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Some classical multiple orthogonal polynomials ☆

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1. Classical orthogonal polynomials

One aspect in the theory of orthogonal polynomials is their study as special functions. Most important orthogonal polynomials can be written as terminating hypergeometric series and during the twentieth century people have been working on a classification of all such hypergeometric orthogonal polynomial and their characterizations.

The *very classical orthogonal polynomials* are those named after Jacobi, Laguerre, and Hermite. In this paper we will always be considering monic polynomials, but in the literature one often uses a different normalization. Jacobi polynomials are (monic) polynomials of degree n which are orthogonal to all lower degree polynomials with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ on $[-1,1]$, where $\alpha,\beta>-1$. The change of variables $x\mapsto 2x-1$ gives Jacobi polynomials on $[0,1]$ for the weight function $w(x)=x^\beta(1-x)^\alpha$, and we will denote these (monic) polynomials by $P_n^{(\alpha,\beta)}(x)$. They are defined by the orthogonality conditions

$$\int_0^1 P_n^{(\alpha,\beta)}(x)x^\beta(1-x)^\alpha x^k\,\mathrm{d}x=0,\quad k=0,1,\ldots,n-1. \tag{1.1}$$

The monic Laguerre polynomials $L_n^{(\alpha)}(x)$ (with $\alpha>-1$) are orthogonal on $[0,\infty)$ to all polynomials of degree less than n with respect to the weight $w(x)=x^\alpha\mathrm{e}^{-x}$ and hence satisfy the orthogonality conditions

$$\int_0^\infty L_n^{(\alpha)}(x)x^\alpha\mathrm{e}^{-x}x^k\,\mathrm{d}x=0,\quad k=0,1,\ldots,n-1. \tag{1.2}$$

Finally, the (monic) Hermite polynomials $H_n(x)$ are orthogonal to all lower degree polynomials with respect to the weight function $w(x)=\mathrm{e}^{-x^2}$ on $(-\infty,\infty)$, so that

$$\int_{-\infty}^\infty H_n(x)\mathrm{e}^{-x^2}x^k\,\mathrm{d}x=0,\quad k=0,1,\ldots,n-1. \tag{1.3}$$

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These three families of orthogonal polynomials can be characterized in a number of ways:

- Their weight functions w satisfy a first order differential equation with polynomial coefficients

$$\sigma(x)w'(x) = \rho(x)w(x), \tag{1.4}$$

  with σ of degree at most two and ρ of degree one. This equation is known as *Pearson's equation* and also appears in probability theory, where the corresponding weights (densities) are known as the beta density (Jacobi), the gamma density (Laguerre), and the normal density (Hermite). Note however that for probability density functions one needs to normalize these weights appropriately. For the Jacobi weight we have $\sigma(x) = x(1-x)$, for the Laguerre weight we have $\sigma(x) = x$, and for the Hermite weight we see that $\sigma(x) = 1$, so that each family corresponds to a different degree of the polynomial σ.
- The derivatives of the very classical polynomials are again orthogonal polynomials of the same family but with different parameters (Sonin, 1887; Hahn, 1949). Indeed, integration by parts of the orthogonality relations and the use of Pearson's equation show that

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x} L_n^{(\alpha)}(x) = nL_{n-1}^{(\alpha+1)}(x),$$

$$\frac{\mathrm{d}}{\mathrm{d}x} H_n(x) = nH_{n-1}(x).$$

  The differential operator $\mathrm{D} = \mathrm{d}/\mathrm{d}x$ therefore acts as a *lowering operator* that lowers the degree of the polynomial.
- Pearson's equation also gives rise to a *raising operator* that raises the degree of the polynomials. Indeed, integration by parts shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^\beta(1-x)^\alpha P_n^{(\alpha,\beta)}(x)] = -(\alpha+\beta+n)x^{\beta-1}(1-x)^{\alpha-1}P_{n+1}^{(\alpha-1,\beta-1)}(x), \tag{1.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x)] = -x^{\alpha-1}\mathrm{e}^{-x}L_{n+1}^{(\alpha-1)}(x), \tag{1.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{e}^{-x^2}H_n(x)] = -2\mathrm{e}^{-x^2}H_{n+1}(x). \tag{1.7}$$

  The raising operator is therefore of the form $\sigma(x)/w(x)\mathrm{D}w(x)$. Using this raising operation repeatedly gives the *Rodrigues formula* for these orthogonal polynomials:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[x^{\beta+n}(1-x)^{\alpha+n}] = (-1)^n(\alpha+\beta+n+1)_n x^\beta(1-x)^\alpha P_n^{(\alpha,\beta)}(x), \tag{1.8}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[x^{\alpha+n}\mathrm{e}^{-x}] = (-1)^n x^\alpha \mathrm{e}^{-x} L_n^{(\alpha)}(x), \tag{1.9}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2} = (-1)^n 2^n \mathrm{e}^{-x^2} H_n(x). \tag{1.10}$$

The Rodrigues formula is therefore of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[\sigma^n(x)w(x)] = C_n w(x)P_n(x),$$

where C_n is a normalization constant (Hildebrandt, 1931).

- Combining the lowering and the raising operator gives a *linear second-order differential equation* for these orthogonal polynomials, of the form

$$\sigma(x)y''(x) + \tau(x)y'(x) = \lambda_n y(x), \tag{1.11}$$

where σ is a polynomial of degree at most 2 and τ a polynomial of degree at most 1, both independent of the degree n, and λ_n is a constant depending on n (Bochner, 1929).

The Laguerre polynomials and the Hermite polynomials are limiting cases of the Jacobi polynomials. Indeed, one has

$$\lim_{\alpha\to\infty} \alpha^n P_n^{(\alpha,\beta)}(x/\alpha) = L_n^{(\beta)}(x), \tag{1.12}$$

and

$$\lim_{\alpha\to\infty} 2^n \alpha^{n/2} P_n^{(\alpha,\alpha)}\left(\frac{x+\sqrt{\alpha}}{2\sqrt{\alpha}}\right) = H_n(x). \tag{1.13}$$

The Hermite polynomials are also a limit case of the Laguerre polynomials:

$$\lim_{\alpha\to\infty} (2\alpha)^{-n/2} L_n^{(\alpha)}(\sqrt{2\alpha}x + \alpha) = H_n(x). \tag{1.14}$$

In this respect the Jacobi, Laguerre and Hermite polynomials are in a hierarchy, with Jacobi leading to Laguerre and Laguerre leading to Hermite, and with a shortcut for Jacobi leading to Hermite. This is just a very small piece in a large table known as Askey's table which also contains classical orthogonal polynomials of a discrete variable (Hahn, Meixner, Kravchuk, and Charlier) for which the differential operator D needs to be replaced by difference operators Δ and ∇ on a linear lattice (a lattice with constant mesh, see [30]). Finally, allowing a quadratic lattice also gives Meixner–Pollaczek, dual Hahn, continuous Hahn, continuous dual Hahn, Racah, and Wilson polynomials, which are all in the Askey table. These polynomials have a number of q-extensions involving the q-difference operator and leading to the q-extension of the Askey table. In [2] Andrews and Askey suggest to define the classical orthogonal polynomials as those polynomials that are a limiting case of the ${}_4\varphi_3$-polynomials

$$R_n(\lambda(x); a,b,c,d; q) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n+1}ab, q^{-x}, q^{x+1}cd \\ aq, bdq, cq \end{matrix}; q, q\right),$$

with $\lambda(x) = q^{-x} + q^{x+1}cd$ and $bdq = q^{-N}$ (these are the q-Racah polynomials) or the ${}_4\varphi_3$-polynomials

$$\frac{a^n W_n(x; a,b,c,d|q)}{(ab;q)_n(ac;q)_n(ad;q)_n} = {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta} \\ ab, ac, ad \end{matrix}; q, q\right),$$

with $x = \cos\theta$ (these are the Askey–Wilson polynomials). All these classical orthogonal polynomials then have the following properties:

- they have a Rodrigues formula,

- an appropriate divided difference operator acting on them gives a set of orthogonal polynomials,
- they satisfy a second-order difference or differential equation in x which is of Sturm–Liouville type.

The classical orthogonal polynomials in this wide sense have been the subject of intensive research during the twentieth century. We recommend the report by Koekoek and Swarttouw [26], the book by Andrews et al. [3], and the books by Nikiforov and Uvarov [31], and Nikiforov, Suslov and Uvarov [30] for more material. Szegő's book [44] is still a very good source for the very classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. For characterization results one should consult a survey by Al-Salam [1].

2. Multiple orthogonal polynomials

Recently, there has been a renewed interest in an extension of the notion of orthogonal polynomials known as multiple orthogonal polynomials. This notion comes from simultaneous rational approximation, in particular from Hermite–Padé approximation of a system of r functions, and hence has its roots in the nineteenth century. However, only recently examples of multiple orthogonal polynomials appeared in the (mostly Eastern European) literature. In this paper we will introduce multiple orthogonal polynomials using the orthogonality relations and we will only use weight functions. The extension to measures is straightforward.

Suppose we are given r weight functions $w_1, w_2, \ldots, w_r$ on the real line and that the support of each w_i is a subset of an interval Δ_i. We will often be using a multi-index $\boldsymbol{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and its length $|\boldsymbol{n}| = n_1 + n_2 + \cdots + n_r$.

- The r-vector of *type I multiple orthogonal polynomials* $(A_{\boldsymbol{n},1}, \ldots, A_{\boldsymbol{n},r})$ is such that each $A_{\boldsymbol{n},i}$ is a polynomial of degree $n_i - 1$ and the following orthogonality conditions hold:

$$\int x^k \sum_{j=1}^{r} A_{\boldsymbol{n},j}(x) w_j(x) \, \mathrm{d}x = 0, \quad k = 0, 1, 2, \ldots, |\boldsymbol{n}| - 2. \tag{2.15}$$

  Each $A_{\boldsymbol{n},i}$ has n_i coefficients so that the type I vector is completely determined if we can find all the $|\boldsymbol{n}|$ unknown coefficients. The orthogonality relations (2.15) give $|\boldsymbol{n}| - 1$ linear and homogeneous relations for these $|\boldsymbol{n}|$ coefficients. If the matrix of coefficients has full rank, then we can determine the type I vector uniquely up to a multiplicative factor.
- The type II multiple orthogonal polynomial $P_{\boldsymbol{n}}$ is the polynomial of degree $|\boldsymbol{n}|$ that satisfies the following orthogonality conditions:

$$\int_{\Delta_1} P_{\boldsymbol{n}}(x) w_1(x) x^k \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_1 - 1, \tag{2.16}$$

$$\int_{\Delta_2} P_{\boldsymbol{n}}(x) w_2(x) x^k \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_2 - 1, \tag{2.17}$$

$$\vdots$$

$$\int_{\Delta_r} P_{\boldsymbol{n}}(x) w_r(x) x^k \, \mathrm{d}x = 0, \quad k = 0, 1, \ldots, n_r - 1. \tag{2.18}$$

This gives $|\boldsymbol{n}|$ linear and homogeneous equations for the $|\boldsymbol{n}|+1$ unknown coefficients of $P_{\boldsymbol{n}}(x)$. We will choose the type II multiple orthogonal polynomials to be monic so that the remaining $|\boldsymbol{n}|$ coefficients can be determined uniquely by the orthogonality relations, provided the matrix of coefficients has full rank.

In this paper the emphasis will be on type II multiple orthogonal polynomials. The unicity of multiple orthogonal polynomials can only be guaranteed under additional assumptions on the r weights. Two distinct cases for which the type II multiple orthogonal polynomials are given are as follows.

1. In an *Angelesco system* (Angelesco, 1918) the intervals Δ_i, on which the weights are supported, are disjoint, i.e., $\Delta_i \cap \Delta_j = \emptyset$ whenever $i \neq j$. Actually, it is sufficient that the open intervals $\mathring{\Delta}_i$ are disjoint, so that the closed intervals Δ_i are allowed to touch.

