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On the q-polynomials: a distributional study

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Abstract

In this paper we present a unified distributional study of the classical discrete q-polynomials (in the Hahn's sense). From the distributional q-Pearson equation we will deduce many of their properties such as the three-term recurrence relations, structure relations, etc. Also several characterizations of such q-polynomials are presented. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

The so-called q-polynomials constitute a very important and interesting set of special functions and more specifically of orthogonal polynomials. They appear in several branches of the natural sciences, e.g., continued fractions, Eulerian series, theta functions, elliptic functions, etc.; see [5,12], quantum groups and algebras [19,20,30], discrete mathematics (combinatorics, graph theory), coding theory, among others (see also [14]).

In 1884, Markov introduced a specific family of these q-polynomials. Later on, Hahn [16] analyzed a more general situation. In fact Hahn was interested to find all orthogonal polynomial sequences such that their q-differences, defined by the linear operator $\Theta f(x) = (f(qx) - f(x))/(q-1)x$ were

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orthogonal. Notice that when $q \to 1$ we recover the characterization of the classical polynomials given by Sonine in 1887 and rediscovered by Hahn in 1937 [15]. Thirty years later the study of such polynomials has known an increasing interest (for a review see [6]). Indeed, this first systematic approach for q-polynomials comes from the fact that they are basic (terminating) hypergeometric series [14]. For a complete set of references on this see [7,14,18].

Another point of view was developed by the Russian (former Soviet) school of mathematicians starting from a work by Nikiforov and Uvarov in 1983 [27]. It was based on the idea that the q-polynomials are the solution of a second-order linear difference equation with certain properties: the so-called difference equation of hypergeometric type on non-uniform lattices. This scheme is usually called the Nikiforov–Uvarov scheme of q-polynomials [28]. For several surveys on this approach see [3,4,7,26,29].

In this work we will present a different approach: It can be considered a pure algebraic approach and constitutes an alternative to the two previous ones, and, in some sense is the continuation of the Hahn's work [16]. Furthermore, we will prove here that the q-classical polynomials are characterized by several relations, analogue to the ones satisfied by the classical "continuous" (Jacobi, Bessel, Laguerre, Hermite) and "discrete" (Hahn, Meixner, Kravchuk and Charlier) orthogonal polynomials [1,13,21,22] and references therein. Besides, our point of view is very different from the previous ones based on the basic hypergeometric series and the difference equation, respectively. In fact we start with the distributional equation that the q-moment functionals satisfy and we will prove all the other characterizations using basically the algebraic theory developed by Maroni [23]. So, somehow, this paper is the natural continuation of the study started in [22,13] for the "continuous" and "discrete" orthogonal polynomials, respectively. Another advantage of this approach is the unified and simple treatment of the q-polynomials where all the information is obtained from the coefficients of the polynomials ϕ and ψ of the distributional or Pearson equation (compare it with the method by the American school [20] or the Russian ones [29]).

Let us point out here that the theory of orthogonal polynomials on the non-uniform lattices is based not on the Pearson equation and on the hypergeometric-type difference equation of the non-uniform lattices as it is shown in papers [7,26,28] and obviously it is possible to derive many properties of the q-classical polynomials from this difference hypergeometric equation. Our purpose is not to show how from the difference equation many properties can be obtained, but to show that some of them *characterize* the q-classical polynomials, i.e., the main *aim* is the proof of several characterizations of these q-families as well as the explicit computations of the corresponding coefficients in a unified way. Some of these results on characterizations (e.g. the Al-Salam-Chihara or Marcellán et al. characterization for classical polynomials) are completely new as far as we know.

Moreover, in our approach there is not any lattice function although the corresponding q-classical polynomials that appear when there exists a positive weight are the corresponding polynomials on q-linear lattices in the Nikiforov et al. approach. Only in this sense our approach is "similar" to the Nikiforov et al. one and, up to now, it is covering only the polynomials corresponding to the aforesaid q-linear lattice (see also [20]). Finally, let us to recall here that we have not dealt with any integral involving these q-polynomials even we have not dealt in any moment with the norm of the polynomials or the weight function. The main reason is that our approach is rather new with respect to the aforesaid two methods since we are working not in the space of functions (or polynomials) but also in its dual distributional space and for this reason it is, as we already pointed out, a pure algebraic approach in the sense developed by Maroni [23].

The structure of the paper is as follows. In Section 1 we introduce some notations and definitions useful for the next ones. In Section 2, starting from the distributional equation $\Theta(\phi \boldsymbol{u}) = \psi \boldsymbol{u}$ that the moment functional $\boldsymbol{u}$, with respect to which the polynomial sequence is orthogonal, satisfies we will obtain five different characterizations of these q-polynomials. They are quoted in Theorems 2.1 and 2.2 and Propositions 2.9 and 2.10, respectively. In Section 3, we deduce the main characteristics of the q-polynomials in terms of the coefficients of the polynomials ϕ and ψ of the distributional equation, i.e., the coefficients of the three-term recurrence relations and of the other characterization relations (those proved in Section 2). In Section 4, all q-classical, according to the Hahn's definition, families of polynomials of the q-Askey Tableu are studied in details including all their characteristics.

1. Preliminaries

In this section we will give a brief survey of the operational calculus that we will use in the rest of the paper.

1.1. Basic concepts and results

Let $\mathbb{P}$ be the linear space of polynomial functions in $\mathbb{C}$ (in the following we will refer to them as *polynomials*) with complex coefficients and $\mathbb{P}^*$ be its algebraic dual space, i.e., $\mathbb{P}^*$ is the linear space of all linear applications $\boldsymbol{u} : \mathbb{P} \to \mathbb{C}$. In the following we will call the elements of $\mathbb{P}^*$ as functionals and we will denote them with bold letters $(\boldsymbol{u}, \boldsymbol{v}, \ldots)$.

Let be $(B_n)_{n\geqslant 0}$ a sequence of polynomials such that $\deg B_n \leqslant n$ for all $n \geqslant 0$. A sequence defined in this way is said to be a basis or a basis sequence of $\mathbb{P}$ if and only if $\deg B_n = n$ for all $n \geqslant 0$. Since the elements of $\mathbb{P}^*$ are linear functionals, it is possible to determine them from their actions on a given basis $(B_n)_{n\geqslant 0}$ of $\mathbb{P}$. We will use here, without loss of generality, the canonical basis of $\mathbb{P}$, $(x^n)_{n\geqslant 0}$. In general, we will represent the action of a functional over a polynomial by

$$\langle \boldsymbol{u}, \pi \rangle, \quad \boldsymbol{u} \in \mathbb{P}^*, \ \pi \in \mathbb{P}.$$

Therefore, a functional is completely determined by a sequence of complex numbers $\langle \boldsymbol{u}, x^n \rangle = u_n$, $n \geqslant 0$, the so-called moments of the functional.

We will use the following definition for an orthogonal polynomial sequence:

Definition 1.1. Let $(P_n)_{n\geqslant 0}$ be a basis sequence of $\mathbb{P}$ such that $\deg P_n = n$. We say that $(P_n)_{n\geqslant 0}$ is an orthogonal polynomial sequence (OPS in short), if and only if there exists a functional $\boldsymbol{u} \in \mathbb{P}^*$ such that

$$\langle \boldsymbol{u}, P_m P_n \rangle = k_n \delta_{mn}, \quad k_n \neq 0, \ n \geqslant 0,$$

where δ_{mn} is the Kronecker delta.

Definition 1.2. Let $\boldsymbol{u} \in \mathbb{P}^*$ be a functional. We say that $\boldsymbol{u}$ is a quasi-definite functional if and only if there exists a polynomial sequence $(P_n)_{n\geqslant 0}$, which is orthogonal with respect to $\boldsymbol{u}$.

