Some statistics on permutations avoiding generalized patterns*

Antonio Bernini† Mathilde Bouvel‡ Luca Ferrari†

Abstract

In the last decade a huge amount of articles has been published studying pattern avoidance on permutations. From the point of view of enumeration, typically one tries to count permutations avoiding certain patterns according to their lengths. Here we tackle the problem of refining this enumeration by considering the statistics "first/last entry". We give complete results for every generalized patterns of type $(1,2)$ or $(2,1)$ as well as for some cases of permutations avoiding a pair of generalized patterns of the above types.

1 Introduction

Let $\pi \in S_n$ and $\tau \in S_k$, where S_t denotes the symmetric group on $[t] = \{1, 2, \ldots, t\}$. We say that π *avoids* τ if there are no subsequences $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ which are order-isomorphic to τ, that is having the entries π_{i_j} in the same relative order of the entries of τ. The permutation τ is called a (classical) *pattern*. We denote the set of all τ-avoiding permutations of S_n with $S_n(\tau)$. In [1], *generalized patterns* were introduced to study the Mahonian statistics on permutations. They are obtained by inserting one or more dashes among the entries of τ. A pattern $\tau = \tau_1 - \tau_2 - \ldots - \tau_k$ with $k-1$ dashes is called of type $(|\tau_1|, |\tau_2|, \ldots |\tau_k|)$, where $|\tau|$ is the length of τ. For instance, $\tau = 13 - 26 - 574$ is a pattern of type $(2,2,3)$. A classical pattern of length k can be seen as a pattern of type $\underbrace{(1,1,\ldots,1)}_{k}$, assuming that a dash is inserted, but not showed, between each pair of consecutive elements of the classical pattern. If $\tau \in S_3$, then generalized patterns deriving from τ are of type $(1,2)$ or $(2,1)$ according

*This work was partially supported by MIUR project: *Automi e linguaggi formali: aspetti matematici e applicativi*.

†Università di Firenze, Dipartimento di Sistemi e Informatica, viale Morgagni 65, 50134 Firenze, Italy `bernini@dsi.unifi.it, ferrari@math.unifi.it`

‡École Normale Supérieure de Cachan, Département informatique, 61 av. du Président Wilson, 94235, Cachan cedèx, France `mathilde.bouvel@dptinfo.ens-cachan.fr`

to the number of elements preceding and following the dash and they are collected in the set

$$\mathcal{P} = \{1-23, 12-3, 1-32, 13-2, 3-12, 31-2, 2-13, 21-3, \\ 2-31, 23-1, 3-21, 32-1\}.$$

A permutation π contains a generalized pattern $p \in \mathcal{P}$ if adjacent elements in p are also adjacent in π. For example $\pi = 7256134$ contains the generalized pattern $13-2$ in its subsequence $\pi_2\pi_3\pi_6 = 253$. Note that it does not contain the pattern $1-32$, but it contains the classical pattern 132 in the subsequences $\pi_2\pi_4\pi_6$ and $\pi_2\pi_4\pi_7$.

Let π be a permutation of S_n, then the *reverse* and *complement* permutations π^r and π^c, respectively, are defined as follows: $\pi_i^r = \pi_{n+1-i}$ and $\pi_i^c = n+1-\pi_i$, for $i = 1, \ldots, n$. We can define the reverse and the complement also for a pattern, regarding the dash as a particular entry in reversing π and leaving it in the same position when the complement π^c is performed. So, for example, if $p = 1-32$, then $p^r = 23-1$ and $p^c = 3-12$. Considering the composition of the reverse and the complement, it is easily seen that $p^{cr} = p^{rc}$. The set $\{p, p^r, p^c, p^{cr}\}$ is called the *symmetry class* of p. Observe that $|S_n(p)| = |S_n(p^r)| = |S_n(p^c)| = |S_n(p^{rc})|$.