Theorem 1. *In an Angelesco system the Type II multiple orthogonal polynomial $P_{\boldsymbol{n}}(x)$ factors into r polynomials $\prod_{j=1}^r q_{n_j}(x)$, where each q_{n_j}* has exactly *n_j zeros on Δ_j.*

Proof. Suppose $P_{\boldsymbol{n}}(x)$ has $m_j < n_j$ sign changes on Δ_j at the points $x_1,\dots,x_{m_j}$. Let $Q_{m_j}(x) = (x-x_1)\cdots(x-x_{m_j})$, then $P_{\boldsymbol{n}}(x)Q_{m_j}(x)$ does not change sign on Δ_j, and hence

$$\int_{\Delta_j} P_{\boldsymbol{n}}(x)Q_{m_j}(x)w_j(x)\,\mathrm{d}x \neq 0.$$

But this is in contradiction with the orthogonality relation on Δ_j. Hence $P_{\boldsymbol{n}}(x)$ has at least n_j zeros on Δ_j. Now all the intervals Δ_j $(j=1,2,\dots,r)$ are disjoint, hence this gives at least $|\boldsymbol{n}|$ zeros of $P_{\boldsymbol{n}}(x)$ on the real line. The degree of this polynomial is precisely $|\boldsymbol{n}|$, so there are exactly n_j zeros on each interval Δ_j. $\square$

2. For an *AT system* all the weights are supported on the same interval Δ, but we require that the $|\boldsymbol{n}|$ functions

$$w_1(x), xw_1(x), \dots, x^{n_1-1}w_1(x), w_2(x), xw_2(x), \dots, x^{n_2-1}w_2(x), \dots, w_r(x), xw_r(x), \dots, x^{n_r-1}w_r(x)$$

form a Chebyshev system on Δ for each multi-index $\boldsymbol{n}$. This means that every linear combination

$$\sum_{j=1}^r Q_{n_j-1}(x)w_j(x),$$

with Q_{n_j-1} a polynomial of degree at most n_j-1, has at most $|\boldsymbol{n}|-1$ zeros on Δ.

Theorem 2. *In an AT system the Type II multiple orthogonal polynomial $P_{\boldsymbol{n}}(x)$ has exactly $|\boldsymbol{n}|$ zeros on Δ. For the Type I vector of multiple orthogonal polynomials, the linear combination $\sum_{j=1}^r A_{\boldsymbol{n},j}(x)w_j(x)$ has exactly $|\boldsymbol{n}|-1$ zeros on Δ.*

Proof. Suppose $P_{\boldsymbol{n}}(x)$ has $m < |\boldsymbol{n}|$ sign changes on Δ at the points $x_1,\dots,x_m$. Take a multi-index $\boldsymbol{m}=(m_1,m_2,\dots,m_r)$ with $|\boldsymbol{m}| = m$ such that $m_i \leqslant n_i$ for every i and $m_j < n_j$ for some j and construct the function

$$Q(x) = \sum_{i=1}^{r} Q_i(x) w_i(x),$$

where each Q_i is a polynomial of degree $m_i - 1$ whenever $i \neq j$, and Q_j is a polynomial of degree m_j, satisfying the interpolation conditions

$$Q(x_k) = 0, \quad k = 1,2,\dots,m,$$

and $Q(x_0) = 1$ for an additional point $x_0 \in \Delta$. This interpolation problem has a unique solution since we are dealing with a Chebyshev system. The function Q has already m zeros, and since we are in a Chebyshev system, it can have no additional sign changes. Furthermore, the function does not vanish identically since $Q(x_0) = 1$. Obviously $P_{\boldsymbol{n}}(x)Q(x)$ does not change sign on Δ, so that

$$\int_\Delta P_{\boldsymbol{n}}(x)Q(x)\,\mathrm{d}x \neq 0,$$

but this is in contrast with the orthogonality relations for the Type II multiple orthogonal polynomial. Hence $P_{\boldsymbol{n}}(x)$ has exactly $|\boldsymbol{n}|$ zeros on Δ.

The proof for the Type I multiple orthogonal polynomials is similar. First of all, since we are dealing with an AT system, the function

$$A(x) = \sum_{j=1}^{r} A_{\boldsymbol{n},j}(x) w_j(x)$$

has at most $|\boldsymbol{n}| - 1$ zeros on Δ. Suppose it has $m < |\boldsymbol{n}| - 1$ sign changes at the points $x_1,x_2,\dots,x_m$, then we use the polynomial $Q_m(x) = (x - x_1)\cdots(x - x_m)$ so that $A(x)Q_m(x)$ does not change sign on Δ, and

$$\int_\Delta A(x)Q(x)\,\mathrm{d}x \neq 0,$$

which is in contradiciton with the orthogonality of the Type I multiple orthogonal polynomial. Hence $A(x)$ has exactly $|\boldsymbol{n}| - 1$ zeros on Δ. □

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation. There are also finite-order recurrences for multiple orthogonal polynomials, and there are quite a few of recurrence relations possible since we are dealing with multi-indices. There is an interesting recurrence relation of order $r + 1$ for the Type II multiple orthogonal polynomials with nearly diagonal multi-indices. Let $n \in \mathbb{N}$ and write it as $n = kr + j$, with $0 \leqslant j < r$. The nearly diagonal multi-index $\boldsymbol{s}(n)$ corresponding to n is then given by

$$\boldsymbol{s}(n) = (\underbrace{k+1, k+1, \dots, k+1}_{j \text{ times}}, \underbrace{k, k, \dots, k}_{r-j \text{ times}}).$$

If we denote the corresponding multiple orthogonal polynomials by

$$P_n(x) = P_{s(n)}(x),$$

then the following recurrence relation holds:

$$xP_n(x) = P_{n+1}(x) + \sum_{j=0}^{r} a_{n,j} P_{n-j}(x), \tag{2.19}$$

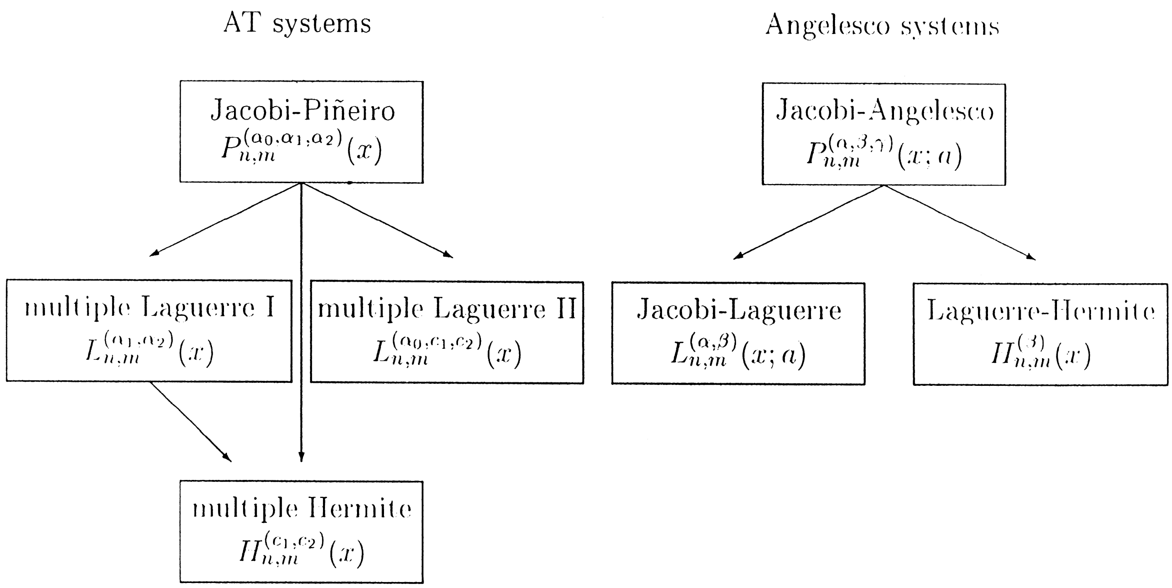
with initial conditions $P_0(x) = 1$, $P_j(x) = 0$ for $j = -1, -2, \ldots, -r$. The matrix

$$\begin{pmatrix}
a_{0,0} & 1 & & & & & & & & \\
a_{1,1} & a_{1,0} & 1 & & & & & & & \\
a_{2,2} & a_{2,1} & a_{2,0} & 1 & & & & & & \\
\vdots & & & \ddots & \ddots & & & & & \\
a_{r,r} & a_{r,r-1} & \cdots & & a_{r,0} & 1 & & & & \\
 & a_{r+1,r} & \ddots & & & a_{r+1,0} & 1 & & & \\
 & & \ddots & \ddots & & & \ddots & \ddots & & \\
 & & & \ddots & \ddots & & & \ddots & 1 \\
 & & & & a_{n,r} & a_{n,r-1} & \cdots & a_{n,1} & a_{n,0}
\end{pmatrix}$$

has eigenvalues at the zeros of $P_{n+1}(x)$, so that in the case of Angelesco systems or AT systems we are dealing with nonsymmetric matrices with real eigenvalues. The infinite matrix will act as an operator on ℓ^2, but this operator is never self-adjoint and furthermore has not a simple spectrum, as is the case for ordinary orthogonal polynomials. Now there will be a set of r cyclic vectors and the spectral theory of this operator becomes more complicated (and more interesting). There are many open problems concerning this nonsymmetric operator.

3. Some very classical multiple orthogonal polynomials

We will now describe seven families of multiple orthogonal polynomials which have the same flavor as the very classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. They certainly deserve to be called classical since they have a Rodrigues formula and there is a first-order differential operator which, when applied to these classical multiple orthogonal polynomials, gives another set of multiple orthogonal polynomials. However, these are certainly not the only families of classical multiple orthogonal polynomials (see Section 4.1). The first four families are AT systems which are connected by limit passages, the last three families are Angelesco systems which are also connected by limit passages. All these families have been introduced in the literature before. We will list some of their properties and give explicit formulas, most of which have not appeared earlier.



3.1. Jacobi–Piñeiro polynomials

The Jacobi–Piñeiro polynomials are multiple orthogonal polynomials associated with an AT system consisting of Jacobi weights on $[0,1]$ with different singularities at 0 and the same singularity at 1. They were first studied by Piñeiro [37] when $\alpha_0 = 0$. The general case appears in [34, p. 162]. Let $\alpha_0 > -1$ and $\alpha_1,\ldots,\alpha_r$ be such that each $\alpha_i > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. The Jacobi–Piñeiro polynomial $P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}$ for the multi-index $\boldsymbol{n} = (n_1,n_2,\ldots,n_r) \in \mathbb{N}^r$ and $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_r)$ is the monic polynomial of degree $|\boldsymbol{n}| = n_1 + n_2 + \cdots + n_r$ that satisfies the orthogonality conditions

$$\int_0^1 P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x)x^{\alpha_1}(1-x)^{\alpha_0}x^k\,dx = 0, \quad k = 0,1,\ldots,n_1-1, \tag{3.20}$$

$$\int_0^1 P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x)x^{\alpha_2}(1-x)^{\alpha_0}x^k\,dx = 0, \quad k = 0,1,\ldots,n_2-1, \tag{3.21}$$

$$\int_0^1 P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x)x^{\alpha_r}(1-x)^{\alpha_0}x^k\,dx = 0, \quad k = 0,1,\ldots,n_r-1. \tag{3.22}$$

Since each weight $w_i(x) = x^{\alpha_i}(1-x)^{\alpha_0}$ satisfies a Pearson equation

$$x(1-x)w_i'(x) = [\alpha_i(1-x) - \alpha_0 x]w_i(x)$$

and the weights are related by

$$w_i(x) = x^{\alpha_i-\alpha_j}w_j(x),$$

one can use integration by parts on each of the r integrals (3.20)–(3.22) to find the following raising operators:

$$\frac{d}{dx}(x^{\alpha_j}(1-x)^{\alpha_0}P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x)) = -(|\boldsymbol{n}|+\alpha_0+\alpha_j)x^{\alpha_j-1}(1-x)^{\alpha_0-1}P_{\boldsymbol{n}+\boldsymbol{e}_j}^{(\alpha_0-1,\boldsymbol{\alpha}-\boldsymbol{e}_j)}(x), \tag{3.23}$$

where $\boldsymbol{e}_j$ is the jth standard unit vector. Repeatedly using this raising operator gives the Rodrigues formula

$$(-1)^{|\boldsymbol{n}|}\prod_{j=1}^{r}(|\boldsymbol{n}|+\alpha_0+\alpha_j+1)_{n_j}P_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{\alpha})}(x) = (1-x)^{-\alpha_0}\prod_{j=1}^{r}\left[x^{-\alpha_j}\frac{d^{n_j}}{dx^{n_j}}x^{n_j+\alpha_j}\right](1-x)^{\alpha_0+|\boldsymbol{n}|}. \tag{3.24}$$

The product of the r differential operators $x^{-\alpha_j}D^{n_j}x^{n_j+\alpha_j}$ on the right-hand side can be taken in any order since these operators are commuting.