Remark 1.3. Given two polynomial sequences, $(P_n)_{n\geqslant 0}$ and $(R_n)_{n\geqslant 0}$, orthogonal with respect to the same linear functional, $\boldsymbol{u}$, i.e.,

$$\left.\begin{array}{l}\langle \boldsymbol{u}, P_m P_n\rangle = k_n\delta_{nm},\ k_n \neq 0,\ n\geqslant 0\\ \langle \boldsymbol{u}, R_m R_n\rangle = \bar{k}_n\delta_{nm},\ \bar{k}_n \neq 0,\ n\geqslant 0\end{array}\right\}\ \text{then there exists } c_n \in \mathbb{C}\setminus\{0\},\quad P_n = c_n R_n,\ \ n\geqslant 0.$$

Moreover, if $(P_n)_{n\geqslant 0}$ is orthogonal with respect to the functionals $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\left.\begin{array}{l}\langle \boldsymbol{u}, P_m P_n\rangle = k_n\delta_{nm},\ k_n \neq 0,\ n\geqslant 0\\ \langle \boldsymbol{v}, P_m P_n\rangle = \bar{k}_n\delta_{nm},\ \bar{k}_n \neq 0,\ n\geqslant 0\end{array}\right\}\ \text{then there exists } c \in \mathbb{C}\setminus\{0\},\quad cv_n = u_n,\ \ n\geqslant 0,$$

where v_n and u_n are the moments corresponding to the functionals $\boldsymbol{v}$ and $\boldsymbol{u}$, respectively. This means that, if we "*normalize*" the OPS in any way, then we have a unique polynomial sequence orthogonal with respect to a given functional.

Definition 1.4. Given a polynomial sequence $(P_n)_{n\geqslant 0}$, we say that $(P_n)_{n\geqslant 0}$ is a monic orthogonal polynomial sequence (MOPS in short) with respect to $\boldsymbol{u}$, and we denote it by $(P_n)_{n\geqslant 0} = \operatorname{mops}\boldsymbol{u}$ if and only if

$$P_n(x) = x^n + \textit{lower degree terms} \quad\text{and}\quad \langle \boldsymbol{u}, P_m P_n\rangle = k_n\delta_{nm},\quad k_n \neq 0,\ \ n\geqslant 0.$$

Since any MOPS $(P_n)_{n\geqslant 0}$ is a basis of $\mathbb{P}$ then, any polynomial π of degree n is a linear combination of $(P_n)_{n\geqslant 0}$:

$$\pi = \sum_{i=0}^{n} c_i P_i,\quad c_n \neq 0\quad \text{where } c_i = k_i^{-1}\langle \boldsymbol{u}, \pi P_i\rangle,\quad k_i = \langle \boldsymbol{u}, P_i^2\rangle,\ \ 0\leqslant i\leqslant n.$$

Thus,

Theorem 1.5. *Let $\boldsymbol{u} \in \mathbb{P}^*$ and $(B_n)_{n\geqslant 0}$ be a basis sequence of $\mathbb{P}$. Then, the following statements are equivalent*

1. $\langle \boldsymbol{u}, B_m B_n\rangle = 0,\ n \neq m$ *if and only if* $\langle \boldsymbol{u}, x^m B_n\rangle = 0,\ 0\leqslant m < n$, *for all* $n\geqslant 0$
2. $\langle \boldsymbol{u}, B_n^2\rangle \neq 0$ *if and only if* $\langle \boldsymbol{u}, x^n B_n\rangle \neq 0$, *for all* $n\geqslant 0$

Also the next theorem will be useful [9, p. 8].

Theorem 1.6. *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a functional with moments $u_n = \langle \boldsymbol{u}, x^n\rangle,\ n\geqslant 0$. Then, $\boldsymbol{u}$ is quasi-definite if and only if the Hankel determinants $H_n := \det(u_{i+j})_{i,j=0}^{n} \neq 0,\ n\geqslant 0$.*

Notice that, given a functional $\boldsymbol{u}$ with moments $(u_n)_{n\geqslant 0}$, the nth monic orthogonal polynomial is

$$P_n = H_{n-1}^{-1}\begin{vmatrix} u_0 & u_1 & \dots & u_n\\ u_1 & u_2 & \dots & u_{n+1}\\ \cdot & \cdot & \dots & \cdot\\ u_{n-1} & u_n & \dots & u_{2n-1}\\ 1 & x & \dots & x^n\end{vmatrix}.$$

Definition 1.7. Let $\boldsymbol{u} \in \mathbb{P}^*$, be a quasi-definite functional. We say that $\boldsymbol{u}$ is positive definite if and only if $H_n > 0$, $\forall n \geqslant 0$.

Theorem 1.8. *Let $(P_n)_{n\geqslant 0}$ be a monic polynomial basis sequence. Then, $(P_n)_{n\geqslant 0}$ is an OPS if and only if there exist two sequences of complex numbers $(d_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 1}$, with $g_n \neq 0$, $n \geqslant 1$ such that*

$$xP_n = P_{n+1} + d_nP_n + g_nP_{n-1}, \quad P_{-1} = 0,\ P_0 = 1, \quad n \geqslant 0, \tag{1.1}$$

where $P_{-1}(x) \equiv 0$ and $P_0(x) \equiv 1$. Moreover, the functional $\boldsymbol{u}$ with respect to which the polynomials $(P_n)_{n\geqslant 0}$ are orthogonal is positive definite if and only if $(d_n)_{n\geqslant 0}$ is a real sequence and $g_n > 0$ for all $n \geqslant 1$.

Remark 1.9. If $(P_n)_{n\geqslant 0} = \text{mops}\,\boldsymbol{u}$, then the sequences $(d_n)_{n\geqslant 0}$ and $(g_n)_{n\geqslant 1}$ are given by

$$d_n = \frac{\langle \boldsymbol{u}, xP_n^2\rangle}{\langle \boldsymbol{u}, P_n^2\rangle}, \quad n \geqslant 0, \qquad \text{and} \qquad g_n = \frac{\langle \boldsymbol{u}, P_n^2\rangle}{\langle \boldsymbol{u}, P_{n-1}^2\rangle}, \quad n \geqslant 1.$$

Theorem 1.8 is usually called *Favard Theorem* [9,11].

1.2. Definition of the operators in $\mathbb{P}$ and $\mathbb{P}^*$

From now on we will use the following notation:

Definition 1.10. Let $\pi \in \mathbb{P}$ and $a \in \mathbb{C}$, $a \neq 0$. The operator

$$H_a : \mathbb{P} \to \mathbb{P}, \quad H_a\pi(x) = \pi(ax).$$

is said to be a dilation of ratio $a \in \mathbb{C} \setminus \{0\}$.

This operator is linear on $\mathbb{P}$ and satisfies $H_a(\pi\rho) = H_a\pi \cdot H_a\rho$. Also notice that for any complex number $a \neq 0$, $H_a \cdot H_{a^{-1}} = \mathrm{I}$, where I is the identity operator on $\mathbb{P}$, i.e., for all $a \neq 0$, H_a has an inverse operator. In the following and for a sake of simplicity we will omit any reference to q in the operators H_q and their inverse $H_{q^{-1}}$. So, $H{:=}H_q$, $H^{-1}{:=}H_{q^{-1}}$.

Next, we will define the so called q-derivative operator, which constitutes a generalization of the Hahn operator for $q \in \mathbb{C} \setminus \{0\}$, see [16]. We will suppose also that $|q| \neq 1$ (although it is possible to weaken this condition).

Definition 1.11. Let $\pi \in \mathbb{P}$ and $q \in \mathbb{C} \setminus \{0\}$, $|q| \neq 1$.