The twelve generalized patterns of $\mathcal{P}$ are organized in three symmetry classes : $\{1-23, 32-1, 3-21, 12-3\}$, $\{3-12, 21-3, 1-32, 23-1\}$ and $\{2-13, 31-2, 2-31, 13-2\}$. If p and p' are two patterns such that $|S_n(p)| = |S_n(p')|$, then p and p' are said to be in the same *Wilf class* [7]. Since in [4] it is shown that

- $|S_n(p)| = B_n$,
  for $p \in \{1-23, 32-1, 3-21, 12-3\} \bigcup \{3-12, 21-3, 1-32, 23-1\}$
- $|S_n(p)| = C_n$,
  for $p \in \{2-13, 31-2, 2-31, 13-2\}$,

where B_n and C_n are the n-th Bell and Catalan numbers, respectively, then we can say that $\mathcal{P}$ is organized in two Wilf classes: $\{1-23, 32-1, 3-21, 12-3, 3-12, 21-3, 1-32, 23-1\}$ and $\{2-13, 31-2, 2-31, 13-2\}$.

In this work, we refine some enumerative results on $S(p)$, $p \in \mathcal{P}$, namely we count p-avoiding permutations, for each p, according to their length and the value of their first or last entry. Next we solve the same problem for some classes of permutations of the kind $S(p, q)$, $p, q \in \mathcal{P}$, and we conclude by proposing to tackle this problem for any remaining pair of generalized patterns of $\mathcal{P}$.

Our results are achieved by using the ECO method together with a graphical representation of permutations. In the following we only briefly recall the ECO construction for (patterns avoiding) permutations, for more details we refer the reader to [2] and [3].

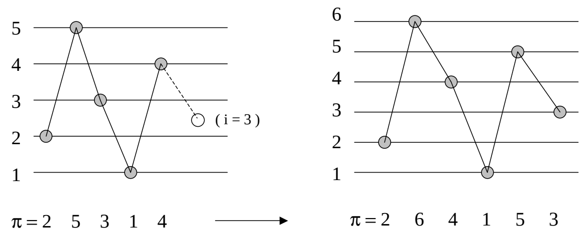

Figure 1: an ECO construction for permutations

Any permutation of length n can be visualized using a path-like representation, as in Figure 1. Note that the plane is divided in $n+1$ strips by the n horizontal lines which are numerated from 1 to n, starting from bottom (in the sequel, we refer to these strips as "regions": region i is included between line $i-1$ and line i, whereas region 1 is the one below line 1 and region $n+1$ is the one above line n). Each entry of the permutation is represented as a "node" lying on the line corresponding to its value, . If $\pi \in S_n$, then $n+1$ permutations belonging to S_{n+1} can be obtained by inserting a new node in each region of the plane. If we wish to generate the permutations in $S_{n+1}(P)$ obtained in such a way from $\pi \in S_n(P)$, where P is a set of forbidden patterns, then the regions the last node can be inserted in form a subset of all the $n+1$ possible regions; in the framework of the ECO method they are called *active sites* [2]. A remarkable feature of this construction is that, if $\pi \in S_n(P)$, then $\pi' \in S_{n+1}$ (which is obtained from π by inserting the last node in one of the regions) does not contain the patterns specified in P in its entries π'_j with $j = 1, \ldots, n$, otherwise π itself would contain some pattern of P. So, to decide if a region i is an active site or not, we just have to check those generalized patterns the last node is involved in.

2 The symmetry class $\{1-23, 32-1, 3-21, 12-3\}$

2.1 ECO construction and generating tree of $S(1-23)$

Let $\pi \in S_n(1-23)$. If $\pi_n = k \neq 1$, then π generates k permutations $\pi^{(i)} \in S_{n+1}(1-23)$, $i = 1, 2, \ldots, k$, by inserting a new node in region i. If $\pi_n = 1$, then π generates $n+1$ permutations by inserting a new node in any region. Note that in this case the number of sons of π is determined by the length of π. If $\pi^{(r)} \in S_{n+1}(1-23)$ denotes the permutation of $S_{n+1}(1-23)$ derived from $\pi \in S_n(1-23)$ by inserting the last node in region r, it is easily seen that $\pi^{(1)}$ generates, in turns, $n+2$ permutations, whereas $\pi^{(r)}$, $r \neq 1$, produces r permutations of $S_{n+2}(1-23)$. This ECO construction can be represented as in Figure 2 and, if we label with (k, n) each permutation of

