P(w)$, the *Durfee square* $D(w)$ is the largest square which fits into the partition $\lambda = \lambda(w)$ so that its northwest corner coincides with the northwest corner of the diagram. Its side length is denoted $d(w)$. In Fig. 4 the Durfee square is shaded, and $d(w) = 2$.

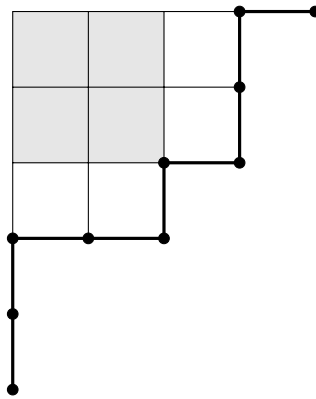


Fig. 4. A lattice path with Ferrers diagram and Durfee square shown.

Our generating function sums will be obtained by decomposing a Ferrers diagram into its Durfee square, the part of the diagram below the square, and the part of the diagram to the right of the square. The part below corresponds to a partition $\beta(w)$ where each part has size at most $d(w)$. Additionally, we allow parts of size 0 in order to incorporate any steps to the north that precede all eastern steps. In Fig. 4, we have $\beta(w) = (2, 0, 0)$. Similarly, if we view the columns to the right of $D(w)$ as parts instead of the rows, we obtain another partition $\rho(w)$. Referring to Fig. 4 again, we see that $\rho(w) = (2, 0)$.

Finally, we will use Foata’s second fundamental bijection $\phi : \mathbb{N}^* \rightarrow \mathbb{N}^*$, where \mathbb{N}^* is the Kleene closure of all finite words formed by elements of \mathbb{N} . Although we do not define this map here, the relevant properties will be stated in the next three lemmas. We shall use v to denote a word in the domain of ϕ , and w a word in the range. Additionally, the length (number of elements) of any word $u \in \mathbb{N}^*$ will be denoted $|u|$. Foata’s bijection originally appeared in [11] and was constructed to have the following properties.

Lemma 5.5 ([11]). For $v \in \mathbb{N}^*$, we have that

$$|v| = |\phi(v)|,$$

and

$$\text{inv } \phi(v) = \text{maj } v. \quad \square$$

The next two lemmas are from Sagan and Savage’s paper [17]. The form for v given in Lemma 5.6 is the general form of a 0–1 word with $\text{des } v = k$. Readers unfamiliar with ϕ may take the Lemma as the definition of ϕ on $\{0, 1\}^*$:

Lemma 5.6 ([17]). Let $v = 0^{m_0} 1^{n_0} 0^{m_1} 1^{n_1} \dots 0^{m_k} 1^{n_k} \in \{0, 1\}^*$, where $m_0, n_k \geq 0$ and $m_i, n_j > 0$ for all other i and j . Then

$$\phi(v) = 0^{m_k-1} 10^{m_{k-1}-1} 1 \dots 0^{m_1-1} 10^{m_0-1} 0 1^{n_1-1} 0 \dots 1^{n_{k-1}-1} 0 1^{n_k}. \quad \square$$

Lemma 5.7 ([17]). If $v \in \{0, 1\}^*$ and $w = \phi(v)$, then

$$\text{des } v = d(w). \quad \square$$

Our first generating function concerns words v that start with a 1, if nonempty. We shall let

$$L = \{\epsilon\} \cup \{1u : u \in \{0, 1\}^*\}$$

be the set of such words, where ϵ is the empty word. The generating function we consider is

$$\sum_{v \in L} q^{\text{maj } v} t^{\text{des } v} x^{|v|}.$$

Applying Foata’s bijection, the above lemmas, and Eq. (9) shows that this sum is equal to

$$\sum_{w \in \phi(L)} q^{|\lambda(w)|} t^{d(w)} x^{|w|}.$$

We claim that a word w is in $\phi(L)$ if and only if $\beta(w)$ has all parts of size less than $k = d(w)$. Keeping the notation of Lemma 5.6, we see that $v \in L$ is equivalent to $m_0 = 0$. So, by the same result, this will happen precisely when the subword of $w = \phi(v)$ following the k th one contains exactly k zeros. This means that the k th east step of w will be the step which

ends at the southeast corner of $D(w)$. And this is equivalent to having part sizes in $\beta(w)$ smaller than $d(w)$. Since zero is allowed as a part in both $\beta(w)$ and $\rho(w)$, standard generating function manipulations show that our desired sum is

$$\sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_k (x)_{k+1}},$$

where

$$(x)_k = (1-x)(1-qx) \cdots (1-q^{k-1}x).$$

Theorem 5.8. *We have*

$$M(231, 321; x) = \sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_k (x)_{k+1}}.$$

