Some Statistics on Permutations avoiding Generalized Patterns

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Outline

1. Introduction
2. $S(1-23)$ and the symmetry class $\{1-23, 32-1, 3-21, 12-3\}$
3. The two other symmetry classes
4. Permutations avoiding a pair of generalized patterns
5. Conclusion and perspectives

Outline

1. Introduction
    - Some definitions and previous results
    - Graphical representation of permutations and ECO construction
2. $S(1-23)$ and the symmetry class $\{1-23, 32-1, 3-21, 12-3\}$
3. The two other symmetry classes
4. Permutations avoiding a pair of generalized patterns
5. Conclusion and perspectives

Classical Pattern Avoidance

$\pi \in S_n$, $\tau \in S_k$ with $k \leq n$

- The permutation π *contains* the pattern τ iff $\exists$ $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ is order-isomorphic to τ : $\pi_{i_p} < \pi_{i_q}$ iff $\tau_p < \tau_q$
- Otherwise, π *avoids* τ
- For example, 1**3**5**6**2**4** contains 132 and avoids 321

Notation :
$S_n(\tau)$ = the set of τ-avoiding permutations of length n
$S(\tau)$ = the set of τ-avoiding permutations

Generalized Pattern Avoidance

Generalized pattern = classical pattern + dashes

- Example : $\tau = 13 - 26 - 574$ is a generalized pattern

Generalized pattern avoidance : classical pattern avoidance + the elements that are adjacent in the pattern must correspond to adjacent elements in the permutation.

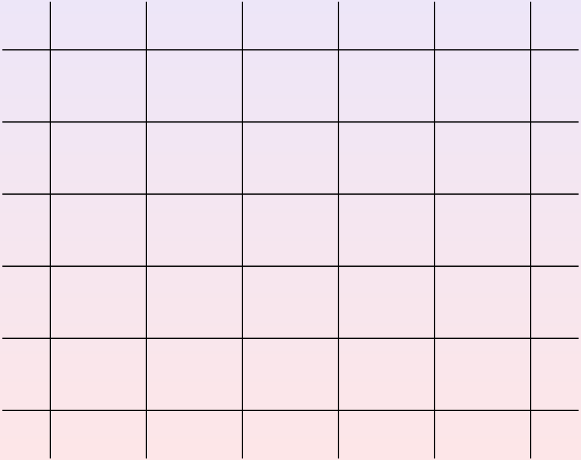
- Example : 7256134 contains $13 - 2$ (7**25**61**3**4) but avoids $1 - 32$

Three Symmetry Classes

- Reverse of a pattern p : $p^r = p$ read from right to left
  Complement of p : $p_i^c = n + 1 - p_i$ (dashes unchanged)
- Generalized patterns of length 3 are organised in 3 symmetry classes $\{p, p^r, p^c, p^{rc}\}$:
  - $\{1-23, 32-1, 3-21, 12-3\}$, $|S_n(p)| = B_n$ (Bell)
  - $\{3-12, 21-3, 1-32, 23-1\}$, $|S_n(p)| = B_n$ (Bell)
  - $\{2-13, 31-2, 2-31, 13-2\}$, $|S_n(p)| = C_n$ (Catalan)

Staff Representation of permutations

Example of 632514



- Staff = portée pentagramma

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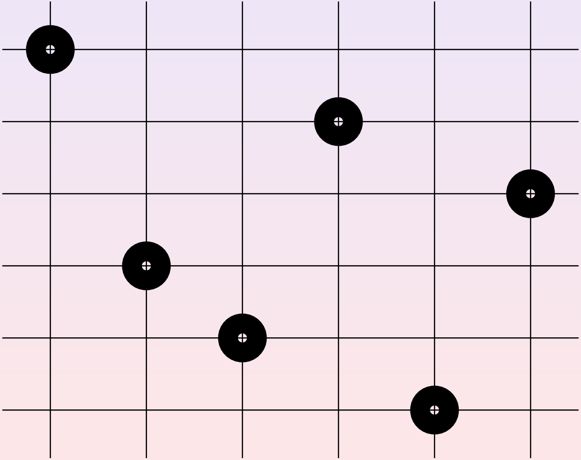

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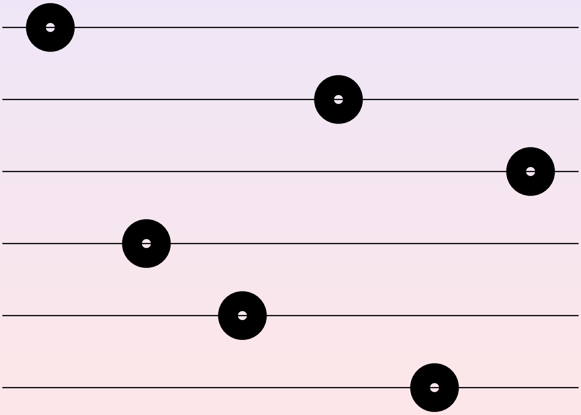