The q-derivative operator Θ, is the operator $\Theta : \mathbb{P} \to \mathbb{P}$,

$$\Theta\pi = \frac{H\pi - \pi}{Hx - x} = \frac{H\pi - \pi}{(q-1)x}.$$

The q^{-1}-derivative operator $\Theta^\star$, is the operator $\Theta^\star : \mathbb{P} \to \mathbb{P}$

$$\Theta^\star\pi = \frac{H^{-1}\pi - \pi}{H^{-1}x - x} = \frac{H^{-1}\pi - \pi}{(q^{-1}-1)x}.$$

In this way, $\Theta\pi$ and $\Theta^\star\pi$ will denote the q-derivative and q^{-1}-derivative of π, respectively.

Obviously, the above two operators Θ and $\Theta^\star$ are linear operators on $\mathbb{P}$. Moreover, since

$$\Theta x^n = \frac{Hx^n - x^n}{(q-1)x} = \frac{(q^n-1)x^n}{(q-1)x} = [n]x^{n-1}, \quad n>0, \quad \Theta 1 = 0, \tag{1.2}$$

then, $\Theta\pi \in \mathbb{P}$. Here $[n]$, $n \in \mathbb{N}$, denotes the basic q-number n defined by

$$[n] = \frac{q^n-1}{q-1} = 1 + q + \cdots + q^{n-1}, \quad n>0, \quad [0]=0. \tag{1.3}$$

It satisfies the following basic arithmetic rule $[n] + q^n[m] = [n+m]$. In the following we will also use the q^{-1}-number $[n]^\star$, defined by

$$[n]^\star = \frac{q^{-n}-1}{q^{-1}-1} = q^{1-n}[n].$$

Remark 1.12. The relation (1.2) is the q-analogue of the property $\mathrm{D}x^n = nx^{n-1}$, where D denotes the standard derivative. For this reason it is natural to choose $(x^n)_{n\geqslant 0}$ as the canonical basis of $\mathbb{P}$.

Notice that $\Theta^\star$ is not the inverse of Θ. In fact they are related by

$$H\Theta^\star = \Theta, \quad H^{-1}\Theta = \Theta^\star.$$

Moreover, using straightforward calculations we get

$$\Theta H = qH\Theta, \quad \Theta H^{-1} = q^{-1}H^{-1}\Theta = q^{-1}\Theta^\star,$$

$$\Theta^\star H = qH\Theta^\star = q\Theta, \quad \Theta^\star H^{-1} = q^{-1}H^{-1}\Theta^\star, \tag{1.4}$$

and

$$\Theta\Theta^\star = \Theta H^{-1}\Theta = q^{-1}H^{-1}\Theta\Theta = q^{-1}\Theta^\star\Theta.$$

Furthermore, the q-derivative satisfies the product rule

$$\Theta(\pi\rho) = \rho\Theta\pi + H\pi\cdot\Theta\rho = H\rho\cdot\Theta\pi + \pi\Theta\rho.$$

Here we will also use the so-called q-factorial power or generalized q-factorial $[n]^{(i)} \equiv [n][n-1]\cdots[n-i+1]$ as well as the q-analogue of the Pochhammer symbol $[n-i+1]_{(i)} \equiv [n-i+1][n-i+2]\cdots[n]$.

Next we will transpose the operations in $\mathbb{P}$ to its dual space $\mathbb{P}^*$.

Definition 1.13. Let $\boldsymbol{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. We define the action of a dilation H_a and the q-derivative Θ on $\mathbb{P}^*$ as follows:

$$H_a : \mathbb{P}^* \to \mathbb{P}^*, \quad \langle H_a\boldsymbol{u}, \pi\rangle = \langle \boldsymbol{u}, H_a\pi\rangle, \qquad \Theta : \mathbb{P}^* \to \mathbb{P}^*, \quad \langle \Theta\boldsymbol{u}, \pi\rangle = -\langle \boldsymbol{u}, \Theta\pi\rangle.$$

Definition 1.14. Let $\boldsymbol{u} \in \mathbb{P}^*$ and $\pi \in \mathbb{P}$. The polynomial modification of a functional $\boldsymbol{u}$, the functional, i.e. $\pi\boldsymbol{u}$, is given by

$$\langle \pi\boldsymbol{u}, \rho\rangle = \langle \boldsymbol{u}, \pi\rho\rangle, \quad \forall \rho \in \mathbb{P}.$$

Notice that we use the same notation for the operators on $\mathbb{P}$ and $\mathbb{P}^*$. Whenever it is not specified the linear space where an operator acts, it will be understood that it acts on the polynomial space $\mathbb{P}$.

2. Characterizations

2.1. Dual bases, q-derivatives, and orthogonality

Since any basis sequence of polynomials $(B_n)_{n\geqslant 0}$ generates a unique basis in $\mathbb{P}^*$, $(\boldsymbol{b}_n)_{n\geqslant 0}$ (the so-called dual basis of $(B_n)_{n\geqslant 0}$), i.e., a sequence of linear functionals $(\boldsymbol{b}_n)_{n\geqslant 0}$ such that

$$\langle \boldsymbol{b}_n, B_m\rangle = \delta_{nm}, \quad n,m\geqslant 0,$$

then, any element of $\mathbb{P}^*$ can be represented in the following way:

$$\boldsymbol{v} = \sum_{n\geqslant 0} v_n \boldsymbol{b}_n, \quad v_n = \langle \boldsymbol{v}, B_n\rangle, \; n\geqslant 0.$$

This leads to the following

Proposition 2.1. *Let $\boldsymbol{u}$, $\boldsymbol{u}\in\mathbb{P}^*$ be a quasi-definite functional. If $(P_n)_{n\geqslant 0}$ is the corresponding monic OPS, and $(\boldsymbol{p}_n)\subset\mathbb{P}^*$ the dual basis of $(P_n)_{n\geqslant 0}$; then, $\boldsymbol{p}_n = k_n^{-1}P_n\boldsymbol{u}$, where $k_n = \langle \boldsymbol{u}, P_n^2\rangle$, $n\geqslant 0$.*

Proof. It follows from the fact that $\langle P_n\boldsymbol{u}, P_m\rangle = \langle \boldsymbol{u}, P_mP_n\rangle = k_n\delta_{nm}$, $n,m\geqslant 0$. $\square$

Proposition 2.2. *Let $(B_n)_{n\geqslant 0}$ be a basis sequence of monic polynomials (not necessary orthogonal) and let $(D_n)_{n\geqslant 0}$ be the sequence of their monic q-derivatives, $D_n = (1/[n+1])\Theta B_{n+1}$. If $(\boldsymbol{b}_n)_{n\geqslant 0}$ and $(\boldsymbol{d}_n)_{n\geqslant 0}$ are the respective dual basis of $(B_n)_{n\geqslant 0}$ and $(D_n)_{n\geqslant 0}$, then*

$$\Theta\boldsymbol{d}_n = -[n+1]\boldsymbol{b}_{n+1}.$$

Proof. It follows from the fact that $\langle \Theta\boldsymbol{d}_m, B_{n+1}\rangle = -\langle \boldsymbol{d}_m, \Theta B_{n+1}\rangle = -[n+1]\langle \boldsymbol{d}_m, D_n\rangle = -[n+1]\delta_{nm}$ and $\Theta\boldsymbol{d}_m = \sum_{n\geqslant 0}\langle \Theta\boldsymbol{d}_m, B_n\rangle \boldsymbol{b}_n$. $\square$

Corollary 2.3. *Let $(P_n)_{n\geqslant 0} = \operatorname{mops}\boldsymbol{u}$ and $(Q_n)_{n\geqslant 0}$ be the sequence of their monic q-derivatives. If $(\boldsymbol{q}_n)_{n\geqslant 0}$ is the dual basis of $(Q_n)_{n\geqslant 0}$ then,*

$$\Theta\boldsymbol{q}_n = -[n+1]k_{n+1}^{-1}P_{n+1}\boldsymbol{u}, \quad k_n = \langle \boldsymbol{u}, P_n^2\rangle, \; n\geqslant 0.$$

Moreover, if $(Q_n)_{n\geqslant 0}$ are orthogonal with respect to the functional $\boldsymbol{v}$, with $v_0 = \langle \boldsymbol{v}, 1\rangle$, then $\Theta\boldsymbol{v} = -v_0k_1^{-1}P_1\boldsymbol{u}$.