The Rodrigues formula allows us to obtain an explicit expression. For the case $r=2$ we write

$$\begin{aligned}&(-1)^{n+m}(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)\\ &=(1-x)^{-\alpha_0}x^{-\alpha_1}\frac{d^n}{dx^n}x^{\alpha_1-\alpha_2+n}\frac{d^m}{dx^m}x^{\alpha_2+m}(1-x)^{\alpha_0+n+m}.\end{aligned} \tag{3.25}$$

The mth derivative can be worked out using the Rodrigues formula (1.8) for Jacobi polynomials and gives

$$(-1)^n(n+m+\alpha_0+\alpha_1+1)_n P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x) = (1-x)^{-\alpha_0}x^{-\alpha_1}\frac{d^n}{dx^n}x^{\alpha_1+n}(1-x)^{\alpha_0+n}P_m^{(\alpha_0+n,\alpha_2)}(x).$$

Now use Leibniz' rule to work out the nth derivative:

$$\begin{aligned}&(-1)^n(n+m+\alpha_0+\alpha_1+1)_n P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x)\\ &=(1-x)^{-\alpha_0}x^{-\alpha_1}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}x^{\alpha_1+n}\frac{d^{n-k}}{dx^{n-k}}(1-x)^{\alpha_0+n}P_m^{(\alpha_0+n,\alpha_2)}(x).\end{aligned}$$

In order to work out the derivative involving the Jacobi polynomial, we will use the following lemma.

Lemma 3. *Let $P_n^{(\alpha,\beta)}(x)$ be the nth degree monic Jacobi polynomial on* $[0,1]$. *Then for $\alpha>0$ and $\beta>-1$*

$$\frac{d}{dx}[(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)] = -(\alpha+n)(1-x)^{\alpha-1}P_n^{(\alpha-1,\beta+1)}(x), \tag{3.26}$$

and

$$\frac{d^m}{dx^m}[(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)] = (-1)^m(\alpha+n-m+1)_m(1-x)^{\alpha-m}P_n^{(\alpha-m,\beta+m)}(x). \tag{3.27}$$

Proof. First of all, observe that

$$\frac{d}{dx}[(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)] = (1-x)^{\alpha-1}(-\alpha P_n^{(\alpha,\beta)}(x)+(1-x)[P_n^{(\alpha,\beta)}(x)]'),$$

so that the right-hand side is $-(\alpha+n)(1-x)^{\alpha-1}Q_n(x)$, with Q_n a monic polynomial of degree n. Integrating by parts gives

$$\begin{aligned}
&-(\alpha+n)\int_0^1 (1-x)^{\alpha-1}x^{\beta+k+1}Q_n(x)\,\mathrm{d}x \\
&\quad = x^{\beta+k+1}(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)|_0^1 - (\beta+k+1)\int_0^1 x^{\beta+k}(1-x)^{\alpha}P_n^{(\alpha,\beta)}(x)\,\mathrm{d}x.
\end{aligned}$$

Obviously, when $\alpha>0$ and $\beta>-1$, then the integrated terms on the right-hand side vanish. The integral on the right-hand side vanishes for $k=0,1,\dots,n-1$ because of orthogonality. Hence Q_n is a monic polynomial which is orthogonal to all polynomials of degree less than n with respect to the weight $x^{\beta+1}(1-x)^{\alpha-1}$, which proves (3.26). The more general expression (3.27) follows by applying (3.26) m times. □

By using this lemma we arrive at

$$\begin{aligned}
&(n+m+\alpha_0+\alpha_1+1)_n P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x) \\
&\quad = n!\sum_{k=0}^{n}\binom{\alpha_1+n}{k}\binom{\alpha_0+m+n}{n-k}x^{n-k}(x-1)^k P_m^{(\alpha_0+k,\alpha_2+n-k)}(x).
\end{aligned}$$

For the Jacobi polynomial we have the expansion

$$(\alpha+\beta+n+1)_n P_n^{(\alpha,\beta)}(x) = n!\sum_{j=0}^{n}\binom{\beta+n}{j}\binom{\alpha+n}{n-j}x^{n-j}(x-1)^j, \tag{3.28}$$

which can easily be obtained from the Rodrigues formula (1.8) by using Leibniz' formula, so that we finally find

$$\begin{aligned}
&(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x) \\
&\quad = n!m!\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{\alpha_1+n}{k}\binom{\alpha_0+m+n}{n-k}\binom{\alpha_2+n+m-k}{j}\binom{\alpha_0+k+m}{m-j}x^{n+m-k-j}(x-1)^{k+j}.
\end{aligned} \tag{3.29}$$

We can explicitly find the first few coefficients of $P_{m,n}^{(\alpha_0,\alpha_1,\alpha_2)}(x)$ from this expression. We introduce the notation

$$\begin{aligned}
K_{n,m} &= \frac{n!m!}{(n+m+\alpha_0+\alpha_1+1)_n(n+m+\alpha_0+\alpha_2+1)_m} \\
&= \binom{\alpha_0+\alpha_1+2n+m}{n}^{-1}\binom{\alpha_0+\alpha_2+2m+n}{m}^{-1}.
\end{aligned}$$

First let us check that the polynomial is indeed monic by working out the coefficient of x^{m+n}. This is given by

$$K_{n,m}\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{\alpha_1+n}{k}\binom{\alpha_0+m+n}{n-k}\binom{\alpha_2+n+m-k}{j}\binom{\alpha_0+k+m}{m-j}.$$

The sum over j can be evaluated using the Chu–Vandermonde identity

$$\sum_{j=0}^{m} \binom{\alpha_2+n+m-k}{j} \binom{\alpha_0+k+m}{m-j} = \binom{\alpha_0+\alpha_2+n+2m}{m},$$

which is independent of k. The remaining sum over k can also be evaluated and gives

$$\sum_{k=0}^{n} \binom{\alpha_1+n}{k} \binom{\alpha_0+m+n}{n-k} = \binom{\alpha_0+\alpha_1+m+2n}{n},$$

and the double sum is therefore equal to $K_{n,m}^{-1}$, showing that this polynomial is indeed monic. Now let us write

$$P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x) = x^{m+n} + A_{n,m}x^{n+m-1} + B_{n,m}x^{n+m-2} + C_{n,m}x^{n+m-3} + \cdots .$$

The coefficient $A_{n,m}$ of x^{m+n-1} is given by

$$-K_{n,m} \sum_{k=0}^{n} \sum_{j=0}^{m} (k+j) \binom{\alpha_1+n}{k} \binom{\alpha_0+m+n}{n-k} \binom{\alpha_2+n+m-k}{j} \binom{\alpha_0+k+m}{m-j}.$$

This double sum can again be evaluated using Chu–Vandermonde and gives

$$A_{n,m} = -\frac{n(\alpha_1+n)(\alpha_0+\alpha_2+n+m) + m(\alpha_2+n+m)(\alpha_0+\alpha_1+2n+m)}{(\alpha_0+\alpha_1+2n+m)(\alpha_0+\alpha_2+n+2m)}.$$

Similarly we can compute the coefficient $B_{n,m}$ of x^{n+m-2} and the coefficient $C_{n,m}$ of x^{m+n-3}, but the computation is rather lengthy. Once these coefficients have been determined, one can compute the coefficients in the recurrence relation

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x) + d_nP_{n-2}(x),$$

where

$$P_{2n}(x) = P_{n,n}^{(\alpha_0,\alpha_1,\alpha_2)}(x), \quad P_{2n+1}(x) = P_{n+1,n}^{(\alpha_0,\alpha_1,\alpha_2)}(x).$$

Indeed, by comparing coefficients we have

$$b_{2n} = A_{n,n} - A_{n+1,n}, \quad b_{2n+1} = A_{n+1,n} - A_{n+1,n+1}, \tag{3.30}$$

which gives

$$\begin{aligned}
b_{2n} = {} & [36n^4 + (48\alpha_0 + 28\alpha_1 + 20\alpha_2 + 38)n^3 \\
& + (21\alpha_0^2 + 8\alpha_1^2 + 4\alpha_2^2 + 30\alpha_0\alpha_1 + 18\alpha_0\alpha_2 + 15\alpha_1\alpha_2 + 39\alpha_0 + 19\alpha_1 + 19\alpha_2 + 9)n^2 \\
& + (3\alpha_0^3 + 10\alpha_0^2\alpha_1 + 4\alpha_0^2\alpha_2 + 6\alpha_0\alpha_1^2 + 2\alpha_0\alpha_2^2 + 11\alpha_0\alpha_1\alpha_2 + 5\alpha_1^2\alpha_2 + 3\alpha_1\alpha_2^2 \\
& + 12\alpha_0^2 + 3\alpha_1^2 + 3\alpha_2^2 + 13\alpha_0\alpha_1 + 13\alpha_0\alpha_2 + 8\alpha_1\alpha_2 + 6\alpha_0 + 3\alpha_1 + 3\alpha_2)n \\
& + \alpha_0^2 + \alpha_0\alpha_1 + \alpha_2\alpha_1^2 + 2\alpha_2\alpha_1^2\alpha_0 + 2\alpha_0^2\alpha_1 + \alpha_1^2\alpha_0 + \alpha_2^2\alpha_0 + \alpha_2^2\alpha_1 + \alpha_0^3\alpha_1 \\
& + \alpha_0^2\alpha_1^2 + \alpha_2^2\alpha_0\alpha_1 + \alpha_2^2\alpha_1^2 + 2\alpha_2\alpha_0^2\alpha_1 + 3\alpha_2\alpha_1\alpha_0 + 2\alpha_2\alpha_0^2 + \alpha_1\alpha_2 + \alpha_0^3 + \alpha_0\alpha_2] \\
& \times (3n+\alpha_0+\alpha_2)^{-1}(3n+\alpha_0+\alpha_1)^{-1}(3n+\alpha_0+\alpha_2+1)^{-1}(3n+\alpha_0+\alpha_1+2)^{-1},
\end{aligned}$$

and

$$\begin{aligned}
b_{2n+1} = {} & [36n^4 + (48\alpha_0 + 20\alpha_1 + 28\alpha_2 + 106)n^3 \\
& +(21\alpha_0^2 + 4\alpha_1^2 + 8\alpha_2^2 + 18\alpha_0\alpha_1 + 30\alpha_0\alpha_2 + 15\alpha_1\alpha_2 + 105\alpha_0 + 41\alpha_1 + 65\alpha_2 + 111)n^2 \\
& +(3\alpha_0^3 + 4\alpha_0^2\alpha_1 + 10\alpha_0^2\alpha_2 + 2\alpha_0\alpha_1^2 + 6\alpha_0\alpha_2^2 + 11\alpha_0\alpha_1\alpha_2 + 3\alpha_1^2\alpha_2 + 5\alpha_1\alpha_2^2 \\
& +30\alpha_0^2 + 5\alpha_1^2 + 13\alpha_2^2 + 23\alpha_0\alpha_1 + 47\alpha_0\alpha_2 + 22\alpha_1\alpha_2 + 72\alpha_0 + 25\alpha_1 + 49\alpha_2 + 48)n \\
& +18\alpha_0\alpha_2 + 8\alpha_2\alpha_0^2 + 4\alpha_1 + 4\alpha_2^2\alpha_1 + 8\alpha_1\alpha_2 + 2\alpha_0^3 + 5\alpha_2^2\alpha_0 + 8\alpha_2\alpha_1\alpha_0 + 12\alpha_2 \\
& +7 + 15\alpha_0 + \alpha_2^2\alpha_1^2 + 10\alpha_0^2 + 6\alpha_0\alpha_1 + 2\alpha_2\alpha_1^2 + 2\alpha_0^2\alpha_1 + \alpha_1^2\alpha_0 + 5\alpha_2^2 + \alpha_2\alpha_0^3 \\
& +\alpha_2^2\alpha_0^2 + \alpha_1^2 + \alpha_2\alpha_1^2\alpha_0 + 2\alpha_2\alpha_0^2\alpha_1 + 2\alpha_2^2\alpha_0\alpha_1] \\
& \times (3n + \alpha_0 + \alpha_2 + 1)^{-1}(3n + \alpha_0 + \alpha_1 + 2)^{-1}(3n + \alpha_0 + \alpha_2 + 3)^{-1}(3n + \alpha_0 + \alpha_1 + 3)^{-1}.
\end{aligned}$$