Proof. From the preceding discussion, it suffices to show that there is a bijection f from $\text{Av}_n(231, 321)$ to $L_n = \{v \in L : |v| = n\}$ which is Des preserving. Recall from the proof of Proposition 4.2 that $\sigma \in \text{Av}_n(231, 321)$ if and only if $\sigma = \iota_j[\check{\iota}_{n_1}, \check{\iota}_{n_2}, \dots, \check{\iota}_{n_j}]$, where the first element of $\check{\iota}_{n_i}$ is a left-right maximum m_i . Clearly, these left-right maxima are the only descents of σ , when not followed immediately by another left-right maximum.

Now we define $f(\sigma) = v = v_1 v_2 \cdots v_n$ where $v_i = 1$ if $\sigma(i)$ is a left-right maximum, and $v_i = 0$ if not. Since the first element of σ is a left-right maximum, $v_1 = 1$ and thus $f(\sigma) \in L_n$. It also follows from the previous paragraph that $\text{Des } \sigma = \text{Des } f(\sigma)$. This map is easily seen to be a bijection, and so we are done. \square

The second generating function is a sum over the set of words which end in a 0 if nonempty, which will be denoted $R = \{u0 : u \in \{0, 1\}^*\} \cup \{\epsilon\}$:

$$\sum_{v \in R} q^{\text{maj } v} t^{\text{des } v} x^{|v|}.$$

Again, to find a formula for this sum we apply ϕ . Lemma 5.6 shows that in fact $\phi(R) = R$, so we obtain

$$\sum_{w \in R} q^{|\lambda(w)|} t^{d(w)} x^{|w|}.$$

Clearly having w end in a zero is equivalent to $\rho(w)$ having all parts positive. Therefore our desired sum is

$$\sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_{k+1} (qx)_k}.$$

Theorem 5.9. *We have*

$$M(312, 321; x) = \sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_{k+1} (qx)_k}.$$

Proof. As usual, it suffices to show that we can find a Des-preserving bijection f from $\text{Av}_n(312, 321)$ to R_n , the length n words in R . Notice that $R_{180}(\{231, 321\}) = \{312, 321\}$. Thus, from the proof of Theorem 5.8, it follows that $\sigma \in \text{Av}_n(312, 321)$ if and only if $\sigma = \iota_j[\hat{\iota}_{n_1}, \hat{\iota}_{n_2}, \dots, \hat{\iota}_{n_j}]$. It follows that σ is uniquely determined by its right-left minima, with the only descents of σ being formed by these right-left minima, when not adjacent.

We create our word $f(\sigma) = v_1 v_2 \cdots v_n$ by letting $v_i = 0$ if $\sigma(i)$ is a right-left minimum, and $v_i = 1$ otherwise. Since $\sigma(n)$ is always a right-left minimum, we have $v_n = 0$ and therefore $v \in R_n$. Also, $\text{Des } \sigma = \text{Des } f(\sigma)$ by construction. This process is clearly reversible, so f is a bijection and we are done. \square

6. The inversion polynomial for triples

As before, we will ignore any Π with $\{123, 321\} \subset \Pi$. Since $\Pi \stackrel{\text{inv}}{\equiv} \Pi'$ forces $\text{inv } \Pi = \text{inv } \Pi'$ for subsets of \mathfrak{S}_3 , we compute $\text{inv } \Pi$ for a representative of the equivalences given by Eq. (1) and, since they are all distinct, deduce that no others exist. Thus we have the following result.