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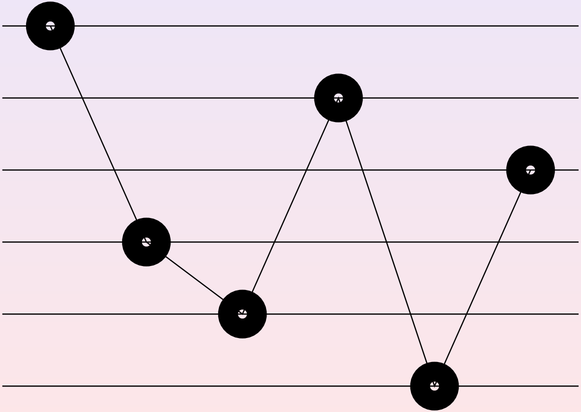
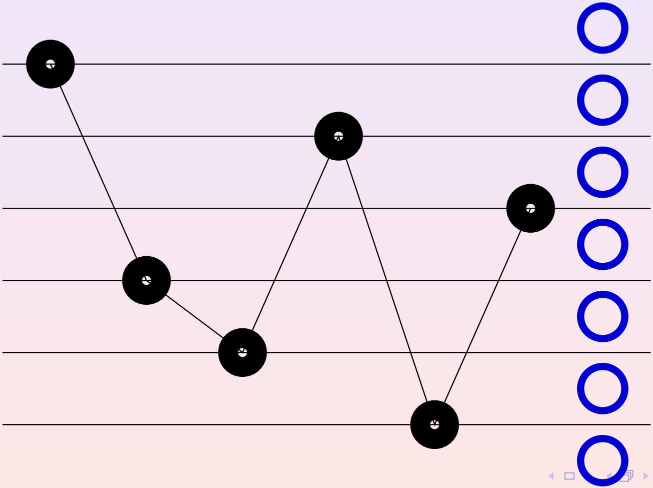


- Staff = portée pentagramma

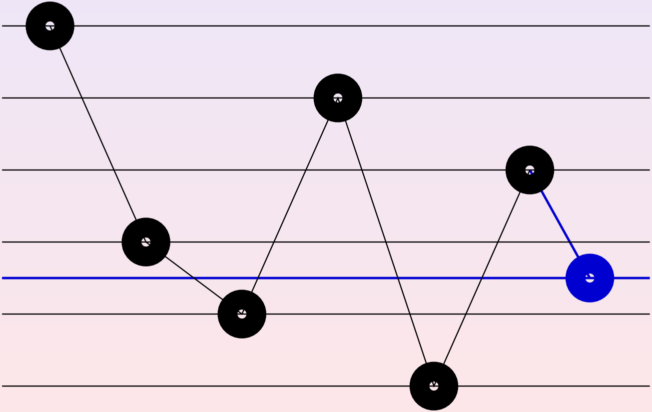
ECO construction on staff representation

Active sites $= n + 1$ regions on the right



ECO construction on staff representation

7426153 is obtained from 632514



A simple but crucial remark

- In this ECO construction, starting from a τ-avoiding permutation, the pattern τ can appear only if it uses the new element inserted.
- It allows us to determine which of the $n+1$ regions are active sites.

Our results

- Enumeration of $S(\tau)$ according to the length and the value of the last (or the first) element for every generalized pattern τ of length 3
- Two examples of extension to permutations avoiding 2 or 3 generalized patterns

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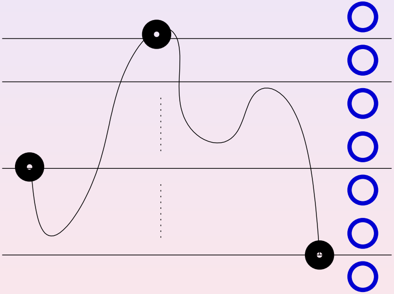
1. Introduction
2. $S(1-23)$ and the symmetry class $\{1-23, 32-1, 3-21, 12-3\}$
   - ECO construction and generating tree for $S(1-23)$
   - Distribution according to the length and the last value
   - The remaining patterns in the symmetry class of $1-23$
3. The two other symmetry classes
4. Permutations avoiding a pair of generalized patterns
5. Conclusion and perspectives