As an immediate consequence of Corollary 2.3: $\Theta\boldsymbol{v} = \psi\boldsymbol{u}$, where $\psi = -v_0k_1^{-1}P_1$ and $\deg\psi = 1$. Next we will show that $\boldsymbol{v} = \phi\boldsymbol{u}$, being $\deg\phi\leqslant 2$. Notice that $\boldsymbol{u} = u_0\boldsymbol{p}_0$, $\boldsymbol{v} = v_0\boldsymbol{q}_0$.

Proposition 2.4. *Let $(P_n)_{n\geqslant 0} = \operatorname{mops}\boldsymbol{u}$ and $(Q_n)_{n\geqslant 0}$ be the sequence of monic q-derivatives. If $(Q_n)_{n\geqslant 0} = \operatorname{mops}\boldsymbol{v}$, then there exists a polynomial ϕ, $\deg\phi\leqslant 2$ such that $\boldsymbol{v} = \phi\boldsymbol{u}$.*

Proof. Since $\Theta(xP_n) = P_n + qx\Theta P_n$, and Corollary 2.3 ($\Theta\boldsymbol{v} = \psi\boldsymbol{u}, \deg\psi = 1$), we get

$$\langle \boldsymbol{v}, P_n\rangle = \langle \boldsymbol{v}, \Theta(xP_n) - qx\Theta P_n\rangle = -\langle \Theta\boldsymbol{v}, xP_n\rangle - q\langle \boldsymbol{v}, x\Theta P_n\rangle = -\langle \boldsymbol{u}, x\psi\cdot P_n\rangle - q[n]\langle \boldsymbol{v}, xQ_{n-1}\rangle.$$

Now, taking into account the orthogonality of $(P_n)_{n\geqslant 0}$ with respect to $\boldsymbol{u}$ as well as the orthogonality of $(Q_n)_{n\geqslant 0}$ with respect to $\boldsymbol{v}$, we obtain

$$\langle \boldsymbol{v}, P_n\rangle = 0 \quad \text{if } n - (\deg\psi + 1) > 0 \quad \text{and} \quad (n-1) - 1 > 0 \quad \text{if and only if } n > 2.$$

Therefore,

$$\boldsymbol{v} = \sum_{n\geqslant 0} \langle \boldsymbol{v}, P_n\rangle \boldsymbol{p}_n = \sum_{i=0}^{2} \langle \boldsymbol{v}, P_i\rangle \boldsymbol{p}_i = \sum_{i=0}^{2} \langle \boldsymbol{v}, P_i\rangle k_i^{-1} P_i \boldsymbol{u} = \phi\boldsymbol{u}, \quad \text{where } \phi = \sum_{i=0}^{2} \langle \boldsymbol{v}, P_i\rangle k_i^{-1} P_i.$$

Thus, $\deg\phi \leqslant 2$. □

So, it is natural to define a q-classical functional as follows:

Definition 2.5. Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional. We say that $\boldsymbol{u}$ is a q-classical functional and its corresponding MOPS $(P_n)_{n\geqslant 0}$ a q-classical MOPS, if and only if there exists a pair of polynomials ϕ and ψ, $\deg\phi \leqslant 2$, $\deg\psi = 1$, such that

$$\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}. \tag{2.1}$$

Remark 2.6. Given the pair of polynomials (ϕ, ψ), the distributional equation (2.1) defines, up to a constant factor, the functional $\boldsymbol{u}$. Thus (2.1) completely determines the corresponding MOPS, and it is also unique.

Furthermore, if

$$\left.\begin{array}{l}\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u} \\ \Theta(\phi\boldsymbol{u}') = \psi\boldsymbol{u}'\end{array}\right\} \quad \text{then there exists } c \in \mathbb{C} \text{ so that } \boldsymbol{u}' = c\boldsymbol{u}.$$

Conversely, if $\boldsymbol{u}$ is q-classical, then polynomials ϕ and ψ associated to its distributional equation are uniquely determined up to a constant factor, i.e., if

$$\left.\begin{array}{l}\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u} \\ \Theta(\phi'\boldsymbol{u}) = \psi'\boldsymbol{u}\end{array}\right\} \quad \text{then there exists } c \in \mathbb{C} \text{ so that } \phi' = c\phi \text{ and } \psi' = c\psi.$$

Notice that the distributional equation (2.1) yields the difference equation that the moments $(u_n)_{n\geqslant 0}$ of the functional satisfy. In fact, if we write the polynomials ϕ and ψ in (2.1)

$$\phi(x) = \hat{a}x^2 + \bar{a}x + \dot{a}, \qquad \psi(x) = \hat{b}x + \bar{b}, \quad \hat{b} \neq 0 \tag{2.2}$$

for all $n \geqslant 0$, we get

$$\begin{aligned}\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u} &\Leftrightarrow \langle \boldsymbol{u}, \phi\Theta x^n + \psi x^n\rangle = 0, \quad n\geqslant 0 \\ &\Leftrightarrow \langle \boldsymbol{u}, ([n]\hat{a} + \hat{b})x^{n+1} + ([n]\bar{a} + \bar{b})x^n + [n]\cdot\dot{a}x^{n-1}\rangle = 0, \\ &\Leftrightarrow ([n]\hat{a} + \hat{b})u_{n+1} + ([n]\bar{a} + \bar{b})u_n + [n]\dot{a}u_{n-1} = 0, \quad u_{-1} = 0.\end{aligned} \tag{2.3}$$

Therefore, the moments $(u_n)_{n\geqslant 0}$ of $\boldsymbol{u}$ satisfy a second-order linear difference equation whose coefficients are polynomials of first degree in $[n]$, with the initial condition u_0. Indeed,

$$u_{-1}=0, \quad \hat{b}u_1+\bar{b}u_0=0 \;\Leftrightarrow\; u_1=-\hat{b}/\bar{b}\cdot u_0.$$

If $[n]\hat{a}+\hat{b}\neq 0$ for every value $n\geqslant 0$, then (2.4) is a non singular second-order difference equation and the moment u_0, as well as the polynomials ϕ and ψ completely determine the sequence $(u_n)_{n\geqslant 1}$. In this way, the distributional equation is very useful in order to generate the moments $(u_n)_{n\geqslant 0}$, while all the information about $\boldsymbol{u}$ is contained in the pair of polynomials (ϕ,ψ).

Remark 2.7. Notice that the condition $[n]\hat{a}+\hat{b}\neq 0$, for $n\geqslant 0$ is satisfied by every quasi-definite functional. In fact it will be a necessary condition for the quasi-definiteness of a q-classical functional $\boldsymbol{u}$. We will prove it later (see Proposition 2.8 and Remark 2.9). Also notice that if $\phi\equiv 0$, then

$$\phi\equiv 0\Rightarrow \hat{b}u_{n+1}+\bar{b}u_n=0, \quad n\geqslant 0, \tag{2.4}$$

which yields $H_n=0$, for $n\geqslant 2$. This fact is not compatible with the quasi-definiteness of $\boldsymbol{u}$.

2.2. The orthogonality of the sequences of derivatives

In this section, we will prove that our definition of q-classical polynomials, which is exclusively developed in the dual space $\mathbb{P}^*$, is equivalent to the Hahn's one for $q\in(0,\infty)\setminus\{1\}$.