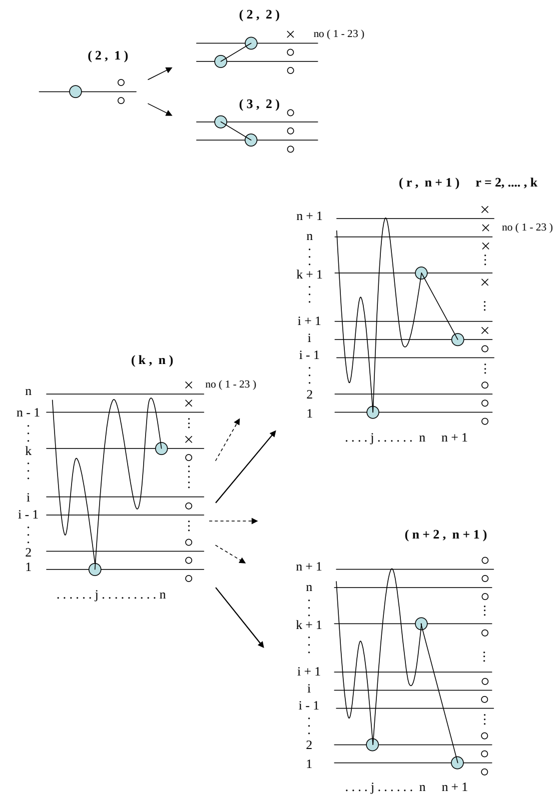

Figure 2: ECO construction of $S(1-23)$

$S_n(1-23)$ having k active sites, it can be encoded by the following succession rule:

$$\Omega : \begin{cases} (2,1) \\ (k,n) \rightsquigarrow (2,n+1)(3,n+1)\cdots(k,n+1)(n+2,n+1) \end{cases} .$$

We now wish to draw the generating tree related the the previous succession rule. For the sake of simplicity and for reasons that will become clear later, we choose to label the nodes of the generating tree using the number of their sons, which correspond to the first element of each label of the succession rule. In Figure 3 we have depicted the first levels of the generating tree of $S(1-23)$. Here the labels in bold character correspond to the labels of the kind $(n+1,n)$ in the succession rule. Observe that the production of each label depends on its level in the generating tree.

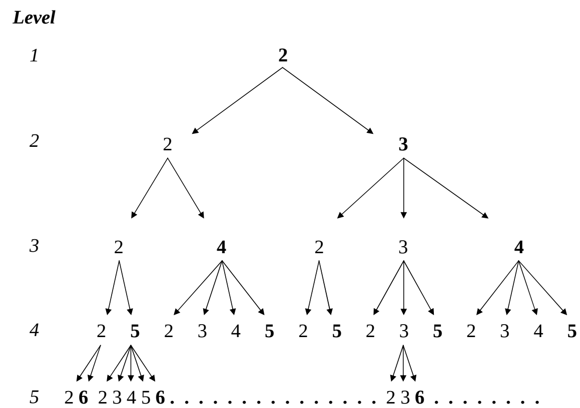

Figure 3: the generating tree of $S(1-23)$

2.2 Distribution according to the length and the last value

Starting from the generating tree of Figure 3, we can consider the matrix $M = (m_{ij})_{i,j\geq 1}$ where $m_{i,j}$ is the number of labels $j+1$ at level i in the generating tree:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 1 & 2 & 0 & 0 & 0 & \vdots \\ 5 & 3 & 2 & 5 & 0 & 0 & \vdots \\ 15 & 10 & 7 & 5 & 15 & 0 & \vdots \\ 52 & 37 & 27 & 20 & 15 & 52 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix}$$

The above matrix M is called the *ECO matrix* of the rule Ω, according to [6]. It is easily seen that M can be recursively described as follows:

1. $m_{1,1} = 1$ (the minimal permutation $\pi = 1$ has two sons);
2. $m_{n,k} = 0$ if $k > n$ (each permutation of length n has at most n sons);
3. $m_{n,k} = \sum_{i=k}^{n-1} m_{n-1,i}$ if $k < n$ (this derives directly from the recursive interpretation of the previous succession rule);
4. $m_{n,n} = m_{n,1}$ (each permutation of length $n-1$ produces precisely one son having label 2 and precisely one son having label $n+1$).

Since $m_{n,1}(= m_{n,n})$ is the sum of all the elements in the $(n-1)$-th row (for $n > 1$), this entry records the total number of $(1-23)$-avoiding permutations of length $n-1$. In other words, $m_{n,1} = B_{n-1}$.