For the recurrence coefficient c_n we have the formulas

$$c_{2n} = B_{n,n} - B_{n+1,n} - b_{2n}A_{n,n}, \quad c_{2n+1} = B_{n+1,n} - B_{n+1,n+1} - b_{2n+1}A_{n+1,n}, \tag{3.31}$$

which after some computation (and using Maple V), gives

$$\begin{aligned}
c_{2n} = {} & n(2n + \alpha_0)(2n + \alpha_0 + \alpha_1)(2n + \alpha_0 + \alpha_2) \\
& \times [54n^4 + (63\alpha_0 + 45\alpha_1 + 45\alpha_2)n^3 \\
& +(24\alpha_0^2 + 8\alpha_1^2 + 8\alpha_2^2 + 42\alpha_0\alpha_1 + 42\alpha_0\alpha_2 + 44\alpha_1\alpha_2 - 8)n^2 \\
& +(3\alpha_0^3 + \alpha_1^3 + \alpha_2^3 + 12\alpha_0^2\alpha_1 + 12\alpha_0^2\alpha_2 + 3\alpha_0\alpha_1^2 + 3\alpha_0\alpha_2^2 + 33\alpha_0\alpha_1\alpha_2 + 8\alpha_1^2\alpha_2 \\
& +8\alpha_1\alpha_2^2 - 3\alpha_0 - 4\alpha_1 - 4\alpha_2)n \\
& +\alpha_0^3\alpha_1 + \alpha_0^3\alpha_2 + 6\alpha_0^2\alpha_1\alpha_2 + \alpha_1^3\alpha_2 + \alpha_1\alpha_2^3 + 3\alpha_0\alpha_1^2\alpha_2 + 3\alpha_0\alpha_1\alpha_2^2 - \alpha_0\alpha_1 - \alpha_0\alpha_2 - 2\alpha_1\alpha_2] \\
& \times (3n + \alpha_0 + \alpha_1 + 1)^{-1}(3n + \alpha_0 + \alpha_2 + 1)^{-1}(3n + \alpha_0 + \alpha_1)^{-2}(3n + \alpha_0 + \alpha_2)^{-2} \\
& (3n + \alpha_0 + \alpha_1 - 1)^{-1}(3n + \alpha_0 + \alpha_2 - 1)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
c_{2n+1} = {} & (2n + \alpha_0 + 1)(2n + \alpha_0 + \alpha_1 + 1)(2n + \alpha_0 + \alpha_2 + 1) \\
& \times [54n^5 + (63\alpha_0 + 45\alpha_1 + 45\alpha_2 + 135)n^4 \\
& + (24\alpha_0^2 + 8\alpha_1^2 + 8\alpha_2^2 + 42\alpha_0\alpha_1 + 42\alpha_0\alpha_2 + 44\alpha_1\alpha_2 + 126\alpha_0 + 76\alpha_1 + 104\alpha_2 + 120)n^3 \\
& + (3\alpha_0^3 + \alpha_1^3 + \alpha_2^3 + 12\alpha_0^2\alpha_1 + 12\alpha_0^2\alpha_2 + 3\alpha_0\alpha_1^2 + 3\alpha_0\alpha_2^2 + 33\alpha_0\alpha_1\alpha_2 + 8\alpha_1^2\alpha_2 \\
& + 8\alpha_1\alpha_2^2 + 36\alpha_0^2 + 5\alpha_1^2 + 19\alpha_2^2 + 54\alpha_0\alpha_1 + 72\alpha_0\alpha_2 + 66\alpha_1\alpha_2 + 87\alpha_0 + 39\alpha_1 \\
& + 81\alpha_2 + 45)n^2 \\
& +(\alpha_0^3\alpha_1 + \alpha_0^3\alpha_2 + 6\alpha_0^2\alpha_1\alpha_2 + \alpha_1^3\alpha_2 + \alpha_1\alpha_2^3 + 3\alpha_0\alpha_1^2\alpha_2 + 3\alpha_0\alpha_1\alpha_2^2 + 3\alpha_0^3 + 2\alpha_2^3 \\
& + 12\alpha_0^2\alpha_1 + 12\alpha_0^2\alpha_2 + 6\alpha_0\alpha_2^2 + 33\alpha_0\alpha_1\alpha_2 + 5\alpha_1^2\alpha_2 + 11\alpha_1\alpha_2^2 + 18\alpha_0^2 + 20\alpha_0\alpha_1 \\
& +38\alpha_0\alpha_2 + 14\alpha_2^2 + 26\alpha_1\alpha_2 + 24\alpha_0 + 6\alpha_1 + 24\alpha_2 + 6)n \\
& +\alpha_0^3\alpha_1 + 3\alpha_0^2\alpha_1\alpha_2 + 3\alpha_0\alpha_1\alpha_2^2 + \alpha_1\alpha_2^3 + \alpha_0^3 + \alpha_2^3 + 3\alpha_0^2\alpha_1 + 3\alpha_0^2\alpha_2 + 6\alpha_0\alpha_1\alpha_2 \\
& +3\alpha_0\alpha_2^2 + 3\alpha_1\alpha_2^2 + 3\alpha_0^2 + 3\alpha_2^2 + 2\alpha_0\alpha_1 + 6\alpha_0\alpha_2 + 2\alpha_1\alpha_2 + 2\alpha_0 + 2\alpha_2] \\
& \times (3n + \alpha_0 + \alpha_1 + 3)^{-1}(3n + \alpha_0 + \alpha_2 + 2)^{-1}(3n + \alpha_0 + \alpha_1 + 2)^{-2}(3n + \alpha_0 + \alpha_2 + 1)^{-2} \\
& (3n + \alpha_0 + \alpha_1 + 1)^{-1}(3n + \alpha_0 + \alpha_2)^{-1}.
\end{aligned}$$

Finally, for d_n we have

$$d_{2n} = C_{n,n} - C_{n+1,n} - b_{2n}B_{n,n} - c_{2n}A_{n,n-1},$$

$$d_{2n+1} = C_{n+1,n} - C_{n+1,n+1} - b_{2n+1}B_{n+1,n} - c_{2n+1}A_{n,n}, \tag{3.32}$$

giving

$$\begin{aligned} d_{2n} = {} & n(2n+\alpha_0)(2n+\alpha_0-1)(2n+\alpha_0+\alpha_1)(2n+\alpha_0+\alpha_1-1) \\ & (2n+\alpha_0+\alpha_2)(2n+\alpha_0+\alpha_2-1)(n+\alpha_1)(n+\alpha_1-\alpha_2) \\ & (3n+1+\alpha_0+\alpha_1)^{-1}(3n+\alpha_0+\alpha_1)^{-2}(3n+\alpha_0+\alpha_2)^{-1}(3n-1+\alpha_0+\alpha_1)^{-2} \\ & (3n-1+\alpha_0+\alpha_2)^{-1}(3n-2+\alpha_0+\alpha_1)^{-1}(3n-2+\alpha_0+\alpha_2)^{-1} \end{aligned}$$

and

$$\begin{aligned} d_{2n+1} = {} & n(2n+1+\alpha_0)(2n+\alpha_0)(2n+\alpha_0+\alpha_1)(2n+1+\alpha_0+\alpha_1) \\ & (2n+1+\alpha_0+\alpha_2)(2n+\alpha_0+\alpha_2)(n+\alpha_2)(n+\alpha_2-\alpha_1) \\ & (3n+2+\alpha_0+\alpha_1)^{-1}(3n+2+\alpha_0+\alpha_2)^{-1}(3n+1+\alpha_0+\alpha_1)^{-1}(3n+1+\alpha_0+\alpha_2)^{-2} \\ & (3n+\alpha_0+\alpha_1)^{-1}(3n+\alpha_0+\alpha_2)^{-2}(3n-1+\alpha_0+\alpha_2)^{-1}. \end{aligned}$$

These formulas are rather lengthy, but explicit knowledge of them will be useful in what follows. Observe that for large n we have

$$\lim_{n\to\infty} b_n = \frac{4}{9} = 3\left(\frac{4}{27}\right),$$

$$\lim_{n\to\infty} c_n = \frac{16}{243} = 3\left(\frac{4}{27}\right)^2,$$

$$\lim_{n\to\infty} d_n = \frac{64}{19683} = \left(\frac{4}{27}\right)^3.$$

3.2. Multiple Laguerre polynomials (first kind)

In the same spirit as for the Jacobi–Piñeiro polynomials, we can consider two different families of multiple Laguerre polynomials. The *multiple Laguerre polynomials of the first kind* $L_{\boldsymbol{n}}^{\boldsymbol{\alpha}}(x)$ are orthogonal on $[0,\infty)$ with respect to the r weights $w_j(x) = x^{\alpha_j}e^{-x}$, where $\alpha_j > -1$ for $j = 1,2,\dots,r$. So these weights have the same exponential decrease at ∞ but have different singularities at 0. Again we assume $\alpha_i - \alpha_j \notin \mathbb{Z}$ in order to have an AT system. These polynomials were first considered by Sorokin [39,41]. The raising operators are given by

$$\frac{d}{dx}(x^{\alpha_j}e^{-x}L_{\boldsymbol{n}}^{\boldsymbol{\alpha}}(x)) = -x^{\alpha_j-1}e^{-x}L_{\boldsymbol{n}+\boldsymbol{e}_j}^{\boldsymbol{\alpha}-\boldsymbol{e}_j}(x), \quad j = 1,\dots,r, \tag{3.33}$$

and a repeated application of these operators gives the Rodrigues formula

$$(-1)^{|\boldsymbol{n}|}L_{\boldsymbol{n}}^{\boldsymbol{\alpha}}(x) = e^x \prod_{j=1}^{r}\left[x^{-\alpha_j}\frac{d^{n_j}}{dx^{n_j}}x^{n_j+\alpha_j}\right]e^{-x}. \tag{3.34}$$

When $r=2$ one can use this Rodrigues formula to obtain an explicit expression for these multiple Laguerre polynomials, from which one can compute the recurrence coefficients in

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

where $P_{2n}(x) = L_{n,n}^{(\alpha_1,\alpha_2)}(x)$ and $P_{2n+1}(x) = L_{n+1,n}^{(\alpha_1,\alpha_2)}(x)$. But having done all that work for Jacobi–Piñeiro polynomials, it is much easier to use the limit relation

$$L_{n,m}^{(\alpha_1,\alpha_2)}(x) = \lim_{\alpha_0\to\infty} \alpha_0^{n+m} P_{n,m}^{(\alpha_0,\alpha_1,\alpha_2)}(x/\alpha_0). \tag{3.35}$$

The recurrence coefficients can then be found in terms of the following limits of the corresponding recurrence coefficients of Jacobi–Piñeiro polynomials:

$$b_n = \lim_{\alpha_0\to\infty} b_n^{(\alpha_0,\alpha_1,\alpha_2)}\alpha_0,$$

$$c_n = \lim_{\alpha_0\to\infty} c_n^{(\alpha_0,\alpha_1,\alpha_2)}\alpha_0^2,$$

$$d_n = \lim_{\alpha_0\to\infty} d_n^{(\alpha_0,\alpha_1,\alpha_2)}\alpha_0^3,$$

giving

$$b_{2n} = 3n + \alpha_1 + 1,$$

$$b_{2n+1} = 3n + \alpha_2 + 2,$$

$$c_{2n} = n(3n + \alpha_1 + \alpha_2),$$

$$c_{2n+1} = 3n^2 + (\alpha_1 + \alpha_2 + 3)n + \alpha_1 + 1,$$

$$d_{2n} = n(n + \alpha_1)(n + \alpha_1 - \alpha_2),$$

$$d_{2n+1} = n(n + \alpha_2)(n + \alpha_2 - \alpha_1).$$

Observe that for large n we have

$$\lim_{n\to\infty} \frac{b_n}{n} = \frac{3}{2} = 3\left(\frac{1}{2}\right),$$

$$\lim_{n\to\infty} \frac{c_n}{n^2} = \frac{3}{4} = 3\left(\frac{1}{2}\right)^2,$$

$$\lim_{n\to\infty} \frac{d_n}{n^3} = \frac{1}{8} = \left(\frac{1}{2}\right)^3.$$

3.3. Multiple Laguerre polynomials (second kind)

Another family of multiple Laguerre polynomials is given by the weights $w_j(x) = x^{\alpha_0}e^{-c_j x}$ on $[0,\infty)$, with $c_j > 0$ and $c_i \neq c_j$ for $i \neq j$. So now the weights have the same singularity at the origin but different exponential rates at infinity. These *multiple Laguerre polynomials of the second kind* $L_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{c})}(x)$ appear already in [34, p. 160]. The raising operators are