Active sites : first case

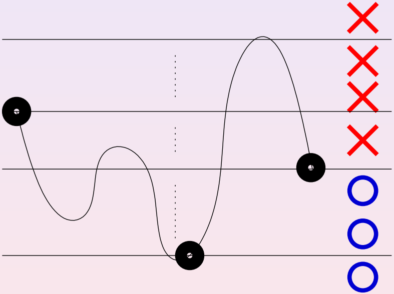
$\pi \in S_n(1-23)$ a permutation that ends with 1



π generates $n+1$ permutations of $S_{n+1}(1-23)$

Active sites : second case

$\pi \in S_n(1-23)$ a permutation that ends with $k \neq 1$



π generates k permutations of $S_{n+1}(1-23)$

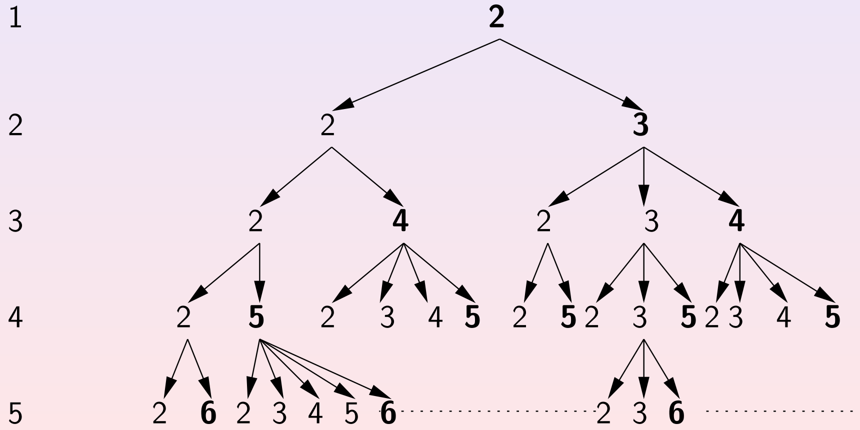
Succession rule

- Each permutation of $S_n(1-23)$ with k active sites is labelled (k, n).
- Succession rule :

$$\begin{cases} (2,1) \\ (k,n) \rightsquigarrow (2,n+1)(3,n+1)\cdots(k,n+1)(n+2,n+1) \end{cases} .$$

Generating tree

Levels



Matrix M

$M = (m_{i,j})_{i,j\geq 1}$

- $m_{i,j}$ is the number of labels $j+1$ at level i in the generating tree.
- i.e. $m_{i,j}$ is the number of permutations of $S_i(1-23)$ with $j+1$ active sites.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 1 & 2 & 0 & 0 & 0 & \vdots \\ 5 & 3 & 2 & 5 & 0 & 0 & \vdots \\ 15 & 10 & 7 & 5 & 15 & 0 & \vdots \\ 52 & 37 & 27 & 20 & 15 & 52 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

Matrix A, known as the *Bell triangle*

$$A = (a_{i,j})_{i,j\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 2 & 0 & 0 & \vdots \\ 15 & 15 & 10 & 7 & 5 & 0 & \vdots \\ 52 & 52 & 37 & 27 & 20 & 15 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

$a_{i,j}$ is the number of $1-23$-avoiding permutations of length i ending with j.

Introducing the *backward difference operator* : ∇

$$\text{for } k \geq 3, a_{n,k} = a_{n,k-1} - a_{n-1,k-1} = \nabla a_{n,k-1}$$

So recursively :

$$\begin{aligned} \text{for } k \geq 3, a_{n,k} &= \nabla a_{n,k-1} \\ &= \nabla^2 a_{n,k-2} \\ &= \cdots \\ &= \nabla^{k-2} a_{n,2} = \nabla^{k-2} B_{n-1} \quad \text{(which holds also for } k = 2 \end{aligned}$$

Stating our first result

The distribution of $1-23$-avoiding permutations according to their length and to the value of their last entry is given by :

$$|\{\pi \in S_n(1-23) : \pi_n = 1\}| = B_{n-1}, \;\; n \geq 1;$$

$$|\{\pi \in S_n(1-23) : \pi_n = k\}| = \nabla^{k-2}(B_{n-1}), \;\; 2 \leq k \leq n.$$

$S(32-1)$: the reverse

If $\pi \in S_n(1-23)$ ends with k, then $\pi^r \in S_n(32-1)$, and $\pi^r_1 = k$.
Consequently :

$$|\{\pi \in S_n(32-1) : \pi_1 = 1\}| = B_{n-1}, \; n \geq 2$$

$$|\{\pi \in S_n(32-1) : \pi_1 = k\}| = \nabla^{k-2}(B_{n-1}), \; 2 \leq k \leq n$$