Moreover, from a careful inspection of M, we have that $m_{n,k-1}$, with $k = 2, \ldots, n$, is the number of permutations of $S_n(1-23)$ ending with k and $m_{n,n}$ is the number of permutations of $S_n(1-23)$ ending with 1. Then, if we move the diagonal of M such that it becomes the first column of the matrix, we obtain a new matrix $A = (a_{i,j})_{i,j\geq 1}$ where $a_{i,j}$ is the number of $(1-23)$-avoiding permutations of length i ending with j.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 2 & 0 & 0 & \vdots \\ 15 & 15 & 10 & 7 & 5 & 0 & \vdots \\ 52 & 52 & 37 & 27 & 20 & 15 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

The matrix A is essentially the *Bell triangle*, which can be found in [10] together with several other references.

The above recursive properties of M can be immediately translated as follows:

1. $a_{1,1} = 1$ (the minimal permutation ends, trivially, with 1);
2. $a_{n,k} = 0$ if $k > n$ (each permutation of length n cannot end with a number greater than n itself);
3. $a_{n,k} = \sum_{i=k}^{n-1} a_{n-1,i} + a_{n-1,1}$ if $2 \leq k \leq n$ (the diagonal of M has been moved in the first column of A);
4. $a_{n,1} = a_{n,2}$ (since $a_{n,1} = m_{n,n} = m_{n,1} = a_{n,2}$).

From 3 we obtain, for $k \geq 3$:

$$a_{n,k} = a_{n,k-1} - a_{n-1,k-1},$$

If we denote by ∇ the usual *backward difference operator*, since $a_{n,2} = B_{n-1}$, we get:

$$\begin{aligned} a_{n,k} &= \nabla a_{n,k-1} \\ &= \nabla^2 a_{n,k-1} \\ &= \cdots \\ &= \nabla^{k-2} a_{n,2} = \nabla^{k-2} B_{n-1} \qquad \text{(which holds also for } k = 2). \end{aligned}$$

Thus we find the following formulas concerning the distribution of $1-23$-avoiding permutations according to their length and to the value of their last entry:

$$|\{\pi \in S_n(1-23) : \pi_n = 1\}| = B_{n-1}, \ n \geq 1;$$

$$|\{\pi \in S_n(1-23) : \pi_n = k\}| = \nabla^{k-2}(B_{n-1}),\ 2 \le k \le n.$$

2.3 The other patterns of the class

The arguments employed for $S(1-23)$ can be easily modified for the other patterns of the symmetry class of $1-23$, obtaining similar results. The ECO construction, in these cases, has to be adapted in order to obtain the same succession rule and the same generating tree we got for $S(1-23)$. The matrices M and A are defined as in the previous section.

1. For the reverse pattern of $1-23$, i.e. $32-1$, we find that $a_{i,j}$ is the number of permutations π of length i such that $\pi_1 = j$, and so:

   - $|\{\pi \in S_n(32-1) : \pi_1 = 1\}| = B_{n-1},\ n \ge 2$;
   - $|\{\pi \in S_n(32-1) : \pi_1 = k\}| = \nabla^{k-2}(B_{n-1}),\ 2 \le k \le n.$

   Note that in this case the ECO construction can be, in some way, "reversed", so that the active sites are not on the right of the diagram of the permutation π but on its left, i.e. before the first entry of π.

2. For the complement pattern $3-21$, we have that $a_{i,j}$ is the number of permutations of length i ending with $i+1-j$:

   - $|\{\pi \in S_n(3-21) : \pi_n = n\}| = B_{n-1}, n \ge 1$;
   - $|\{\pi \in S_n(3-21) : \pi_n = k\}| = \nabla^{n-k-1}(B_{n-1}),\ 1 \le k \le n-1.$

3. For the reverse-complement pattern $12-3$, $a_{i,j}$ is the number of permutations π of length i such that $\pi_1 = i+1-j$, and so:

   - $|\{\pi \in S(12-3) : \pi_1 = n\}| = B_{n-1},\ n \ge 1$;
   - $|\{\pi \in S(12-3) : \pi_1 = k\}| = \nabla^{n-k-1}(B_{n-1}),\ 1 \le k \le n-1.$