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\alpha_0}\mathrm{e}^{-c_jx}L_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{c})}(x)) = -c_jx^{\alpha_0-1}\mathrm{e}^{-c_jx}L_{\boldsymbol{n}+\boldsymbol{e}_j}^{(\alpha_0-1,\boldsymbol{c})}(x), \quad j=1,\ldots,r, \tag{3.36}$$

and a repeated application of these operators gives the Rodrigues formula

$$(-1)^{|\boldsymbol{n}|}\prod_{j=1}^{r}c_j^{n_j}L_{\boldsymbol{n}}^{(\alpha_0,\boldsymbol{c})}(x) = x^{-\alpha_0}\prod_{j=1}^{r}\left[\mathrm{e}^{c_jx}\frac{\mathrm{d}^{n_j}}{\mathrm{d}x^{n_j}}\mathrm{e}^{-c_jx}\right]x^{|\boldsymbol{n}|+\alpha_0}. \tag{3.37}$$

These polynomials are also a limit case of the Jacobi–Piñeiro polynomials. For the case $r=2$ we have

$$L_{n,m}^{(\alpha_0,c_1,c_2)}(x) = \lim_{\alpha\to\infty}(-\alpha)^{n+m}P_{n,m}^{(\alpha_0,c_1\alpha,c_2\alpha)}(1-x/\alpha). \tag{3.38}$$

The recurrence coefficients can be obtained from the corresponding recurrence coefficients of Jacobi–Piñeiro polynomials by

$$b_n = \lim_{\alpha\to\infty}(1-b_n^{(\alpha_0,c_1\alpha,c_2\alpha)})\alpha,$$

$$c_n = \lim_{\alpha\to\infty}c_n^{(\alpha_0,c_1\alpha,c_2\alpha)}\alpha^2,$$

$$d_n = \lim_{\alpha\to\infty}-d_n^{(\alpha_0,c_1\alpha,c_2\alpha)}\alpha^3,$$

giving

$$b_{2n} = \frac{n(c_1+3c_2)+c_2+\alpha_0c_2}{c_1c_2},$$

$$b_{2n+1} = \frac{n(3c_1+c_2)+2c_1+c_2+\alpha_0c_1}{c_1c_2},$$

$$c_{2n} = \frac{n(2n+\alpha_0)(c_1^2+c_2^2)}{c_1^2c_2^2},$$

$$c_{2n+1} = \frac{2n^2(c_1^2+c_2^2)+n[c_1^2+3c_2^2+\alpha_0(c_1^2+c_2^2)]+c_2^2+\alpha_0c_2^2}{c_1^2c_2^2},$$

$$d_{2n} = \frac{n(2n+\alpha_0)(2n+\alpha_0-1)(c_2-c_1)}{c_1^3c_2},$$

$$d_{2n+1} = \frac{n(2n+\alpha_0)(2n+\alpha_0+1)(c_1-c_2)}{c_1c_2^3}.$$

Observe that for large n we have

$$\lim_{n\to\infty}\frac{b_n}{n} = \begin{cases}\dfrac{c_1+3c_2}{2c_1c_2} & \text{if } n\equiv 0 \,(\mathrm{mod}\, 2),\\[2ex] \dfrac{3c_1+c_2}{2c_1c_2} & \text{if } n\equiv 1 \,(\mathrm{mod}\, 2),\end{cases}$$

$$\lim_{n\to\infty} \frac{c_n}{n^2} = \frac{c_1^2 + c_2^2}{2c_1^2 c_2^2},$$

$$\lim_{n\to\infty} \frac{d_n}{n^3} = \begin{cases} \dfrac{c_2 - c_1}{2c_1^3 c_2} & \text{if } n \equiv 0 \,(\operatorname{mod} 2), \\ \dfrac{c_1 - c_2}{2c_1 c_2^3} & \text{if } n \equiv 1 \,(\operatorname{mod} 2). \end{cases}$$

3.4. Multiple Hermite polynomials

Finally we can consider the weights $w_j(x) = e^{-x^2 + c_j x}$ on $(-\infty, \infty)$, for $j = 1, 2, \ldots, r$ and c_i different real numbers. The *multiple Hermite polynomials* $H_{\mathbf{n}}^{\mathbf{c}}(x)$ once more have raising operators and a Rodrigues formula, and they are also limiting cases of the Jacobi–Piñeiro polynomials, but also of the multiple Laguerre polynomials of the second kind. For $r = 2$ this is

$$H_{n,m}^{(c_1,c_2)}(x) = \lim_{\alpha\to\infty} (2\sqrt{\alpha})^{n+m} P_{n,m}^{(\alpha,\alpha+c_1\sqrt{\alpha},\alpha+c_2\sqrt{\alpha})}\left(\frac{x+\sqrt{\alpha}}{2\sqrt{\alpha}}\right), \tag{3.39}$$

so that the recurrence coefficients can be obtained from the Jacobi–Piñeiro case by

$$b_n = \lim_{\alpha\to\infty} 2(b_n^{(\alpha,\alpha+c_1\sqrt{\alpha},\alpha+c_2\sqrt{\alpha})} - \tfrac{1}{2})\sqrt{\alpha},$$

$$c_n = \lim_{\alpha\to\infty} 4c_n^{(\alpha,\alpha+c_1\sqrt{\alpha},\alpha+c_2\sqrt{\alpha})}\alpha,$$

$$d_n = \lim_{\alpha\to\infty} 8d_n^{(\alpha,\alpha+c_1\sqrt{\alpha},\alpha+c_2\sqrt{\alpha})}(\sqrt{\alpha})^3.$$

This gives

$$b_{2n} = c_1/2,$$

$$b_{2n+1} = c_2/2,$$

$$c_n = n/2,$$

$$d_{2n} = n(c_1 - c_2)/4,$$

$$d_{2n+1} = n(c_2 - c_1)/4.$$

Alternatively, we can use the limit transition from the multiple Laguerre polynomials of the first kind:

$$H_{n,m}^{(c_1,c_2)}(x) = \lim_{\alpha\to\infty} \alpha_0^{n+m} L_{n,m}^{(\alpha+c_1\sqrt{\alpha/2},\alpha+c_2\sqrt{\alpha/2})}(\sqrt{2\alpha}x + \alpha). \tag{3.40}$$

The recurrence coefficients are then also given in terms of the following limits of the recurrence coefficients of the multiple Laguerre polynomials of the first kind

$$b_n = \lim_{\alpha\to\infty} \left(b_n^{(\alpha+c_1\sqrt{\alpha/2},\alpha+c_2\sqrt{\alpha/2})} - \alpha\right)/\sqrt{2\alpha},$$

$$c_n = \lim_{\alpha\to\infty} c_n^{(\alpha+c_1\sqrt{\alpha/2},\alpha+c_2\sqrt{\alpha/2})}/(2\alpha),$$

$$d_n = \lim_{\alpha\to\infty} d_n^{(\alpha+c_1\sqrt{\alpha/2},\alpha+c_2\sqrt{\alpha/2})}/(\sqrt{2\alpha})^3,$$

which leads to the same result. Observe that for large n we have

$$\lim_{n\to\infty} \frac{b_n}{\sqrt{n}} = 0,$$

$$\lim_{n\to\infty} \frac{c_n}{n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{d_n}{(\sqrt{n})^3} = 0.$$

3.5. Jacobi–Angelesco polynomials

The following system is probably the first that was investigated in detail [20,25]. It is an Angelesco system with weights $w_1(x)=|h(x)|$ on $[a,0]$ (with $a<0$) and $w_2(x)=|h(x)|$ on $[0,1]$, where $h(x)=(x-a)^\alpha x^\beta(1-x)^\gamma$ and $\alpha,\beta,\gamma>-1$. Hence the same weight is used for both weights w_1 and w_2 but on two touching intervals. The Jacobi–Angelesco polynomials $P_{n,m}^{(\alpha,\beta,\gamma)}(x;a)$ therefore satisfy the orthogonality relations

$$\int_a^0 P_{n,m}^{(\alpha,\beta,\gamma)}(x;a)(x-a)^\alpha|x|^\beta(1-x)^\gamma x^k\,\mathrm{d}x = 0, \quad k=0,1,2,\ldots,n-1, \tag{3.41}$$

$$\int_0^1 P_{n,m}^{(\alpha,\beta,\gamma)}(x;a)(x-a)^\alpha x^\beta(1-x)^\gamma x^k\,\mathrm{d}x = 0, \quad k=0,1,2,\ldots,m-1. \tag{3.42}$$

The function $h(x)$ satisfies a Pearson equation

$$(x-a)x(1-x)h'(x) = [\alpha x(1-x)+\beta(x-a)(1-x)-\gamma(x-a)x]h(x),$$

where $\sigma(x)=(x-a)x(1-x)$ is now a polynomial of degree 3. Using this relation, we can integrate the orthogonality relations by part to see that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}[(x-a)^\alpha x^\beta(1-x)^\gamma P_{n,m}^{(\alpha,\beta,\gamma)}(x;a)]\\ &\quad = -(\alpha+\beta+\gamma+n+m)(x-a)^{\alpha-1}x^{\beta-1}(1-x)^{\gamma-1}P_{n+1,m+1}^{(\alpha-1,\beta-1,\gamma-1)}(x;a),\end{aligned} \tag{3.43}$$

which raises both indices of the multi-index (n,m). Repeated use of this raising operation gives the Rodrigues formula

$$\begin{aligned}&\frac{\mathrm{d}^m}{\mathrm{d}x^m}[(x-a)^{\alpha+m}x^{\beta+m}(1-x)^{\gamma+m}P_{k,0}^{(\alpha+m,\beta+m,\gamma+m)}(x;a)]\\ &\quad =(-1)^m(\alpha+\beta+\gamma+k+2m+1)_m(x-a)^\alpha x^\beta(1-x)^\gamma P_{m+k,m}^{(\alpha,\beta,\gamma)}(x;a).\end{aligned} \tag{3.44}$$

For $k=0$ and $m=n$, this then gives

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x-a)^{\alpha+n}x^{\beta+n}(1-x)^{\gamma+n}] \\ =(-1)^n(\alpha+\beta+\gamma+2n+1)_n(x-a)^\alpha x^\beta(1-x)^\gamma P_{n,n}^{(\alpha,\beta,\gamma)}(x;a). \tag{3.45}$$

Use Leibniz' formula to find

$$(-1)^n(\alpha+\beta+\gamma+2n+1)_n(x-a)^\alpha x^\beta(1-x)^\gamma P_{n,n}^{(\alpha,\beta,\gamma)}(x;a) \\ =\sum_{k=0}^n \binom{n}{k}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^{\beta+n}(1-x)^{\gamma+n}\right)\left(\frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}(x-a)^{\alpha+n}\right).$$

Now use the Rodrigues formula for the Jacobi polynomials (1.8) to find

$$\binom{\alpha+\beta+\gamma+3n}{n}P_{n,n}^{(\alpha,\beta,\gamma)}(x;a) \\ =\sum_{k=0}^n(-1)^{n-k}\binom{\beta+\gamma+2n}{k}\binom{\alpha+n}{n-k}(x-a)^k x^{n-k}(1-x)^{n-k}P_k^{(\gamma+n-k,\beta+n-k)}(x).$$

Use of the expansion (3.28) for the Jacobi polynomial gives

$$\binom{\alpha+\beta+\gamma+3n}{n}P_{n,n}^{(\alpha,\beta,\gamma)}(x;a) \\ =\sum_{k=0}^n\sum_{j=0}^k\binom{\alpha+n}{n-k}\binom{\beta+n}{j}\binom{\gamma+n}{k-j}(x-a)^k x^{n-j}(x-1)^{n-k+j} \tag{3.46}$$

$$=\sum_{k=0}^n\sum_{j=0}^{n-k}\binom{\alpha+n}{k}\binom{\beta+n}{j}\binom{\gamma+n}{n-k-j}(x-a)^{n-k}x^{n-j}(x-1)^{k+j}, \tag{3.47}$$

where the last equation follows by the change of variable $k\mapsto n-k$. If we write this in terms of Pochhammer symbols, then

$$\binom{\alpha+\beta+\gamma+3n}{n}P_{n,n}^{(\alpha,\beta,\gamma)}(x;a) \\ =\frac{(\gamma+1)_n}{n!}\sum_{k=0}^n\sum_{j=0}^{n-k}\frac{(-n)_{k+j}(-\alpha-n)_k(-\beta-n)_j}{(\gamma+1)_{k+j}k!j!}(x-a)^{n-k}(x-1)^{k+j}x^{n-j} \\ =x^n(x-a)^n\binom{\gamma+n}{n}F_1\left(-n,-\alpha-n,-\beta-n,\gamma+1;\frac{x-1}{x-a},\frac{x-1}{x}\right),$$

where

$$F_1(a,b,b',c;x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}}\frac{x^m y^n}{m!n!}$$

is the first of Appell's hypergeometric functions of two variables.