Theorem 6.1. *We have*

$$\begin{aligned} [123, 132, 231]_{\text{inv}} &= \{ \{123, 132, 231\}, \{123, 132, 312\}, \{123, 213, 231\}, \{123, 213, 312\} \}, \\ [132, 231, 321]_{\text{inv}} &= \{ \{132, 231, 321\}, \{132, 312, 321\}, \{213, 231, 321\}, \{213, 312, 321\} \}, \\ [132, 213, 231]_{\text{inv}} &= \{ \{132, 213, 231\}, \{132, 213, 312\} \}, \\ [132, 231, 312]_{\text{inv}} &= \{ \{132, 231, 312\}, \{213, 231, 312\} \}. \end{aligned}$$

All other inv-Wilf equivalence classes for $\Pi \subset \mathfrak{S}_3$ with $\#\Pi = 3$ and $\{123, 321\} \not\subset \Pi$ consist of a single triple. \square

As usual, in the above theorem, the results are listed so that each $[\Pi]_{\text{inv}}$ is on a line adjacent to $[\Pi^c]_{\text{inv}}$, and we only compute $I_n(\Pi; q)$ for one representative of each complementary pair.

Proposition 6.2. For all $n \geq 0$ we have

$$\begin{aligned}
 I_n(132, 213, 321; q) &= \sum_{k=1}^n q^{k(n-k)}, \\
 I_n(132, 231, 312; q) &= \sum_{k=1}^n q^{\binom{k}{2}}, \\
 I_n(132, 231, 321; q) &= [n]_q, \\
 I_n(231, 312, 321; q) &= \sum_{k=0}^n \binom{n-k}{k} q^k.
 \end{aligned}$$

Proof. The first three identities all follow from a straightforward count of inversions in the following characterizations of the permutations under consideration, where in each case k varies over $[n]$: $\sigma \in \text{Av}_n(132, 213, 321)$ if and only if $\sigma = 21[\iota_k, \iota_{n-k}]$; $\sigma \in \text{Av}_n(132, 231, 312)$ if and only if $\sigma = 12[\delta_k, \iota_{n-k}]$; and finally $\sigma \in \text{Av}_n(132, 231, 321)$ if and only if $\sigma = 213[1, \iota_{n-k}]$.

For the fourth identity, we know from the proof of Proposition 4.2 that the permutations avoiding 231 and 321 are those of the form $\sigma = \iota_j[\iota_{n_1}, \iota_{n_2}, \dots, \iota_{n_j}]$. And to avoid 312 we must have $n_i \leq 2$ for all i . Thus $\sigma \in \text{Av}_n(231, 312, 321)$ if and only if $\sigma = \iota_j[\delta_{n_1}, \delta_{n_2}, \dots, \delta_{n_j}]$ where $|\delta_{n_i}| \leq 2$ for all i . With this characterization, the only inversions of σ are descents within components of the form δ_2 . Now $\text{inv } \sigma = k$ if and only if there are k components of this form, and moreover $j = n - k$ since there are k length two components and the rest are length one. So σ is determined by choosing k of the $n - k$ components to be the ones of form δ_2 . The desired result follows. \square

Consider the Fibonacci numbers defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Simion and Schmidt [18] showed that $\#\text{Av}_n(231, 312, 321) = F_n$. So the fourth equation in Proposition 6.2 is a q -analogue of the well-known identity

$$F_n = \sum_{k=0}^n \binom{n-k}{k}.$$

A simple substitution of variables transforms $I_n(231, 312, 321)$ into the Fibonacci polynomials introduced by Morgan-Voyce [16]. These have since been studied by many authors, with special attention given to their roots.

7. The major index polynomial for triples

Again, we will omit any Π with $\{123, 321\} \subset \Pi$. We have the following theorem for the remaining 16 triples and, as is our custom, the results are listed so that each $[\Pi]_{\text{maj}}$ is adjacent to $[\Pi^c]_{\text{maj}}$.