3 The symmetry class $\{3-12, 21-3, 1-32, 23-1\}$

3.1 ECO construction and generating tree of $S(3-12)$

Let $\pi \in S_n(3-12)$. If $\pi_n = k-1 \ne n$, then π generates k permutations $\pi^{(i)} \in S_{n+1}(3-12),\ i = 1, 2, \ldots, k-1, n+1$, by inserting a new node in region i. If $\pi_n = n$, then π generates $n+1$ permutations by inserting a new node in any region. As it happened for the class $S(1-23)$, note that the number of sons of π is determined by the length of π. It is easily seen that $\pi^{(n+1)}$ generates, in turns, $n+2$ permutations, whereas $\pi^{(i)}$ $(i \ne n+1)$ produces $i+1$ permutations. This ECO construction is illustrated in Figure 4. If each permutation of $S_n(3-12)$ with k active sites is labelled (k, n), then such a construction can be encoded using the following succession rule:

$$\begin{cases} (2,1) \\ (k,n) \rightsquigarrow (2,n+1)(3,n+1)\cdots(k,n+1)(n+2,n+1) \end{cases} .$$

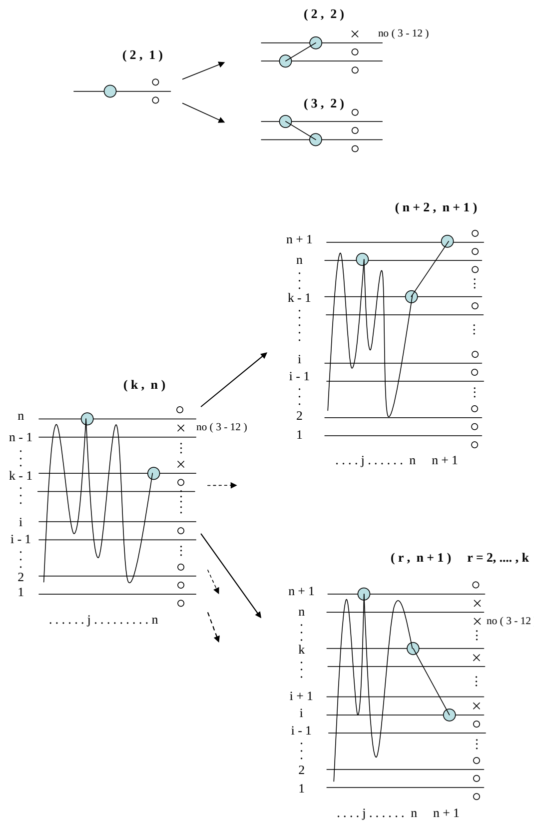

Figure 4: ECO construction of $S(3-12)$

Since it is the same succession rule we got for $S(1-23)$, the generating tree for $S(3-12)$ can be obtained in the same way.

3.2 Distribution according to the length and the last value

Defining the matrix $M = (m_{ij})_{i,j\geq 1}$ as in Section 2.2, it can be easily deduced that $m_{n,k}$ is the number of permutations of $S_n(3-12)$ ending with k. Note that in this case we do not need to move the diagonal of M to obtain the final matrix. Therefore, using again the backward difference operator ∇, the entries of M have the form:

$$m_{n,k} = \nabla^{k-1}(B_{n-1})$$

whence:

$$|\{\pi \in S_n(3-12) : \pi_n = n\}| = B_{n-1},\ n \geq 2\ ;$$
$$|\{\pi \in S_n(3-12) : \pi_n = k\}| = \nabla^{k-1}(B_{n-1}),\ 1 \leq k \leq n-1\ .$$

3.3 The other patterns of the class

Proceeding as in Section 2.3, we get:

- $|\{\pi \in S_n(21-3) : \pi_1 = n\}| = B_{n-1},\ n \geq 2;$
- $|\{\pi \in S_n(21-3) : \pi_1 = k\}| = \nabla^{k-1}(B_{n-1}),\ 1 \leq k \leq n-1;$
- $|\{\pi \in S_n(1-32) : \pi_n = 1\}| = B_{n-1},\ n \geq 2;$
- $|\{\pi \in S_n(1-32) : \pi_n = k\}| = \nabla^{n-k}(B_{n-1}),\ 2 \leq k \leq n\ ;$
- $|\{\pi \in S_n(23-1) : \pi_1 = 1\}| = B_{n-1},\ n \geq 2\ ;$
- $|\{\pi \in S_n(23-1) : \pi_1 = k\}| = \nabla^{n-k}(B_{n-1}),\ 2 \leq k \leq n.$