For the polynomial $P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a)$ we have the Rodrigues formula

$$\begin{aligned}&\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(x-a)^{\alpha+n}x^{\beta+n}(1-x)^{\gamma+n}P_{1,0}^{(\alpha+n,\beta+n,\gamma+n)}(x;a)]\\ &\quad=(-1)^n(\alpha+\beta+\gamma+2n+2)_n(x-a)^{\alpha}x^{\beta}(1-x)^{\gamma}P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a),\end{aligned}\tag{3.48}$$

where $P_{1,0}^{(\alpha+n,\beta+n,\gamma+n)}(x;a)=x-X_n^{(\alpha,\beta,\gamma)}$ is the monic orthogonal polynomial of first degree for the weight $(x-a)^{\alpha+n}|x|^{\beta+n}(1-x)^{\gamma+n}$ on $[a,0]$. If we write down the orthogonality of this polynomial to the constant function,

$$\int_a^0 (x-X_n^{(\alpha,\beta,\gamma)})(x-a)^{\alpha+n}|x|^{\beta+n}(1-x)^{\gamma+n}\,\mathrm{d}x=0,$$

then we see that

$$X_n^{(\alpha,\beta,\gamma)}=\frac{\int_a^0 x(x-a)^{\alpha+n}|x|^{\beta+n}(1-x)^{\gamma+n}\,\mathrm{d}x}{\int_a^0(x-a)^{\alpha+n}|x|^{\beta+n}(1-x)^{\gamma+n}\,\mathrm{d}x}.$$

A standard saddle point method gives the asymptotic behavior

$$\lim_{n\to\infty}X_n^{(\alpha,\beta,\gamma)}=x_1,\tag{3.49}$$

where x_1 is the zero of $\sigma'(x)$ in $[a,0]$, where $\sigma(x)=(x-a)x(1-x)$. Combining the Rodrigues equation in (3.48) with the Rodrigues equation (3.45) shows that

$$P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a)=xP_{n,n}^{(\alpha,\beta+1,\gamma)}(x;a)-X_n^{(\alpha,\beta,\gamma)}\frac{\alpha+\beta+\gamma+2n+1}{\alpha+\beta+\gamma+3n+1}P_{n,n}^{(\alpha,\beta,\gamma)}(x;a).\tag{3.50}$$

In order to compute the coefficients of the recurrence relation

$$xP_n(x)=P_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x)+d_nP_{n-2}(x),$$

where

$$P_{2n}(x)=P_{n,n}^{(\alpha,\beta,\gamma)}(x;a),\qquad P_{2n+1}(x)=P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a),$$

we will compute the first few coefficients of the polynomials

$$P_{n,m}^{(\alpha,\beta,\gamma)}(x;a)=x^{m+n}+A_{n,m}x^{n+m-1}+B_{n,m}x^{m+n-2}+C_{n,m}x^{n+m-3}+\cdots.$$

First we take $n=m$. In order to check that our polynomial is monic, we see from (3.46) that the leading coefficient is given by

$$\binom{\alpha+\beta+\gamma+3n}{n}^{-1}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{\alpha+n}{n-k}\binom{\beta+n}{j}\binom{\gamma+n}{k-j}.$$

Chu–Vandermonde gives

$$\sum_{j=0}^{k} \binom{\beta+n}{j} \binom{\gamma+n}{k-j} = \binom{\beta+\gamma+2n}{k},$$

and also

$$\sum_{k=0}^{n} \binom{\alpha+n}{n-k} \binom{\beta+\gamma+2n}{k} = \binom{\alpha+\beta+\gamma+3n}{n},$$

so that the leading coefficient is indeed 1. The coefficient $A_{n,n}$ of x^{2n-1} is equal to

$$-\binom{\alpha+\beta+\gamma+3n}{n}^{-1} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{\alpha+n}{n-k} \binom{\beta+n}{j} \binom{\gamma+n}{k-j} (ak+n-k+j).$$

Working out this double sum gives

$$A_{n,n}^{(\alpha,\beta,\gamma)} = \frac{-n[\alpha+\beta+2n+a(\beta+\gamma+2n)]}{\alpha+\beta+\gamma+3n}. \tag{3.51}$$

For $P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a)$ the coefficient $A_{n+1,n}$ of x^{2n} can be obtained from (3.50)

$$A_{n+1,n}^{(\alpha,\beta,\gamma)} = A_{n,n}^{(\alpha,\beta+1,\gamma)} - X_n^{(\alpha,\beta,\gamma)} \frac{\alpha+\beta+\gamma+2n+1}{\alpha+\beta+\gamma+3n+1}. \tag{3.52}$$

The coefficient b_n in the recurrence relation can now be found from (3.30)

$$b_{2n} = \frac{n[n+\gamma+a(n+\alpha)]}{(\alpha+\beta+\gamma+3n)(\alpha+\beta+\gamma+3n+1)} + X_n^{(\alpha,\beta,\gamma)} \frac{2n+\alpha+\beta+\gamma+1}{3n+\alpha+\beta+\gamma+1},$$

$$\begin{aligned} b_{2n+1} = {} & (5n^2+(4\alpha+4\beta+3\gamma+7)n+(\alpha+\beta+\gamma+1)(\alpha+\beta+2) \\ & + a[5n^2+(3\alpha+4\beta+4\gamma+7)n+(\alpha+\beta+\gamma+1)(\beta+\gamma+2)]) \\ & \times (\alpha+\beta+\gamma+3n+1)^{-1}(\alpha+\beta+\gamma+3n+3)^{-1} \\ & - X_n^{(\alpha,\beta,\gamma)} \frac{2n+\alpha+\beta+\gamma+1}{3n+\alpha+\beta+\gamma+1}. \end{aligned}$$

The coefficient $B_{n,n}$ of x^{2n-2} in $P_{n,n}^{(\alpha,\beta,\gamma)}(x;a)$ is given by

$$\begin{aligned} B_{n,n}^{(\alpha,\beta,\gamma)} = {} & \frac{an(\alpha+\beta+\gamma+2n)(\beta+n)}{(\alpha+\beta+\gamma+3n)(\alpha+\beta+\gamma+3n-1)} \\ & + \frac{n(n-1)}{2(\alpha+\beta+\gamma+3n)(\alpha+\beta+\gamma+3n-1)} \\ & \times [(\alpha+\beta+2n)(\alpha+\beta+2n-1)+2a(\alpha+\beta+2n)(\beta+\gamma+2n) \\ & + a^2(\beta+\gamma+2n)(\beta+\gamma+2n-1)], \end{aligned}$$

and from (3.50) we also find

$$B_{n+1,n}^{(\alpha,\beta,\gamma)} = B_{n,n}^{(\alpha,\beta+1,\gamma)} - X_n^{(\alpha,\beta,\gamma)} A_{n,n}^{(\alpha,\beta,\gamma)} \frac{\alpha+\beta+\gamma+2n+1}{\alpha+\beta+\gamma+3n+1}.$$

Using (3.31) then gives

$$\begin{aligned} c_{2n} = {} & \frac{n(\alpha+\beta+\gamma+2n)}{(\alpha+\beta+\gamma+3n-1)(\alpha+\beta+\gamma+3n)^2(\alpha+\beta+\gamma+3n-1)} \\ & ((\alpha+\beta+2n)(\gamma+n) - 2a(\alpha+n)(\gamma+n) + a^2(\beta+\gamma+2n)(\alpha+n)). \end{aligned}$$

and

$$\begin{aligned} c_{2n+1} = {} & \frac{\alpha+\beta+\gamma+2n+1}{(\alpha+\beta+\gamma+3n+3)(\alpha+\beta+\gamma+3n+2)(\alpha+\beta+\gamma+3n+1)^2(\alpha+\beta+\gamma+3n)} \\ & \times (n(n+\gamma)(\alpha+\beta+2n+1)(\alpha+\beta+\gamma+3n+3) \\ & - a[24n^4 + (29\alpha+41\beta+29\gamma+48)n^3 \\ & + (10\alpha^2+39\alpha\beta+26\alpha\gamma+29\beta^2+39\beta\gamma+10\gamma^2+44\alpha+62\beta+44\gamma+30)n^2 \\ & + (\alpha^3+11\alpha^2\beta+5\alpha^2\gamma+19\alpha\beta^2+24\alpha\beta\gamma+5\alpha\gamma^2+9\beta^3+19\beta^2\gamma+11\beta\gamma^2+\gamma^3 \\ & + 11\alpha^2+39\alpha\beta+28\alpha\gamma+28\beta^2+39\beta\gamma+11\gamma^2+19\alpha+25\beta+19\gamma+6)n \\ & + (\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)(\alpha+\beta+\gamma+2)(\beta+1)] \\ & + a^2 n(n+\alpha)(\beta+\gamma+2n+1)(\alpha+\beta+\gamma+3n+3)) \\ & + \frac{\alpha+\beta+\gamma+2n+1}{(\alpha+\beta+\gamma+3n+3)(\alpha+\beta+\gamma+3n+1)^2(\alpha+\beta+\gamma+3n)} X_n^{(\alpha,\beta,\gamma)} \\ & \times (12n^3 + (16\alpha+16\beta+10\gamma+18)n^2 \\ & + [(\alpha+\beta+\gamma)(7\alpha+7\beta+2\gamma)+16\alpha+16\beta+10\gamma]n \\ & + (\alpha+\beta+\gamma)^2(\alpha+\beta) + (\alpha+\beta+\gamma)(3\alpha+3\beta+2\gamma+2) \\ & + a[12n^3 + (10\alpha+16\beta+16\gamma+18)n^2 \\ & + [(\alpha+\beta+\gamma)(2\alpha+7\beta+7\gamma)+10\alpha+16\beta+16\gamma]n \\ & + (\alpha+\beta+\gamma)^2(\beta+\gamma) + (\alpha+\beta+\gamma)(2\alpha+3\beta+3\gamma+2)]) \\ & - \frac{(\alpha+\beta+\gamma+2n+1)^2}{(\alpha+\beta+\gamma+3n+1)^2} (X_n^{(\alpha,\beta,\gamma)})^2. \end{aligned}$$

The coefficient $C_{n,n}$ of x^{2n-3} in $P_{n,n}^{(\alpha,\beta,\gamma)}(x;a)$ can be computed in a similar way, and the coefficient $C_{n+1,n}$ of x^{2n-2} in $P_{n+1,n}^{(\alpha,\beta,\gamma)}(x;a)$ is given by

$$C_{n+1,n}^{(\alpha,\beta,\gamma)} = C_{n,n}^{(\alpha,\beta+1,\gamma)} - X_n^{(\alpha,\beta,\gamma)} B_{n,n}^{(\alpha,\beta,\gamma)} \frac{\alpha+\beta+\gamma+2n+1}{\alpha+\beta+\gamma+3n+1}.$$

A lengthy but straightforward calculation, using (3.32), then gives

$$d_{2n}=\frac{-an(n+\beta)(\alpha+\beta+\gamma+2n)(\alpha+\beta+\gamma+2n-1)[n+\gamma+a(n+\alpha)]}{(\alpha+\beta+\gamma+3n-2)(\alpha+\beta+\gamma+3n-1)(\alpha+\beta+\gamma+3n)^2(\alpha+\beta+\gamma+3n+1)}$$
$$+\frac{n(\alpha+\beta+\gamma+2n)(\alpha+\beta+\gamma+2n-1)X_{n-1}^{(\alpha,\beta,\gamma)}}{(\alpha+\beta+\gamma+3n-2)(\alpha+\beta+\gamma+3n-1)(\alpha+\beta+\gamma+3n)^2(\alpha+\beta+\gamma+3n+1)}$$
$$\times[(n+\gamma)(\alpha+\beta+2n)-2a(n+\gamma)(n+\alpha)+a^2(n+\alpha)(\beta+\gamma+2n)],$$

and

$$d_{2n+1}=\frac{n(\alpha+\beta+\gamma+2n+1)(\alpha+\beta+\gamma+2n)}{(\alpha+\beta+\gamma+3n+2)(\alpha+\beta+\gamma+3n+1)^2(\alpha+\beta+\gamma+3n)^2(\alpha+\beta+\gamma+3n-1)}$$
$$\times((n+\gamma)(\alpha+\beta+2n)(\alpha+\beta+2n+1)$$
$$-a(n+\alpha)(n+\gamma)(2\alpha+2\beta-\gamma+3n+1)$$
$$-a^2(n+\alpha)(n+\gamma)(-\alpha+2\beta+2\gamma+3n+1)$$
$$+a^3(n+\alpha)(\beta+\gamma+2n)(\beta+\gamma+2n+1))$$
$$-\frac{n(\alpha+\beta+\gamma+2n+1)(\alpha+\beta+\gamma+2n)X_n^{(\alpha,\beta,\gamma)}}{(\alpha+\beta+\gamma+3n+1)^2(\alpha+\beta+\gamma+3n)^2(\alpha+\beta+\gamma+3n-1)}$$
$$\times[(n+\gamma)(\alpha+\beta+2n)-2a(n+\alpha)(n+\gamma)+a^2(n+\alpha)(\beta+\gamma+2n)].$$