Theorem 7.1. We have

$$\begin{aligned}
 [123, 132, 312]_{\text{maj}} &= \{\{123, 132, 312\}, \{123, 213, 231\}, \{123, 231, 312\}\}, \\
 [132, 213, 321]_{\text{maj}} &= \{\{132, 213, 321\}, \{132, 312, 321\}, \{213, 231, 321\}\}, \\
 [132, 213, 231]_{\text{maj}} &= \{\{132, 213, 231\}, \{132, 231, 312\}\}, \\
 [132, 213, 312]_{\text{maj}} &= \{\{132, 213, 312\}, \{213, 231, 312\}\}.
 \end{aligned}$$

All other maj-Wilf equivalence classes for $\Pi \subset \mathfrak{S}_3$ with $\#\Pi = 3$ and $\{123, 321\} \not\subset \Pi$ consist of a single triple.

As before, it will suffice to prove the equivalences in the second and third equivalence classes above which we do in the next proposition.

Proposition 7.2. For all $n \geq 0$ and $\Pi_1 = \{132, 213, 321\}, \{132, 312, 321\}$, or $\{213, 231, 321\}$ we have

$$M_n(\Pi_1; q, t) = 1 + \sum_{k=1}^{n-1} q^k t = 1 + qt[n-1]_q.$$

For $\Pi_2 = \{132, 213, 231\}$ or $\{132, 231, 312\}$ we have

$$M_n(\Pi_2; q, t) = 1 + \sum_{k=2}^n q^{\binom{k}{2}} t^{k-1}.$$

We also have

$$M_n(213, 312, 321; q, t) = 1 + (n - 1)q^{n-1}t,$$

$$M_n(132, 231, 321; q, t) = 1 + (n - 1)qt.$$

Proof. The major index counts follow easily from the characterizations of the various avoidance classes given in the following table:

Class	Elements
$Av_n(132, 213, 321)$	$21[\iota_k, \iota_{n-k}]$
$Av_n(132, 312, 321)$	$213[\iota_k, 1, \iota_{n-k+1}]$
$Av_n(213, 231, 321)$	$132[\iota_{k-1}, 1, \iota_{n-k}]$
$Av_n(132, 213, 231)$	$21[\delta_{k-1}, \iota_{n-k+1}]$
$Av_n(132, 231, 312)$	$12[\delta_k, \iota_{n-k}]$
$Av_n(213, 312, 321)$	$132[\iota_{k-1}, \iota_{n-k}, 1]$
$Av_n(132, 231, 321)$	$213[1, \iota_{k-1}, \iota_{n-k}]$

where, as usual, k ranges over $[n]$. □

For the last triple, there is no known closed form for its major index polynomial, so we will compute the generating function. Consider the set of words which are either the empty word or contain no consecutive ones and end in zero, denoted by P . The generating function we will investigate is

$$\sum_{v \in P} q^{\text{maj } v} t^{\text{des } v} x^{|v|}.$$

Applying Foata's bijection, and using Lemmas 5.5 and 5.7 as well as Eq. (9), we know the sum is equal to

$$\sum_{w \in \phi(P)} q^{|\lambda(w)|} t^{d(w)} x^{|w|}.$$

We claim that a word w is in $\phi(P)$ if and only if $\rho(w)$, the portion to the right of the Durfee square $D(w)$, is empty. This is equivalent to $w = \phi(v)$ ending in at least as many consecutive zeros as there are total ones in w . Using the notation of Lemma 5.6, we see that $v \in P$ if and only if $n_i = 1$ for $i < k$ and $n_k = 0$. The equation for $w = \phi(v)$ shows that there are at least k zeros after the k th one since $1^{n_i-1} = 1^0$. This gives us the desired property for $\phi(P)$. Also, zero is allowed as a part in $\beta(w)$, the portion below $D(w)$, and thus our sum becomes

$$\sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_{k+1}}.$$

Theorem 7.3. We have

$$M(231, 312, 321; x) = \sum_{k \geq 0} \frac{q^{k^2} t^k x^{2k}}{(x)_{k+1}}.$$

Proof. From the above discussion, it suffices to show that there is a Des-preserving bijection f from $Av_n(231, 312, 321)$ to the set $P_n = \{v \in P : |v| = n\}$. Recall from the proof of Proposition 6.2 that $\sigma \in Av_n(231, 312, 321)$ if and only if $\sigma = \iota_j[\delta_{n_1}, \delta_{n_2}, \dots, \delta_{n_j}]$, where $n_i \leq 2$ for all i . Thus the right-left minima of σ are formed by the last elements of each δ_{n_i} . And the only descents of σ are formed by the first elements of the δ_{n_i} where $n_i = 2$ forcing no two to be adjacent.