$S(3-21)$ and $S(12-3)$

- Complement :

$$|\{\pi \in S_n(3-21) : \pi_n = n\}| = B_{n-1}, n \geq 1$$

$$|\{\pi \in S_n(3-21) : \pi_n = k\}| = \nabla^{n-k-1}(B_{n-1}),\ 1 \leq k \leq n-1$$

- Reverse-complement :

$$|\{\pi \in S(12-3) : \pi_1 = n\}| = B_{n-1},\ n \geq 1$$

$$|\{\pi \in S(12-3) : \pi_1 = k\}| = \nabla^{n-k-1}(B_{n-1}),\ 1 \leq k \leq n-1$$

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   - The symmetry class $\{3-12, 21-3, 1-32, 23-1\}$
   - The symmetry class $\{2-13, 31-2, 2-31, 13-2\}$
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Same ideas

- One pattern in the class
- Succession rule
- Matrix of the distribution
- Recursive relation defining the entries of the matrix
- Extension to the remaining patterns in the symmetry class

M strikes again

The distribution of permutations avoiding $3-12$ according to their length (row index) and their last value (column index) is given by :

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 1 & 2 & 0 & 0 & 0 & \vdots \\ 5 & 3 & 2 & 5 & 0 & 0 & \vdots \\ 15 & 10 & 7 & 5 & 15 & 0 & \vdots \\ 52 & 37 & 27 & 20 & 15 & 52 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

Catalan triangle

The distribution of permutations avoiding $2-13$ according to their length (row index) and their last value (column index) is given by :

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 1 & 0 & 0 & \vdots \\ 14 & 14 & 9 & 4 & 1 & 0 & \vdots \\ 42 & 42 & 28 & 14 & 5 & 1 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

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4. Permutations avoiding a pair of generalized patterns
   - $S(1-23, 1-32)$: an easy case
   - $S(1-23, 21-3) = S(1-23, 21-3, 12-3)$: a not so easy case
5. Conclusion and perspectives

Avoiding more than one pattern

- Claesson and Mansour [2003] : enumeration of permutations avoiding any pair of generalized patterns of length 3, according to their length
- Bernini, Ferrari and Pinzani [2005] : enumeration of permutations avoiding any triple of generalized patterns of length 3, according to their length

Refine those enumerations according to the first or last entry ?
Two examples.

Labelling and succession rule

- $|S_n(1-23, 1-32)| = I_n$ n-th involution number

$\pi \in S(1-23, 1-32)$ is labelled (k, n) where k is the number of active sites of π.

- $k = 1$ when $\pi_n \neq 1$
- $k = n+1$ when $\pi_n = 1$

Succession rule :

$$\begin{cases} (2,1) \\ (1,n) \rightsquigarrow (n+2, n+1) \\ (n+1, n) \rightsquigarrow (1, n+1)^n (n+2, n+1) \end{cases}$$

Subsequent matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \vdots \\ 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & \vdots \\ 16 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & \vdots \\ 50 & 0 & 0 & 0 & 0 & 0 & 26 & 0 & \vdots \\ 156 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

Main steps

$|S_n(1-23, 21-3)| = |S_n(1-23, 21-3, 12-3)| = M_n$ n-th Motzkin number

- Succession rule with coloured labels.
- Generating tree.
- Matrix recording the number of labels at each level in the tree.
- Interpretation of this matrix as the distribution of $S(1-23, 21-3)$ according to the length and the last value
- Recursive description of the entries of the matrix.
- Generating function of each column of the matrix.

Distribution of $S(1-23, 21-3)$ according to the length and the last value

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \vdots \\
1 & 1 & 0 & 0 & 0 & 0 & \vdots \\
2 & 2 & 0 & 0 & 0 & 0 & \vdots \\
4 & 4 & 1 & 0 & 0 & 0 & \vdots \\
9 & 9 & 3 & 0 & 0 & 0 & \vdots \\
21 & 21 & 8 & 1 & 0 & 0 & \vdots \\
51 & 51 & 21 & 4 & 0 & 0 & \vdots \\
127 & 127 & 55 & 13 & 1 & 0 & \vdots \\
323 & 323 & 145 & 39 & 5 & 0 & \vdots \\
835 & 835 & 385 & 113 & 19 & 1 & \vdots \\
\dots & \dots & \dots & \dots & \dots & \dots & \ddots
\end{pmatrix}$$

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The end

- For any generalized pattern p of length 3, distribution of the p-avoiding permutations according to the length and the value of the first or last element
- Similar distributions for two sets of patterns

Can we get such a distribution for other sets of up to 3 patterns ? for all of them ?