Proposition 2.8. *Let $\boldsymbol{u}\in\mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geqslant 0}=\mathrm{mops}\,\boldsymbol{u}$. Then the following statements are equivalent*:

(a) *$\boldsymbol{u}$ is a q-classical functional* (*see Definition* 2.5), (b) *(ΘP_{n+1}) is an OPS* (*Hahn*).

Moreover, if $\boldsymbol{u}$ satisfies $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$, then $(Q_n)_{n\geqslant 0}=\mathrm{mops}\,\boldsymbol{v}$, where $Q_n=(1/[n+1])\Theta P_{n+1}$ and $\boldsymbol{v}=\phi\boldsymbol{u}$.

Proof. (a) $\Rightarrow$ (b): We start from the monic sequence $(Q_n)_{n\geqslant 0}$. Using Theorem 1.5 we will show that $(Q_n)_{n\geqslant 0}$ is an MOPS associated to $\phi\,\boldsymbol{u}$:

$$\langle\phi\boldsymbol{u},x^mQ_n\rangle=\frac{1}{[n+1]}\langle\phi\boldsymbol{u},x^m\Theta P_{n+1}\rangle.$$

Taking into account that $q^mx^m\Theta P_{n+1}=\Theta(x^mP_{n+1})-[m]x^{m-1}P_{n+1}$, as well as $\deg\phi\leqslant 2$ and $\deg\psi=1$, we get

$$\begin{aligned}\langle\phi\boldsymbol{u},x^mQ_n\rangle&=\frac{1}{q^m[n+1]}(\langle\phi\boldsymbol{u},\Theta(x^mP_{n+1})\rangle-[m]\langle\phi\boldsymbol{u},x^{m-1}P_{n+1}\rangle)\\&=\frac{1}{q^m[n+1]}(-\langle\Theta(\phi\boldsymbol{u}),x^mP_{n+1}\rangle+[m]\langle\phi\boldsymbol{u},\,x^{m-1}P_{n+1}\rangle)\\&=\frac{-1}{q^m[n+1]}(\langle\boldsymbol{u},\,\underbrace{x^m\psi}_{\deg=m+1}\cdot P_{n+1}\rangle+[m]\langle\boldsymbol{u},\underbrace{x^{m-1}\phi}_{\deg\leqslant m+1}\cdot P_{n+1}\rangle)\\&=0\quad\text{if } m+1<n+1\Leftrightarrow m<n, \quad n\geqslant 0.\end{aligned}\tag{2.5}$$

Now we need to check that $\langle \phi\boldsymbol{u}, Q_n^2\rangle \neq 0$, or equivalently, $\langle \phi\boldsymbol{u}, x^nQ_n\rangle \neq 0$, $n\geqslant 0$. In order to do this, we will consider $m=n$ in (2.5). Thus,

$$\langle \phi\boldsymbol{u}, x^nQ_n\rangle = \frac{-1}{q^n[n+1]}\langle \boldsymbol{u}, (x^n\psi + [n]x^{n-1}\phi)P_{n+1}\rangle. \tag{2.6}$$

Thus two situations appear: (1) $\deg\phi < 2$, (2) $\deg\phi = 2$. In the first case

$$\deg\phi < 2 \Rightarrow \deg(x^n\psi + [n]x^{n-1}\phi) = n+1 \Rightarrow \langle \phi\boldsymbol{u}, x^nQ_n\rangle \neq 0, \quad n\geqslant 0.$$

In the second one, if $\phi = \hat{a}x^2 + \bar{a}x + \dot{a}$, $\psi = \hat{b}x + \bar{b}$, $\hat{a} \neq 0 \neq \hat{b}$, then, assuming $[n]\hat{a} + \hat{b} \neq 0$, for every $n\geqslant 0$, from (2.6)

$$\deg(x^n\psi + [n]x^{n-1}\phi) = n+1 \Rightarrow \langle \phi\boldsymbol{u}, x^nQ_n\rangle \neq 0.$$

On the other hand, if there exists $n_0 \geqslant 0$, such that $[n_0]\hat{a} + \hat{b} = 0$, then

$$\langle \boldsymbol{v}, x^nQ_n\rangle \neq 0, \ \ n \neq n_0 \quad \text{and} \quad \langle \boldsymbol{v}, x^{n_0}Q_{n_0}\rangle = 0, \tag{2.7}$$

and $\boldsymbol{v}$ is not quasi-definite. We will show that this fact yields $\boldsymbol{u}$ which is not a quasi-definite functional. Let us consider the polynomial $(\phi Q_n)P_{n+2}$. Since $(Q_n)_{n\geqslant 0}$ is a basis of $\mathbb{P}$, $P_{n+2} = \sum_{i=0}^{n+2} a_{n+2,i}Q_i$. Then,

$$\langle \boldsymbol{u}, (\phi Q_n)P_{n+2}\rangle = \left\langle \phi\boldsymbol{u}, \sum_{i=0}^{n+2} a_{n+2,i}Q_i \cdot Q_n \right\rangle = a_{n+2,n}\langle \phi\boldsymbol{u}, Q_n^2\rangle = a_{n+2,n}\langle \phi\boldsymbol{u}, x^nQ_n\rangle.$$

For $n = n_0$ we get

$$\langle \boldsymbol{u}, (\phi Q_{n_0})P_{n_0+2}\rangle = a_{n_0+2,n_0}\langle \phi\boldsymbol{u}, x^{n_0}Q_{n_0}\rangle = 0. \tag{2.8}$$

On the other hand, ϕQ_{n_0} is a polynomial of degree n_0+2 with leading coefficient $\hat{a} \neq 0$. Thus,

$$\langle \boldsymbol{u}, (\phi Q_{n_0})P_{n_0+2}\rangle = \hat{a}\langle \boldsymbol{u}, x^{n_0+2}P_{n_0+2}\rangle = \hat{a}k_{n_0+2}.$$

So (2.8) leads to $\langle \boldsymbol{u}, P_{n_0+2}^2\rangle = 0$, contradicting the quasi-definiteness of $\boldsymbol{u}$ and the condition that $(P_n)_{n\geqslant 0}$ is an OPS with respect to $\boldsymbol{u}$. So (2.5) holds.

(b) $\Rightarrow$ (a): This is a straightforward consequence of Corollary 2.3 and Proposition 2.4. □

Remark 2.9. From the above proof follows that the condition $[n]\hat{a} + \hat{b} \neq 0$, $n\geqslant 0$ where $\phi = \hat{a}x^2 + \bar{a}x + \dot{a}$, $\hat{a} \neq 0$, $\psi = \hat{b}x + \bar{b}$, $\hat{b} \neq 0$, is a necessary condition for the quasi-definiteness of $\boldsymbol{u}$, satisfying the distributional equation (2.1).