The asymptotic behavior of these recurrence coefficients can easily be found using (3.49), giving

$$\lim_{n\to\infty} b_{2n}=\frac{a+1}{9}+\frac{2x_1}{3},\qquad \lim_{n\to\infty} b_{2n+1}=\frac{5(a+1)}{9}-\frac{2x_1}{3},$$

$$\lim_{n\to\infty} c_{2n}=\frac{4}{81}(a^2-a+1),\qquad \lim_{n\to\infty} c_{2n+1}=-\frac{4}{9}x_1^2+\frac{8}{27}x_1+\frac{1}{81}(4a^2-a+4),$$

$$\lim_{n\to\infty} d_{2n}=\frac{4}{243}[2(a^2-a+1)x_1-a(a+1)],$$

$$\lim_{n\to\infty} d_{2n+1}=\frac{4}{729}(4a^3-3a^2-3a+4)-\frac{8x_1}{243}(a^2-a+1),$$

where x_1 is the zero of $\sigma'(x)$ in $[a,0]$ and $\sigma(x)=(x-a)x(x-1)$. These formulas can be made more symmetric by also using the zero x_2 of $\sigma'(x)$ in $[0,1]$ and using the fact that $x_1+x_2=2(a+1)/3$:

$$\lim_{n\to\infty} b_{2n}=\frac{a+1}{9}+\frac{2x_1}{3},\qquad \lim_{n\to\infty} b_{2n+1}=\frac{a+1}{9}-\frac{2x_2}{3},$$

$$\lim_{n\to\infty} c_n=\frac{4}{81}(a^2-a+1),$$

$$\lim_{n\to\infty} d_{2n}=-\frac{4}{27}\sigma(x_1),\qquad \lim_{n\to\infty} d_{2n}=-\frac{4}{27}\sigma(x_2).$$

3.6. Jacobi–Laguerre polynomials

When we consider the weights $w_1(x) = (x-a)^\alpha |x|^\beta e^{-x}$ on $[a,0]$, with $a<0$, and $w_2(x) = (x-a)^\alpha |x|^\beta e^{-x}$ on $[0,\infty)$, then we are again using one weight but on two touching intervals, one of which is the finite interval $[a,0]$ (Jacobi part), the other the infinite interval $[0,\infty)$ (Laguerre part). This system was considered by Sorokin [38]. The corresponding *Jacobi–Laguerre polynomials* $L_{n,m}^{(\alpha,\beta)}(x;a)$ satisfy the orthogonality relations

$$\int_a^0 L_{n,m}^{(\alpha,\beta)}(x;a)(x-a)^\alpha |x|^\beta e^{-x} x^k \, dx = 0, \quad k = 0,1,\ldots,n-1,$$

$$\int_0^\infty L_{n,m}^{(\alpha,\beta)}(x;a)(x-a)^\alpha x^\beta e^{-x} x^k \, dx = 0, \quad k = 0,1,\ldots,m-1.$$

The raising operator is

$$\frac{d}{dx}[(x+a)^\alpha x^\beta e^{-x} L_{n,m}^{(\alpha,\beta)}(x;a)] = -(x-a)^{\alpha-1} x^{\beta-1} e^{-x} L_{n+1,m+1}^{(\alpha-1,\beta-1)}(x;a), \tag{3.53}$$

from which the Rodrigues formula follows:

$$\frac{d^m}{dx^m}[(x-a)^{\alpha+m} x^{\beta+m} e^{-x} L_{k,0}^{(\alpha+m,\beta+m)}(x;a)] = (-1)^m (x-a)^\alpha x^\beta e^{-x} L_{m+k,m}^{(\alpha,\beta)}(x;a). \tag{3.54}$$

From this Rodrigues formula we can proceed as before to find an expression of the polynomials, but it is more convenient to view these Jacobi–Laguerre polynomials as a limit case of the Jacobi–Angelesco polynomials

$$L_{n,m}^{(\alpha,\beta)}(x;a) = \lim_{\gamma\to\infty} \gamma^{n+m} P_{n,m}^{(\alpha,\beta,\gamma)}(x/\gamma; a/\gamma), \tag{3.55}$$

so that (3.47) gives

$$L_{n,n}^{(\alpha,\beta)}(x;a) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{\alpha+n}{k} \binom{\beta+n}{j} \frac{(-1)^{k+j}(x-a)^{n-k} x^{n-j}}{(n-k-j)!}. \tag{3.56}$$

For the recurrence coefficients in

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x) + d_n P_{n-2}(x),$$

where $P_{2n}(x) = L_{n,n}^{(\alpha,\beta)}(x;a)$ and $P_{2n+1}(x) = L_{n+1,n}^{(\alpha,\beta)}(x;a)$ we have in terms of the corresponding recurrence coefficients of the Jacobi–Angelesco polynomials

$$b_n = \lim_{\gamma\to\infty} \gamma b_n^{(\alpha,\beta,\gamma)}(a/\gamma),$$

$$c_n = \lim_{\gamma\to\infty} \gamma^2 c_n^{(\alpha,\beta,\gamma)}(a/\gamma),$$

$$d_n = \lim_{\gamma\to\infty} \gamma^3 d_n^{(\alpha,\beta,\gamma)}(a/\gamma),$$

and

$$\lim_{\gamma\to\infty} \gamma X_n^{(\alpha,\beta,\gamma)}(a/\gamma) = \frac{\int_a^0 x(x-a)^{\alpha+n}|x|^{\beta+n} e^{-x}\, dx}{\int_a^0 (x-a)^{\alpha+n}|x|^{\beta+n} e^{-x}\, dx} := X_n^{(\alpha,\beta)}.$$

This gives

$$
\begin{aligned}
&b_{2n} = n + X_n^{(\alpha,\beta)},\\
&b_{2n+1} = 3n + \alpha + \beta + 2 + a - X_n^{(\alpha,\beta)},\\
&c_{2n} = n(\alpha + \beta + 2n),\\
&c_{2n+1} = n(\alpha + \beta + 2n + 1) - a(n + \beta + 1) + (\alpha + \beta + 2n + 2 + a)X_n^{(\alpha,\beta)} - (X_n^{(\alpha,\beta)})^2,\\
&d_{2n} = -an(\beta + n) + n(\alpha + \beta + 2n)X_{n-1}^{(\alpha,\beta)},\\
&d_{2n+1} = n[(\alpha + \beta + 2n)(\alpha + \beta + 2n + 1) + a(n + \alpha)] - n(\alpha + \beta + 2n)X_n^{(\alpha,\beta)}.
\end{aligned}
$$

For large n we have $X_n^{(\alpha,\beta)} = a/2 + o(1)$ so that

$$
\lim_{n\to\infty} \frac{b_n}{n} = \begin{cases} 1/2 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2),\\ 3/2 & \text{if } n \equiv 1 \,(\mathrm{mod}\, 2),\end{cases}
$$

$$
\lim_{n\to\infty} \frac{c_n}{n^2} = 1/2,
$$

$$
\lim_{n\to\infty} \frac{d_n}{n^3} = \begin{cases} 0 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2),\\ 1/2 & \text{if } n \equiv 1 \,(\mathrm{mod}\, 2).\end{cases}
$$

3.7. Laguerre–Hermite polynomials

Another limit case of the Jacobi–Angelesco polynomials are the multiple orthogonal polynomials $H_{n,m}^{(\beta)}(x)$ for which

$$
\begin{aligned}
&\int_{-\infty}^{0} H_{n,m}^{(\beta)}(x)|x|^{\beta} \mathrm{e}^{-x^2} x^k \,\mathrm{d}x = 0, \quad k = 0,1,\ldots,n-1,\\
&\int_{0}^{\infty} H_{n,m}^{(\beta)}(x) x^{\beta} \mathrm{e}^{-x^2} x^k \,\mathrm{d}x = 0, \quad k = 0,1,\ldots,m-1.
\end{aligned}
$$

We call these *Laguerre–Hermite polynomials* because both weights are supported on semi-infinite intervals (Laguerre) with a common weight that resembles the Hermite weight. These polynomials were already considered (for general r) by Sorokin [40]. The limit case is obtained by taking

$$
H_{n,m}^{(\beta)}(x) = \lim_{\alpha\to\infty} (\sqrt{\alpha})^{n+m} P_{n,m}^{(\alpha,\beta,\alpha)}(x/\sqrt{\alpha}; -1). \tag{3.57}
$$

This allows us to obtain the raising operator, the Rodrigues formula, an explicit expression, and the recurrence coefficients by taking the appropriate limit passage in the formulas for the Jacobi–Angelesco polynomials. For the recurrence coefficients this gives

$$
b_n = \lim_{\alpha\to\infty} \sqrt{\alpha} b_n^{(\alpha,\beta,\alpha)}(a = -1),
$$

$$
c_n = \lim_{\alpha\to\infty} \alpha c_n^{(\alpha,\beta,\alpha)}(a = -1),
$$

$$
d_n = \lim_{\alpha\to\infty} (\sqrt{\alpha})^3 d_n^{(\alpha,\beta,\alpha)}(a = -1),
$$

and

$$\lim_{\alpha\to\infty} \sqrt{\alpha} X_n^{(\alpha,\beta,\alpha)}(a=-1) = \frac{\int_{-\infty}^{0} x|x|^{\beta+n}\mathrm{e}^{-x^2}\,\mathrm{d}x}{\int_{-\infty}^{0} |x|^{\beta+n}\mathrm{e}^{-x^2}\,\mathrm{d}x} := X_n^{(\beta)},$$

from which we find

$$b_{2n} = X_n^{(\beta)},$$

$$b_{2n+1} = -X_n^{(\beta)},$$

$$c_{2n} = n/2,$$

$$c_{2n+1} = \frac{2n+\beta+1}{2} - (X_n^{(\beta)})^2,$$

$$d_{2n} = \frac{n}{2} X_{n-1}^{(\beta)},$$

$$d_{2n+1} = \frac{-n}{2} X_n^{(\beta)}.$$

For large n we have

$$X_n^{(\beta)} = -\sqrt{\frac{\beta+n}{2}} + \mathrm{o}(\sqrt{n}),$$

so that

$$\lim_{n\to\infty} \frac{b_n}{\sqrt{n}} = \begin{cases} -1/2 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2), \\ 1/2 & \text{if } n \equiv 1 \,(\mathrm{mod}\, 2), \end{cases}$$

$$\lim_{n\to\infty} \frac{c_n}{n} = 1/4,$$

$$\lim_{n\to\infty} \frac{d_n}{(\sqrt{n})^3} = \begin{cases} -1/8 & \text{if } n \equiv 0 \,(\mathrm{mod}\, 2), \\ 1/8 & \text{if } n \equiv 1 \,(\mathrm{mod}\, 2). \end{cases}$$

4. Open research problems

In the previous sections we gave a short description of multiple orthogonal polynomials and a few examples. For a more detailed account of multiple orthogonal polynomials we refer to Aptekarev [4] and Chapter 4 of the book of Nikishin and Sorokin [34]. Multiple orthogonal polynomials arise naturally in Hermite–Padé approximation of a system of (Markov) functions. For this kind of simultaneous rational approximation we refer to Mahler [28] and de Bruin [9,10]. Hermite–Padé approximation goes back to the nineteenth century, and many algebraic aspects have been investigated since then: existence and uniqueness, recurrences, normality of indices, etc. A study of Type II multiple orthogonal polynomials based on the recurrence relation can be found in Maroni

[29]. The more detailed analytic investigation of the zero distribution, the nth root asymptotics, and the strong asymptotics is more recent and mostly done by researchers from the schools around Nikishin [32,33] and Gonchar [18,19]. See in particular the work of Aptekarev [4], Kalyagin [20,25], Bustamante and López [11], and also the work by Driver and Stahl [15,16] and Nuttall [35]. First, one needs to understand the analysis of ordinary orthogonal polynomials, and then one has a good basis for studying this extension, for which there are quite a few possibilities for research.