4 The symmetry class $\{2-13, 31-2, 2-31, 13-2\}$

The permutations of $S(2-13)$ are enumerated by Catalan numbers [4]. As far as the ECO construction of $S(2-13)$ is concerned, we just note that, if $\pi \in S_n(2-13)$ is such that $\pi_n = k$, then region i, for $i = 1, 2, \ldots, k+1$, is an active site for π. The succession rule encoding this construction is:

$$\left\{ \begin{array}{l} (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k+1) \end{array} \right.$$

Defining the matrix M as in the preceding sections, we obtain

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 1 & 0 & 0 & 0 & \vdots \\ 5 & 5 & 3 & 1 & 0 & 0 & \vdots \\ 14 & 14 & 9 & 4 & 1 & 0 & \vdots \\ 42 & 42 & 28 & 14 & 5 & 1 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

which is the well-known Catalan Triangle whose entries $m_{i,j} = \frac{j}{i}\binom{2i-j-1}{i-1}$ are the ballot numbers and whose properties can be found, for example, in [8].

In the following, we present the results for all the patterns of the class, which can be derived as in the previous sections (the $m_{n,k}$'s are defined as before):

- $|\{\pi \in S_n(2-13) : \pi_n = k\}| = m_{n,k} = \frac{k}{n}\binom{2n-k-1}{n-1}\ ;$
- $|\{\pi \in S_n(31-2) : \pi_1 = k\}| = m_{n,k} = \frac{k}{n}\binom{2n-k-1}{n-1}\ ;$

- $|\{\pi \in S_n(2-31) : \pi_n = k\}| = m_{n,n-k+1} = \frac{n-k+1}{n}\binom{n+k-2}{n-1}$;
- $|\{\pi \in S_n(13-2) : \pi_1 = k\}| = m_{n,n-k+1} = \frac{n-k+1}{n}\binom{n+k-2}{n-1}$.

5 Permutations avoiding a pair of generalized patterns of type $(1,2)$ or $(2,1)$

In [5] Claesson and Mansour counted permutations avoiding a pair of generalized patterns of type (1,2) or (2,1). Similarly to what we have done in the previous sections, we can study the distribution of the statistic "first/last entry" on permutations avoiding two or more generalized patterns. Here, we consider only two special examples, the former being quite easy, whereas the latter is surely more interesting. All the remaining cases are left to the readers as open problems for future research.

5.1 An easy case

We first deal with the permutations of $S(1-23, 1-32)$. This class is enumerated by the number I_n of involutions in S_n (see [5]). An ECO construction of this class can be encoded by the following succession rule :

$$\Omega : \begin{cases} (2,1) \\ (1,n) \rightsquigarrow (n+2,n+1) \\ (n+1,n) \rightsquigarrow (1,n+1)^n(n+2,n+1) \end{cases}$$

where the first element in the label is the number of active sites of the permutation and the second one is its length. This can be checked by representing permutations by means of the usual path-like representation: indeed, if a permutation ends with 1, then an element can be inserted on its right in any region, whereas if a permutation ends with $k \neq 1$, then the only element which can be inserted must be placed in region 1 on the right. The reader is invited to complete the details, so to obtain the construction described precisely by the succession rule Ω.

From the generating tree of Ω, the matrix M whose entry $m_{i,j}$ is the number of vertices with label j at level i $(i,j \geq 1)$ can be constructed as in the preceding cases:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \vdots \\ 6 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & \vdots \\ 16 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & \vdots \\ 50 & 0 & 0 & 0 & 0 & 0 & 26 & 0 & \vdots \\ 156 & 0 & 0 & 0 & 0 & 0 & 0 & 76 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix} .$$

The entries can be immediately computed as follows:

- $m_{1,1} = 0$, $m_{1,2} = 1$;
- $m_{n,1} = (n-1)m_{n-1,n}$, $n \geq 2$;
- $m_{n,n+1} = m_{n-1,1} + m_{n-1,n}$, $n \geq 2$;
- $m_{i,j} = 0$ in all the other cases.