Furthermore, the sequence of q-derivatives $(Q_n)_{n\geqslant 0}$ is also a q-classical sequence since the functional $\boldsymbol{v}$, with respect to which they are orthogonal satisfies a distributional equation of the same type (2.1) (see Definition 2.5). Furthermore,

Lemma 2.10. *Let $\boldsymbol{u} \in \mathbb{P}^*$ and $\phi, \psi \in \mathbb{P}$, $\deg\phi \leqslant 2$ and $\deg\psi = 1$ such that $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$. If $\boldsymbol{v} = \phi\boldsymbol{u}$ then $\Theta(\phi^{(1)}\boldsymbol{v}) = \psi^{(1)}\boldsymbol{v}$ where $\phi^{(1)} = H\phi$ and $\psi^{(1)} = \psi + \Theta\phi$. Moreover, if $\boldsymbol{v}^{(k)} = H^{(k)}\phi \cdot \boldsymbol{u}$, $H^{(k)}\phi := \phi \cdot H\phi \cdot H^2\phi \cdots H^{k-1}\phi = \prod_{i=1}^{k} H^{i-1}\phi$, $k\geqslant 1$, then $\Theta(\phi^{(k)}\boldsymbol{v}^{(k)}) = \psi^{(k)}\boldsymbol{v}^{(k)}$, where*

$$\phi^{(k)} = H^k\phi, \quad \psi^{(k)} = \psi + \Theta(\phi + H\phi + \cdots + H^{k-1}\phi) = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi.$$

Proof. We start from the expression

$$\langle \Theta(\phi^{(1)}\boldsymbol{v}), \pi\rangle = -\langle \boldsymbol{v}, \phi^{(1)}\Theta\pi\rangle, \quad \pi \in \mathbb{P}. \tag{2.9}$$

We want to find polynomials $\phi^{(1)}$ and $\psi^{(1)}$ such that $\Theta(\phi^{(1)}\boldsymbol{v})=\psi^{(1)}\boldsymbol{v}$ holds. In order to do that we will substitute in (2.9) $\phi^{(1)}\Theta\pi=\Theta(\phi^{(1)}\pi)-H\pi\cdot\Theta\phi^{(1)}$, and without loss of generality put $\phi^{(1)}=H\tilde{\phi}$. Thus,

$$\phi^{(1)}\Theta\pi = H\tilde{\phi}\cdot\Theta\pi = \Theta(\tilde{\phi}\pi) - \pi\Theta\tilde{\phi}.$$

Then,

$$\begin{aligned}\langle \Theta(\phi^{(1)}\boldsymbol{v}), \pi\rangle &= -\langle \boldsymbol{v}, \Theta(\tilde{\phi}\pi) - \pi\Theta\tilde{\phi}\rangle = -(\langle \boldsymbol{v}, \Theta(\tilde{\phi}\pi)\rangle + \langle \boldsymbol{v}, \pi\Theta\tilde{\phi}\rangle)\\ &= -(\langle \underbrace{\Theta\boldsymbol{v}}_{\psi\boldsymbol{u}}, \tilde{\phi}\pi\rangle + \langle \Theta\tilde{\phi}\cdot\boldsymbol{v}, \pi\rangle) = -(\langle \tilde{\phi}\boldsymbol{u}, \psi\pi\rangle + \langle \Theta\tilde{\phi}\cdot\boldsymbol{v}, \pi\rangle).\end{aligned}$$

So if we impose that $\tilde{\phi}=\phi$ and $\tilde{\phi}\boldsymbol{u}=\boldsymbol{v}$ the distributional equation for $\boldsymbol{v}$ is

$$\langle \Theta(\phi^{(1)}\boldsymbol{v}), \pi\rangle = \langle \psi\phi\boldsymbol{u}, \pi\rangle + \langle \Theta\phi\cdot\boldsymbol{v}, \pi\rangle = \langle (\underbrace{\psi + \Theta\phi}_{\psi^{(1)}})\boldsymbol{v}, \pi\rangle, \quad \pi \in \mathbb{P},$$

and therefore,

$$\Theta(\phi^{(1)}\boldsymbol{v}) = \psi^{(1)}\boldsymbol{v}, \quad \phi^{(1)} = H\phi \quad \text{and} \quad \psi^{(1)} = \psi + \Theta\phi.$$

The second part of the lemma follows by induction just following an analogous procedure. □

Remark 2.11. The above lemma is the distributional analogue of the hypergeometricity of the q-classical polynomials (see the next section Theorem 2.18).

Theorem 2.12. *Let $\boldsymbol{u} \in \mathbb{P}^*$ be a quasi-definite functional, $(P_n)_{n\geqslant 0} = \text{mops}\,\boldsymbol{u}$ and $Q_n^{(k)} = (1/[n+1]_{(k)})\Theta^k P_{n+k}$. The following statements are equivalent*:

(a) $(P_n)_{n\geqslant 0}$ *is q-classical*, (b) $(Q_n^{(k)})_{n\geqslant 0}$ *is q-classical*, $k\geqslant 1$.

Moreover, if $\boldsymbol{u}$ satisfies the equation $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$, $\deg\phi\leqslant 2$ and $\deg\psi=1$, then $(Q_n^{(k)})$ is orthogonal with respect to $\boldsymbol{v}^{(k)} = H^{(k)}\phi\cdot\boldsymbol{u}$, $H^{(k)} = \prod_{i=1}^{k} H^{i-1}\phi$, and

$$\Theta(\phi^{(k)}\boldsymbol{v}^{(k)}) = \psi^{(k)}\boldsymbol{v}^{(k)}, \quad \deg\phi^{(k)}\leqslant 2 \quad \textit{and} \quad \deg\psi^{(k)} = 1 \quad \textit{where}$$

$$\phi^{(k)} = H^k\phi \quad \textit{and} \quad \psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi.$$

If $\phi = \hat{a}x + \bar{a}x + \dot{a}$ and $\psi = \hat{b}x + \bar{b}$ then

$$\phi^{(k)} = q^{2k}\hat{a}x^2 + q^k\bar{a}x + \dot{a}, \quad \psi^{(k)} = ([2k]\hat{a} + \hat{b})x + ([k]\bar{a} + \bar{b}). \tag{2.10}$$

Proof. The proof is a simple consequence of the previous lemma. Next, since

$$\phi^{(k+1)} = H\phi^{(k)} = H(H\phi^{(k-1)}) = \cdots = H^{k+1}\phi,$$

$$\begin{aligned}\psi^{(k+1)} &= \psi^{(k)} + \Theta\phi^{(k)} = (\psi^{(k-1)} + \Theta\phi^{(k-1)}) + \Theta\phi^{(k)} = \cdots = \psi + \Theta\phi + \Theta\phi^{(1)} + \cdots + \Theta\phi^{(k)}\\ &= \psi + \Theta\phi + \Theta H\phi + \cdots + \Theta H^k\phi = \psi + \Theta\sum_{i=0}^{k} H^i\phi,\end{aligned}$$

then, $\phi^{(k)} = H^k\phi = \hat{a}H^kx^2 + \bar{a}H^kx + \dot{a} = q^{2k}\hat{a}x^2 + q^k\bar{a}x + \dot{a}$, and

$$\psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi = \psi + \sum_{i=0}^{k-1}\Theta H^i\phi = \psi + \sum_{i=0}^{k-1} q^iH^i\Theta\phi = \psi + \sum_{i=0}^{k-1} q^iH^i([2]\hat{a}x + \bar{a})$$

$$= \psi + \sum_{i=0}^{k-1} q^i([2]q^i\hat{a}x + \bar{a}) = \psi + \sum_{i=0}^{k-1} q^{2i}[2]\hat{a}x + \sum_{i=0}^{k-1} q^i\bar{a} = \psi + [2k]\hat{a}x + [k]\bar{a}.$$

Therefore, $\psi^{(k)} = ([2k]\hat{a} + \hat{b})x + ([k]\bar{a} + \bar{b})$. □

Remark 2.13. From the above proposition, we get

$$[n]\hat{a}^{(k)} + \hat{b}^{(k)} = [n]\cdot q^{2k}\hat{a} + ([2k]\hat{a} + \hat{b}) = [2k+n]\hat{a} + \hat{b}.$$

So, the condition $[n]\hat{a}^{(k)} + \hat{b}^{(k)} \neq 0$ for all $k, n \in \mathbb{N}$, for the quasi-definiteness of $\boldsymbol{v}^{(k)}$ follows from the quasi-definiteness condition of $\boldsymbol{u}$.

2.3. The q-Sturm–Liouville operator

In this section we will study another characterization of the q-classical polynomials: They are the unique polynomial eigenfunctions of a certain Sturm–Liouville operator on $\mathbb{P}$.