4.1. Special functions

The research of orthogonal polynomials as special functions has now led to a classification and arrangement of various important (basic hypergeometric) orthogonal polynomials. In Section 3 we gave a few multiple orthogonal polynomials of the same flavor as the very classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. Regarding these very classical multiple orthogonal polynomials, a few open problems arise:

(1) Are the polynomials given in Section 3 the only possible very classical multiple orthogonal polynomials? The answer very likely is no. First one needs to make clear what the notion of classical multiple orthogonal polynomial means. A possible way is to start from a Pearson type equation for the weights. If one chooses one weight but restricted to disjoint intervals, as we did for the Jacobi–Angelesco, Jacobi–Laguerre, and Laguerre–Hermite polynomials, then Aptekarev et al. [7] used the Pearson equation for this weight as the starting point of their characterization. For several weights it is more natural to study a Pearson equation for the vector of weights $(w_1, w_2, \ldots, w_r)$. Douak and Maroni [13,14] have given a complete characterization of all Type II multiple orthogonal polynomials for which the derivatives are again Type II multiple orthogonal polynomials (Hahn's characterization for the Jacobi, Laguerre, and Hermite polynomials, and the Bessel polynomials if one allows moment functionals which are not positive definite). They call such polynomials classical d-orthogonal polynomials, where d corresponds to our r, i.e., the number of weights (functionals) needed for the orthogonality. Douak and Maroni show that this class of multiple orthogonal polynomials is characterized by a Pearson equation of the form

$$(\Phi \boldsymbol{w})' + \Psi \boldsymbol{w} = \boldsymbol{0},$$

where $\boldsymbol{w} = (w_1, \ldots, w_r)^t$ is the vector of weights, and Ψ and Φ are $r \times r$ matrix polynomials:

$$\Psi(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r-1 \\ \psi(x) & c_1 & c_2 & \cdots & c_{r-1} \end{pmatrix},$$

with $\psi(x)$ a polynomial of degree one and $c_1, \ldots, c_{r-1}$ constants, and

$$\Phi(x) = \begin{pmatrix} \phi_{1,1}(x) & \phi_{1,2}(x) & \cdots & \phi_{1,r}(x) \\ \phi_{2,1}(x) & \phi_{2,2}(x) & \cdots & \phi_{2,r}(x) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{r,1}(x) & \phi_{r,2}(x) & \cdots & \phi_{r,r}(x) \end{pmatrix},$$

where $\phi_{i,j}(x)$ are polynomials of degree at most two. In fact only $\phi_{r,1}$ can have degree at most two and all other polynomials are constant or of degree one, depending on their position in the matrix Φ. Douak and Maroni actually investigate the more general case where orthogonality is given by r linear functionals, rather than by r positive measures. We believe that Hahn's characterization is not the appropriate property to define classical multiple orthogonal polynomials, but gives a more restricted class. None of the seven families, given in the present paper, belong to the class studied by Douak and Maroni, but their class certainly contains several interesting families of multiple orthogonal polynomials. In fact, the matrix Pearson equation could result from a single weight (and its derivatives) satisfying a higher-order differential equation with polynomial coefficients. As an example, one can have multiple orthogonal polynomials with weights $w_1(x) = 2x^{\alpha+v/2}K_v(2\sqrt{x})$ and $w_2(x)=2x^{\alpha+(v+1)/2}K_{v+1}(2\sqrt{x})$ on $[0,\infty)$, where $K_v(x)$ is a modified Bessel function and $\alpha > -1, v \geqslant 0$ (see [47,12]).

(2) The polynomials of Jacobi, Laguerre, and Hermite all satisfy a linear second-order differential equation of Sturm–Liouville type. A possible way to extend this characterizing property is to look for multiple orthogonal polynomials satisfying a linear differential equation of order $r+1$. Do the seven families in this paper have such a differential equation? If the answer is yes, then an explicit construction would be desirable. We only worked out in detail the case where $r=2$, so the search is for a third-order differential equation for all the polynomials considered in Section 3. Such a third-order equation has been found for certain Jacobi–Angelesco systems in [25]. For the Angelesco systems in Section 3 this third-order differential equation indeed exists and it was constructed in [7]. The existence (and construction) is open for the AT systems. A deeper problem is to characterize all the multiple orthogonal polynomials satisfying a third order (order $r+1$) differential equation, extending Bochner's result for ordinary orthogonal polynomials. Observe that we already know appropriate raising operators for the seven systems described in Section 3. If one can construct lowering operators as well, then a combination of the raising and lowering operators will give the differential equation, which will immediately be in factored form. Just differentiating will usually not be sufficient (except for the class studied by Douak and Maroni): if we take $P'_{n,m}(x)$, then this is a polynomial of degree $n+m-1$, so one can write it as $P_{n-1,m}(x)+$ lower order terms, but also as $P_{n,m-1}(x)+$ lower-order terms. So it is not clear which of the multi-indices has to be lowered. Furthermore, the lower-order terms will not vanish in general since there usually are not enough orthogonality conditions to make them disappear.

(3) In the present paper we only considered the Type II multiple orthogonal polynomials. Derive explicit expressions and relevant properties of the corresponding vector $(A_{n,m}(x), B_{n,m}(x))$ of Type I multiple orthogonal polynomials. Type I and Type II multiple orthogonal polynomials are connected by

$$P_{n,m}(x) = \text{const.} \begin{vmatrix} A_{n+1,m}(x) & B_{n+1,m}(x) \\ A_{n,m+1}(x) & B_{n,m+1}(x) \end{vmatrix},$$

but from this it is not so easy to obtain the Type I polynomials.

(4) So far we limited ourselves to the very classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. Discrete orthogonal polynomials, such as those of Charlier, Kravchuk, Meixner, and Hahn, can also be considered and several kinds of discrete multiple orthogonal polynomials can be worked out. It would not be a good idea to do this case by case, since these polynomials are

all connected by limit transitions, with the Hahn polynomials as the starting family. At a later stage, one could also consider multiple orthogonal polynomials on a quadratic lattice and on the general exponential lattice, leading to q-polynomials. Again, all these families are related with the Askey–Wilson polynomials as the family from which all others can be obtained by limit transitions. Do these polynomials have a representation as a (basic) hypergeometric function? Recall that we needed an Appell hypergeometric function of two variables for the Jacobi–Angelesco polynomials, so that one may need to consider (basic) hypergeometric functions of several variables.

(5) Multiple orthogonal polynomials arise naturally in the study of Hermite–Padé approximation, which is simultaneous rational approximation to a vector of r functions. In this respect it is quite natural to study multiple orthogonal polynomials as orthogonal vector polynomials. This approach is very useful in trying to extend results for the case $r = 1$ to the case $r > 1$ by looking for an appropriate formulation using vector algebra. Van Iseghem already used this approach to formulate a vector QD-algorithm for multiple orthogonal polynomials [48]. Several algebraic aspects of multiple orthogonal polynomials follow easily from the vector orthogonality [42,27]. A further generalization is to study matrix orthogonality, where the matrix need not be a square matrix [43]. Orthogonal polynomials and Padé approximants are closely related to certain continued fractions (J-fractions and S-fractions). For multiple orthogonal polynomials there is a similar relation with vector continued fractions and the Jacobi–Perron algorithm [36]. The seven families which we considered in this paper lead to seven families of vector continued fractions, which could be studied in more detail in the framework of continued fractions. Finally, one may wonder whether it is possible to use hypergeometric functions of matrix argument in the study of multiple orthogonal polynomials.

4.2. Non-symmetric banded operators

In Section 2 the connection between multiple orthogonal polynomials and banded Hessenberg operators of the form

$$\begin{pmatrix}
a_{0,0} & 1 & & & & & & & & & \\
a_{1,1} & a_{1,0} & 1 & & & & & & & & \\
a_{2,2} & a_{2,1} & a_{2,0} & 1 & & & & & & & \\
\vdots & & & \ddots & \ddots & & & & & & \\
a_{r,r} & a_{r,r-1} & \cdots & & a_{r,0} & 1 & & & & & \\
 & a_{r+1,r} & \ddots & & & a_{r+1,0} & 1 & & & & \\
 & & \ddots & \ddots & & & \ddots & \ddots & & & \\
 & & & \ddots & \ddots & & & \ddots & 1 & & \\
 & & & & a_{n,r} & a_{n,r-1} & \cdots & a_{n,1} & a_{n,0} & \ddots & \\
 & & & & & \ddots & \cdots & & \cdots & \ddots &
\end{pmatrix}$$

was explained. For ordinary orthogonal polynomials the operator is tridiagonal and can always be made symmetric, and often it can be extended in a unique way to a self-adjoint operator (e.g., when all the coefficients are bounded). The spectrum of this tridiagonal operator corresponds to the support of the orthogonality measure, and the spectral measure is precisely the orthogonality measure.

Each tridiagonal matrix with ones on the upper diagonal and positive coefficients on the lower diagonal, corresponds to a system of orthogonal polynomials on the real line (Favard's theorem). Some preliminary work on the spectral theory of the higher-order operators ($r > 1$) was done by Kalyagin [21–23,5], but there are still quite a few open problems here.

(1) What is the proper extension of Favard's theorem for these higher-order banded Hessenberg operators? Not every banded Hessenberg operator corresponds to a system of multiple orthogonal polynomials with orthogonality relations on the real line. There needs to be additional structure, but so far this additional structure is still unknown. There is a weak version of the Favard theorem that gives multiple orthogonality with respect to linear functionals [48,24], but a stronger version that gives positive measures on the real line is needed. How do we recognize an Angelesco system, an AT system, or one of the combinations considered in [19] from the recurrence coefficients (from the operator)? The special case where all the diagonals are zero, except for the upper diagonal (which contains 1's) and the lower diagonal, has been studied in detail in [6]. They show that when the lower diagonal contains positive coefficients, the operator corresponds to multiple orthogonal polynomials on an $(r + 1)$-star in the complex plane. Using a symmetry transformation, similar to the quadratic transformation that transforms Hermite polynomials to Laguerre polynomials, this also gives an AT system of multiple orthogonal polynomials on $[0, \infty)$.

(2) The asymptotic behavior of the recurrence coefficients of the seven systems described above is known. Each of the limiting operators deserves to be investigated in more detail. The limiting operator for the Jacobi–Piñeiro polynomials is a Toeplitz operator, and hence can be investigated in more detail. See, e.g., [46] for this case. Some of the other limiting operators are block Toeplitz matrices and can be investigated as well. Are there any multiple orthogonal polynomials having such recurrence coefficients? The Chebyshev polynomials of the second kind have this property when one deals with tridiagonal operators.

(3) The next step would be to work out a perturbation theory, where one allows certain perturbations of the limiting matrices. Compact perturbations would be the first step, trace class perturbations would allow us to give more detailed results.

4.3. Applications

(1) Hermite–Padé approximation was introduced by Hermite for his proof of the transcendence of e. More recently it became clear that Apéry's proof of the irrationality of $\zeta(3)$ relies on an AT system of multiple orthogonal polynomials with weights $w_1(x) = 1$, $w_2(x) = -\log(x)$ and $w_3(x) = \log^2(x)$ on [0,1]. These multiple orthogonal polynomials are basically limiting cases of Jacobi–Piñeiro polynomials where $\alpha_0 = 0 = \alpha_1 = \alpha_2$. A very interesting problem is to prove irrationality of other remarkable constants, such as $\zeta(5)$, Catalan's constant, or Euler's constant. Transcendence proofs will even be better. See [4,45] for the connection between multiple orthogonal polynomials, irrationality, and transcendence.

(2) In numerical analysis one uses orthogonal polynomials when one constructs Gauss quadrature. In a similar way one can use multiple orthogonal polynomials to construct optimal quadrature formulas for jointly approximating r integrals of the same function f with respect to r weights $w_1, \ldots, w_r$. See, e.g., Borges [8], who apparently is not aware that he is using multiple orthogonal polynomials. Gautschi [17] has summarized some algorithms for computing recurrence coefficients, quadrature nodes (zeros of orthogonal polynomials) and quadrature weights (Christoffel numbers)

for ordinary Gauss quadrature. A nice problem is to modify these algorithms so that they compute recurrence coefficients, zeros of multiple orthogonal polynomials (eigenvalues of banded Hessenberg operators) and quadrature weights for simultaneous Gauss quadrature.

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