From the ECO construction it easily appears that the first column of M counts the permutations π of $S_n(1-23, 1-32)$ such that $\pi_{n-1} = 1$ (or, which is the same, $\pi_n \neq 1$), whereas the super-diagonal sequence $m_{n,n+1}$ ($n \geq 1$) shows the number of π ending with 1. Since if $\pi \in S_n(1-23, 1-32)$, then $\pi_{n-1} = 1$ or $\pi_n = 1$, we deduce that the super-diagonal satisfies $m_{n,n+1} = I_{n-1}$ ($n \geq 1$).

5.2 A not so easy case

Our second example concerns the permutations of the class $S(1-23, 21-3)$, which also coincide with those of $S(1-23, 21-3, 12-3)$(see [3]) and are enumerated by Motzkin numbers. We will find the distribution of these permutations according to their length and their last entry; moreover, we will be able to derive the generating function of the sequences enumerating the permutations of this class whose last entry is k, for $k = 1, 2, \ldots$. We start by recalling the coloured succession rule Φ encoding an ECO construction for the above set of permutations (which can be found in [3]):

$$\Phi : \begin{cases} (\bar{2}) \\ (\bar{k}) \rightsquigarrow (\bar{2})(2)(3)\cdots(k) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(\overline{k+1}) \end{cases} .$$

In Figure 5, the first levels of the corresponding generating tree are presented.

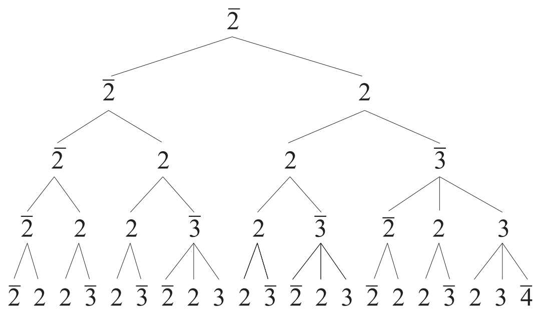

Figure 5: the generating tree of $S(1-23, 21-3)$

As in the preceding examples, we construct a matrix $A = (a_{i,j})_{i,j\geq 1}$ recording in its entries the number of labels at each level of the tree: namely, $a_{i,1}$ is the number of coloured label $\bar{k}$, $k \geq 2$, at level i of the tree and $a_{i,j}$, $j \geq 2$, is the number of labels j at level i. The first lines of A are:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \vdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \vdots \\ 2 & 2 & 0 & 0 & 0 & 0 & \vdots \\ 4 & 4 & 1 & 0 & 0 & 0 & \vdots \\ 9 & 9 & 3 & 0 & 0 & 0 & \vdots \\ 21 & 21 & 8 & 1 & 0 & 0 & \vdots \\ 51 & 51 & 21 & 4 & 0 & 0 & \vdots \\ 127 & 127 & 55 & 13 & 1 & 0 & \vdots \\ 323 & 323 & 145 & 39 & 5 & 0 & \vdots \\ 835 & 835 & 385 & 113 & 19 & 1 & \vdots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{pmatrix} .$$

As usual, we can find a recursive description of the entries of A:

- each label at level $i-1$ produces, among its sons, precisely one coloured label at level i, and so:

$$a_{i,1} = \sum_{r\geq 1} a_{i-1,r} \quad ;$$

- each label $j \geq 2$ at level i is generated either by a label $k \geq j$ at level $i-1$ or by a coloured label $\bar{k}$, with $k \geq j$ at level $i-1$, which, in turn, is generated by the label $k-1$ at level $i-2$, then:

$$a_{i,j} = \sum_{k\geq j} a_{i-1,k} + \sum_{k\geq j-1} a_{i-2,k} \quad \text{for} \quad j \geq 2 \; ; \tag{1}$$

- it is easily seen that, in the above generating tree, the coloured label $\bar{k}$ first appears at the odd level $2k-3$, whereas the label k first appears at the even level $2k-2$, whence:

$$a_{i,j} = 0 \quad \text{for} \quad j \geq \lfloor i/2 \rfloor + 2 \ .$$

The ECO construction of $S(1-23, 21-3)$ shows that, if a permutation has label k, then it ends with k, while if it has a coloured label $\bar{k}$, then its last entry is 1. Therefore, the entry $a_{i,j}$ is the number of permutations with length i and ending with the element j.