In the following we will use, among all possible q-analogues of the classical Sturm–Liouville operator $\phi\mathrm{D}^2 + \psi\,\mathrm{D}$, the operator

$$\mathscr{SL} : \mathbb{P} \to \mathbb{P}, \quad \mathscr{SL} := \phi\Theta\Theta^\star + \psi\,\Theta^\star.$$

There are two reasons for this choice. First, when $q \to 1$ the operator $\mathscr{SL}$ becomes the classical one $\phi\mathrm{D}^2 + \psi\mathrm{D}$. Second, $\mathscr{SL}$ involves the same ϕ and ψ as in the distributional equation (2.1).

Lemma 2.14. *Let $\boldsymbol{u} \in \mathbb{P}^*$ and the q-Sturm–Liouville operator $\mathscr{SL} = \phi\Theta\Theta^\star + \psi\Theta^\star$, $\phi, \psi \in \mathbb{P}$. If $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$, then $\langle \phi\boldsymbol{u}, \Theta\pi\Theta\rho\rangle = -\langle \boldsymbol{u}, \pi\mathscr{SL}\rho\rangle$, $\pi, \rho \in \mathbb{P}$.*

Proof. Since $\Theta(\pi\Theta^\star\rho) = \pi\Theta\Theta^\star\rho + \Theta\pi\Theta\rho$, $\pi, \rho \in \mathbb{P}$, then

$$\langle \phi\boldsymbol{u}, \Theta\pi\Theta\rho\rangle = \langle \phi\boldsymbol{u}, \Theta(\pi\Theta^\star\rho) - \pi\Theta\Theta^\star\rho\rangle = -\langle \Theta(\phi\boldsymbol{u}), \pi\Theta^\star\rho\rangle - \langle \phi\boldsymbol{u}, \pi\Theta\Theta^\star\rho\rangle$$

$$\overset{hip.}{=} -(\langle \psi\boldsymbol{u}, \pi\Theta^\star\rho\rangle + \langle \phi\boldsymbol{u}, \pi\Theta\Theta^\star\rho\rangle) = -\langle \boldsymbol{u}, \pi(\phi\Theta\Theta^\star\rho + \psi\Theta^\star\rho)\rangle. \qquad \square$$

This lemma leads to a q-analogue of the Bochner's characterization [8,9].

Proposition 2.15. *Let $\boldsymbol{u} \in \mathbb{P}^*$, be a quasi-definite functional, $(P_n)_{n\geqslant 0} = \operatorname{mops}\boldsymbol{u}$, $\phi, \psi \in \mathbb{P}$, $\deg\phi \leqslant 2, \deg\psi = 1$. Then, the following statements are equivalent:*

(a) *$\boldsymbol{u}$ satisfies the equation*

$$\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u} \tag{2.11}$$

(b) *there exists* $\hat{\lambda}_n \in \mathbb{C}$, $\hat{\lambda}_n \neq 0$, $n \geqslant 1$ *and* $\hat{\lambda}_0 = 0$, *such that*

$$\phi\Theta\Theta^\star P_n + \psi\ \Theta^\star P_n = \hat{\lambda}_n P_n, \quad n = 0, 1, 2, \ldots \ . \tag{2.12}$$

Proof. Let $\boldsymbol{v}:=\phi\boldsymbol{u}$, and let $(Q_n)_{n\geqslant 0}$ be the monic sequence of derivatives, $Q_n := (1/[n+1])\Theta P_{n+1}$.

(a) $\Rightarrow$ (b): First of all, according to the previous lemma

$$\langle \boldsymbol{v}, x^m Q_n\rangle = \frac{1}{[m+1][n+1]}\langle \phi\boldsymbol{u}, \Theta x^{m+1}\Theta P_{n+1}\rangle = -\langle \boldsymbol{u}, x^{m+1}\mathscr{S}\mathscr{L}P_{n+1}\rangle, \quad m, n \geqslant 0.$$

Since, $(Q_n)_{n\geqslant 0} = \text{mops}\,\phi\boldsymbol{u}$ (see Proposition 2.8).

$$\langle \boldsymbol{v}, x^m Q_n\rangle = k'_n\delta_{nm}, \quad k'_n = \langle \boldsymbol{v}, Q_n^2\rangle \neq 0,$$

we get

$$-\langle \boldsymbol{u}, x^{m+1}\mathscr{S}\mathscr{L}P_{n+1}\rangle = k'_n\delta_{nm}, \quad k'_n \neq 0, \ n \geqslant 0. \tag{2.13}$$

Let now $(R_n)_{n\geqslant 0}$ be the sequence of polynomials $R_n = \mathscr{S}\mathscr{L}P_n$, $n \geqslant 1$, $R_0 = 1$. Notice that $[n]\hat{a} + \hat{b} \neq 0$, $n \geqslant 0$, $\deg R_n = \deg \mathscr{S}\mathscr{L}P_n = n$, $n \geqslant 0$ being $(P_n)_{n\geqslant 0} = \text{mops}\,\boldsymbol{u}$. As before, $\hat{a}$ and $\hat{b}$ are the coefficients in x^2 and x of ϕ and ψ, respectively.

Moreover, since $\boldsymbol{u}$ is quasi-definite, $\langle \boldsymbol{u}, 1\rangle = u_0 \neq 0$, and

$$\langle \boldsymbol{u}, \mathscr{S}\mathscr{L}P_n\rangle = \langle \boldsymbol{u}, \phi\Theta\Theta^\star P_n + \psi\Theta^\star P_n\rangle = -\langle \Theta(\phi\boldsymbol{u}) - \psi\boldsymbol{u}, \Theta^\star P_n\rangle = 0,$$

then,

$$\langle \boldsymbol{u}, 1 \cdot R_n\rangle = \langle \boldsymbol{u}, \mathscr{S}\mathscr{L}P_n\rangle = 0, \quad n \geqslant 1. \tag{2.14}$$

Thus, $(R_n)_{n\geqslant 0}$ is a basis sequence and according to Theorem 1.5, they are orthogonal with respect to $\boldsymbol{u}$. Therefore, there exists $\hat{\lambda}_n \in \mathbb{C}$, $\hat{\lambda}_n \neq 0$, such that $R_n = \hat{\lambda}_n P_n$, for $n \geqslant 1$. Furthermore, since $R_n = \mathscr{S}\mathscr{L}P_n$, $n \geqslant 1$, thus $\hat{\lambda}_n \neq 0$ for all $n \geqslant 1$. On the other hand, for $n=0$ the equation $\phi\cdot 0 + \psi\cdot 0 = \hat{\lambda}_0 \cdot 1$ leads to $\hat{\lambda}_0 = 0$.

(b) $\Rightarrow$ (a): To prove this part, we will consider the basis sequence $(Q_n^\star)$, not necessarily orthogonal, defined by

$$Q_n^\star = \frac{1}{[n+1]^\star}\Theta^\star P_{n+1},$$

and we will compute the action of the functional $\Theta(\phi\boldsymbol{u})$ in this basis. Thus,

$$\langle \Theta(\phi\boldsymbol{u}), Q_n^\star\rangle = -\langle \boldsymbol{u}, \phi\Theta Q_n^\star\rangle = \frac{-1}{[n+1]^\star}\langle \boldsymbol{u}, \phi\Theta\Theta^\star P_{n+1}\rangle = \frac{-1}{[n+1]^\star}\langle \boldsymbol{u}, \hat{\lambda}_{n+1}P_{n+1} - \psi\Theta^\star P_{n+1}\rangle$$

$$= \frac{-1}{[n+1]^\star}(\lambda_{n+1}\underbrace{\langle \boldsymbol{u}, P_{n+1}\rangle}_{=0} - \langle \psi\boldsymbol{u}, \Theta^\star P_{n+1}\rangle) = \langle \psi\boldsymbol{u}, Q_n^\star\rangle.$$

So, $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$. □

Remark 2.16. Notice that if $(P_n)_{n\geqslant 0}$ is an MOPS then ψ and ϕ are coprime polynomials, i.e., they have no common roots. In fact if there exists a real number a such that $\phi(a) = \psi(a) = 0$, then from (2.12) we get that $P_n(a) = 0$ for all $n \geqslant 1$. Thus, the TTRR gives $g_1 = 0$ which is a contradiction with the quasi-definiteness of the functional $\boldsymbol{u}$.