We define $f(\sigma) = v = v_1 v_2 \dots v_n$, where $v_i = 0$ if $\sigma(i)$ is a right-left minimum and $v_i = 1$ otherwise. Since the last element of σ is always a right-left minimum, $v_n = 0$. Also, there can be no consecutive ones in v because of the description of the descents of σ in the previous paragraph. Hence, $f(\sigma) \in P_n$. By construction $\text{Des } \sigma = \text{Des } f(\sigma)$ and this process can be reversed to create f^{-1} . This completes the proof. □

It turns out that the polynomials $M_n(231, 312, 321; q)$ are precisely the q -Fibonacci numbers introduced by Carlitz [2] and subsequently studied by a number of authors. In particular, they are closely related to pattern avoidance in set partitions as was first explored by Goyt and Sagan [12].

8. Concluding remarks

8.1. Other $\Pi \subseteq \mathfrak{S}_3$

We wish to complete the classification of the inv - and maj -Wilf equivalence classes for all $\Pi \subseteq \mathfrak{S}_3$. To do this, we must consider Π with $\Pi \supseteq \{123, 321\}$ or with $\#\Pi \geq 4$.

Table 1
Mahonian pairs.

Π	{132, 213}, {132, 312}, {213, 231}, {231, 312}
Π'	{132, 231}, {132, 312}, {213, 231}, {213, 312}
Π	{132, 213, 231}, {132, 231, 312}
Π'	{132, 231, 312}, {213, 231, 312}
Π	{132, 213, 312}, {213, 231, 312}
Π'	{132, 213, 231}, {132, 213, 312}
Π	{123, 132, 312}, {123, 213, 231}, {123, 231, 312}
Π'	{123, 132, 231}, {123, 132, 312}, {123, 213, 231}, {123, 213, 312}
Π	{132, 213, 321}, {132, 312, 321}, {213, 231, 321}
Π'	{132, 231, 321}, {132, 312, 321}, {213, 231, 321}, {213, 312, 321}

As already remarked, if $\Pi \supseteq \{123, 321\}$ then $Av_n(\Pi) = \emptyset$ for $n \geq 5$. So these cases can be handled by mere computation. It turns out that for certain Π we also have $Av_4(\Pi) = \emptyset$ and this leads to three sporadic maj-Wilf equivalences.

If $\Pi \not\supseteq \{123, 321\}$, then we use the following (slightly modified) result of Simion and Schmidt [18]. In it, we let

$$\tilde{\iota}_n = 132[\iota_{n-2}, 1, 1].$$

Proposition 8.1 ([18]). *If $\{123, 321\} \not\subseteq \Pi \subseteq \mathfrak{S}_3$ and $\#\Pi = 4$ then, for $n \geq 2$, $Av_n(\Pi)$ consists of exactly two elements taken from $\iota_n, \hat{\iota}_n, \tilde{\iota}_n$ and their rotations. \square*

We note that $\text{maj } R(\hat{\iota}_n) = \text{maj } R(\tilde{\iota}_n)$ for any rotation $R \in D_4$. This causes four more maj-Wilf equivalences.

For $\#P = 5, 6$, at most ι_n or δ_n survives in $Av_n(\Pi)$, and so these sets are easy to classify. Putting everything together, it is a simple (if tedious) matter to verify the following theorem.