Our next aim is to find the generating function for the sequences displayed in the columns of the matrix A, which are the sequences enumerating the permutations of $S(1-23, 21-3)$ with last entry $j = 1, 2, \ldots$, according to their length. It is convenient to change a little bit the notation: from now on, we will index the lines of A starting from 0 instead of 1. First of all, we derive a simple recurrence for the entries of A: using (1), simple calculations show that

$$a_{n,k} = a_{n,k-1} - a_{n-1,k-1} - a_{n-2,k-2} \ , \quad \text{for} \quad k \geq 2 \ , \ n \geq 0 \ . \tag{2}$$

Let $C_k(x)$ be the generating function of the k-th column of A:

$$C_k(x) = \sum_{n \geq 0} a_{n,k} x^n \ .$$

Using (2), we find the following recurrence relation for $C_k(x)$:

$$C_{k+2}(x) = (1-x)C_{k+1}(x) - x^2 C_k(x), \quad k \geq 0 \ . \tag{3}$$

From the succession rule Φ (or from the ECO construction for $S(1-23, 21-3)$), it is easy to check that

$$C_0(x) = M(x), \quad C_1(x) = M(x) - 1 \ ,$$

where

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$$

is the generating function of Motzkin numbers $\{M_n\}_{n \geq 0}$. In order to find a closed form for $C_k(x)$, we define a linear operator L on the vector space $\mathcal{V}$ of formal power series of odd order. The set $(C_k(x))_{k \geq 1}$ is a basis of $\mathcal{V}$, so L can be defined as follows:

$$L(C_k(x)) = C_{k+1}(x) \quad \text{for} \quad k \geq 1. \tag{4}$$

From (3) it is:

$$L^2(C_k(x)) = (1-x)L(C_k(x)) - x^2 C_k(x)$$

which is the same of

$$(L^2 - (1-x)L + x^2)C_k(x) = 0 \ .$$

Therefore the operator $L^2 - (1-x)L + x^2$ must vanish on $\mathcal{V}$. Solving the equation $L^2 - (1-x)L + x^2 = 0$, leads to

$$L = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2} = x^2 M(x) \ .$$

Now, from (4), we obtain the desired closed form for $C_k(x)$:

$$C_k(x) = x^2 M(x) C_{k-1}(x) = \cdots = x^{2(k-1)} M^{k-1}(x)(M(x) - 1), \ k \geq 1 \ .$$

References

[1] E. Babson, E. Steingrímsson *Generalized permutation patterns and a classification of the Mahonian statistics*, Sém. Lothar. Combin. 44 (2000) Art. B44b, 18 pp. (electronic).

[2] E. Barcucci, A. Del Lungo, E. Pergola, R. Pinzani *ECO: A Methodology for the Enumeration of Combinatorial Objects*, J. Difference Equ. Appl. 5 (1999) 435-490.

[3] A. Bernini, L. Ferrari, R. Pinzani *Enumerating permutations avoiding three Babson-Steingrímsson patterns*, Ann. Comb. 9 (2005) 137-162.

[4] A. Claesson *Generalized pattern avoidance*, European J. Combin. 22 (2001) 961-971.

[5] A. Claesson, T. Mansour *Enumerating permutations avoiding a pair of Babson-Steingrímsson patterns*, Ars Combinatoria 77 (2005) 17-31.

[6] E. Deutsch, L. Ferrari, S. Rinaldi *Production matrices*, Adv. Appl. Math. 34 (2005) 101-122.

[7] T. Mansour *Permutations avoiding a pattern from S_k and at least two patterns from S_3*, Ars Combinatoria 62 (2001) 227-239.

[8] J. Noonan, D. Zeilberger *The Enumeration of Permutations with a Prescribed Number of 'Forbidden' Patterns*, Adv. in Appl. Math. 17 (1996) 381-407.

[9] R. Simion, F. W. Schmidt *Restricted Permutations*, European J. Combin. 6 (1985) 383-406.

[10] E. W. Weisstein. "Bell Triangle." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/BellTriangle.html