Theorem 8.2. *Suppose $\Pi \subseteq \mathfrak{S}_3$ and that either $\Pi \supseteq \{123, 321\}$ or $\#\Pi \geq 4$.*

1. All inv-Wilf equivalences among such Π are trivial.
2. We have the following maj-Wilf equivalence classes:

$$\begin{aligned} [123, 132, 213, 231]_{\text{maj}} &= \{\{123, 132, 213, 231\}, \{123, 132, 231, 312\}\}, \\ [123, 132, 213, 312]_{\text{maj}} &= \{\{123, 132, 213, 312\}, \{123, 213, 231, 312\}\}, \\ [123, 132, 312, 321]_{\text{maj}} &= \{\{123, 132, 312, 321\}, \{123, 213, 231, 321\}\}, \\ [132, 213, 231, 321]_{\text{maj}} &= \{\{132, 213, 231, 321\}, \{132, 231, 312, 321\}\}, \\ [132, 213, 312, 321]_{\text{maj}} &= \{\{132, 213, 312, 321\}, \{213, 231, 312, 321\}\}, \\ [123, 132, 213, 231, 321]_{\text{maj}} &= \{\{123, 132, 213, 231, 321\}, \{123, 132, 231, 312, 321\}\}, \\ [123, 132, 213, 312, 321]_{\text{maj}} &= \{\{123, 132, 213, 312, 321\}, \{123, 213, 231, 312, 321\}\}, \end{aligned}$$

and all other maj-Wilf equivalence classes consist of a single set. \square

8.2. Mahonian pairs

In [17], Sagan and Savage introduced the notion of a Mahonian pair. Let S and T be finite subsets of \mathbb{N}^* . Then (S, T) is a Mahonian pair if the distribution of maj over S is the same as the distribution of inv over T , equivalently,

$$\sum_{s \in S} q^{\text{maj } s} = \sum_{t \in T} q^{\text{inv } t}.$$

So, for example, a special case of a celebrated theorem of MacMahon [15, pp. 508–549 and pp. 556–563] can be restated as saying that $(\mathfrak{S}_n, \mathfrak{S}_n)$ is a Mahonian pair and such pairs where maj and inv are equidistributed over $S = T$ have been the object of intense study. But [17] is the first investigation of the case $S \neq T$. However, all the pairs found in that paper are using words over $\{0, 1\}^*$ and the authors asked if there were interesting pairs for larger alphabets. We can now answer that question in the affirmative. The following result follows immediately by comparing the first two results in Section 4 with the first two in Section 5, as well as the first two results in Section 6 with the first two in Section 7.

Theorem 8.3. *Consider any Π, Π' picked from corresponding rows in the same box of Table 1. For all $n \geq 0$, if $S = Av_n(\Pi)$ and $T = Av_n(\Pi')$ then (S, T) is a Mahonian pair. \square*

8.3. Future work

There are many directions in which this research could be taken. One could look at Π containing patterns of length 4 or more. These would give q -analogues of other famous combinatorial sequences which appear in pattern avoidance. For

example, it was proved by West [19] that $\#Av_n(3142, 2413) = S_{n-1}$, a large Schröder number. These numbers are closely connected to the Catalan numbers and have many interesting properties.

One could investigate other statistics. Just as inv and maj are equidistributed over \mathfrak{S}_n , so are another pair of well-known statistics. One of them, des , we have already met. The other is the number of excedences

$$\text{exc } \sigma = \#\{i \mid \sigma(i) > i\}.$$

Statistics with the same distribution as des or exc are called *Eulerian*. Interesting results about the distribution of exc have been found by Elizalde [7,6], Elizalde and Deutsch [8], and Elizalde and Pak [9]. And of course we already have results for des in the polynomials $M_n(\mathbb{T}; q, t)$. But there are a host of other statistics which might give interesting results. See the paper of Babson and Steingrímsson [1] for a partial list.

Rather than classical avoidance, one could look at generalized pattern avoidance. A generalized pattern π is one where, in order to have a copy of π in σ certain elements of the diagram of the copy must be adjacent either horizontally or vertically. This concept was introduced in the paper of Babson and Steingrímsson just cited. They are of interest, among other reasons, because many statistics on permutations can be expressed as linear combinations of the avoidance statistics for these generalized patterns.

Finally, one could look at pattern avoidance in other combinatorial structures. As previously mentioned, the q -Fibonacci numbers of Carlitz can also be obtained by avoiding patterns in set partitions.

Note added in proof. Recently Cheng et al. [4] proved [Conjecture 3.2](#) and also answered [Question 3.4](#) in the affirmative. Also, Killpatrick [14] has proved [Conjecture 3.6](#).

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