Remark 2.17. Both the distributional equation (2.11) and the Sturm–Liouville equation (2.12), characterize a q-classical functional and its corresponding OPS by means of ϕ and ψ. The first one is a differential equation of first order which is *easier* to use than the second one which is of second order. Nevertheless, the Sturm–Liouville equation has the advantage that is an equation in the space of polynomials and combined with the TTRR (1.1) gives an alternative method to prove the quasi-definiteness of the functional instead of the analysis of the Hankel determinants (see Theorem 1.6) as we already pointed out in the previous remark.

Theorem 2.18. *Let* $\mathscr{SL} = \phi\Theta\Theta^\star + \psi\Theta^\star$ *be the q-Sturm–Liouville operator and* ϕ *and* ψ *the polynomials* $\phi=\hat{a}x^2+\bar{a}x+\dot{a}$ *and* $\psi=\hat{b}x+\bar{b}$, *respectively. Then*, $(P_n)_{n\geqslant 0}=\text{mops}\,\boldsymbol{u}$ *are the eigenfunctions of* $\mathscr{SL}$ *corresponding to the eigenvalues* $\hat{\lambda}_n$, *i.e.*,

$$\phi\Theta\Theta^\star P_n + \psi\Theta^\star P_n = \hat{\lambda}_n P_n, \quad n\geqslant 0, \tag{2.15}$$

and they are of the hypergeometric type, i.e., the sequence of their kth order q-derivatives $(Q_n^{(k)})$,

$$Q_n^{(k)} = \frac{\Theta^k P_{n+k}}{[n+1]_{(k)}}, \quad k\geqslant 0,$$

satisfies a second-order difference equation of the same type, namely

$$\phi^{(k)}\Theta\Theta^\star Q_n^{(k)} + \psi^{(k)}\Theta^\star Q_n^{(k)} = \hat{\lambda}_n^{(k)} Q_n^{(k)}, \quad n\geqslant 0,\ k\geqslant 1, \tag{2.16}$$

where $\phi^{(k)} = H^k\phi$ *and* $\psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi$.

Eq. (2.15) is usually called the second-order q-difference equation of hypergeometric type [26].

Proof. This theorem is the analogue of Theorem 2.12 but in $\mathbb{P}$ (see also Proposition 2.15). Here, we will present its proof developed in $\mathbb{P}$.

The first part was already stated in Proposition 2.15. To prove the second part, we apply the operator Θ to $\mathscr{SL}^{(k)}$. So,

$$\begin{aligned}\Theta\mathscr{SL}^{(k)} &= \Theta\phi^{(k)}\cdot\Theta\Theta^\star + H\phi^{(k)}\cdot\Theta\Theta\Theta^\star + \Theta\psi^{(k)}\cdot H\Theta^\star + \psi^{(k)}\cdot\Theta\Theta^\star \\ &= q^{-1}(\Theta\phi^{(k)}\cdot\Theta^\star + H\phi^{(k)}\cdot\Theta\Theta^\star + \psi^{(k)}\Theta^\star)\Theta + \Theta\psi^{(k)}\cdot\Theta \\ &= q^{-1}(H\phi^{(k)}\cdot\Theta\Theta^\star + (\psi^{(k)} + \Theta\phi^{(k)})\Theta^\star)\Theta + \Theta\psi^{(k)}\cdot\Theta.\end{aligned}$$

Since the statement is valid for $k=0$, and if we suppose that it is valid for some k, i.e.,

$$\mathscr{SL}^{(k)}Q_n^{(k)} = \hat{\lambda}_n^{(k)} Q_n^{(k)}, \quad n\geqslant 0 \quad \text{with } \phi^{(k)} = H^k\phi \quad \text{and} \quad \psi^{(k)} = \psi + \Theta\sum_{i=0}^{k-1} H^i\phi,$$

then, applying in the above expression the operator Θ we find

$$\begin{aligned}&q^{-1}(H\phi^{(k)}\cdot\Theta\Theta^\star + (\psi^{(k)} + \Theta\phi^{(k)})\Theta^\star)\Theta Q_n^{(k)} + \Theta\psi^{(k)}\cdot\Theta Q_n^{(k)} = \hat{\lambda}_n^{(k)}\Theta Q_n^{(k)} \\ &\Leftrightarrow q^{-1}(\phi^{(k+1)}\Theta\Theta^\star + \psi^{(k+1)}\Theta^\star)\Theta Q_n^{(k)} = (\hat{\lambda}_n^{(k)} - \Theta\psi^{(k)})\Theta Q_n^{(k)} \\ &\Leftrightarrow \mathscr{SL}^{(k+1)}Q_{n-1}^{(k+1)} = q(\hat{\lambda}_n^{(k)} - \Theta\psi^{(k)})Q_{n-1}^{(k+1)}.\end{aligned}$$

Thus $(Q_n^{(k+1)})$ are the eigenfunctions of $\mathscr{S}\mathscr{L}^{(k+1)}$, and then the result follows for $k+1$. Notice that the polynomials $\phi^{(k)}$ and $\psi^{(k)}$ are those of the distributional equation $\Theta(\phi^{(k)}\boldsymbol{v}^{(k)})=\psi^{(k)}\boldsymbol{v}^{(k)}$ of Theorem 2.12. □

Remark 2.19. From Proposition 2.2 it follows that the condition of $\phi^{(k)}$ and $\psi^{(k)}$ to be coprime polynomials is a necessary condition for the quasi-definiteness of the functional. Moreover, this condition together with the condition $\hat{a}[n]+\hat{b}\neq 0$ for all $n\geqslant 0$ is also a sufficient condition for the quasi-definiteness of $\boldsymbol{u}$ (see Appendix A).

Proposition 2.20. *Let* $\mathscr{S}\mathscr{L}=\phi\Theta\Theta^\star+\psi\Theta^\star$, $\phi,\psi\in\mathbb{P}$ *be the q-Sturm–Liouville operator. Let* $(B_n)_{n\geqslant 0}$ *be a basis sequence of eigenvectors of* $\mathscr{S}\mathscr{L}$ *and* $(\boldsymbol{b}_n)_{n\geqslant 0}$ *the dual basis of* $(B_n)_{n\geqslant 0}$, *i.e.*, $\langle \boldsymbol{b}_n, B_m\rangle=\delta_{nm}$. *Then, the functional* $\boldsymbol{u}=c\boldsymbol{b}_0$, $c\in\mathbb{C}$, *satisfies the equation*

$$\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}.$$

Proof. Since $(B_n)_{n\geqslant 0}$ is a basis sequence on $\mathbb{P}$, then $(Q_n^\star)_{n\geqslant 0}$ where $Q_n^\star=\Theta^\star B_{n+1}$ is also a basis of $\mathbb{P}$. Thus,

$$\mathscr{S}\mathscr{L}B_n=\hat{\lambda}_n B_n \;\Leftrightarrow\; \phi\Theta\Theta^\star B_n+\psi\Theta^\star B_n=\hat{\lambda}_n B_n \;\Leftrightarrow\; \phi\Theta Q_{n-1}^\star+\psi Q_{n-1}^\star=\hat{\lambda}_n B_n,\quad n\geqslant 1.$$

Next,

$$\langle \boldsymbol{u},\mathscr{S}\mathscr{L}B_n\rangle=\langle \boldsymbol{u},\phi\Theta Q_{n-1}^\star+\psi Q_{n-1}^\star\rangle=\langle -\Theta(\phi\boldsymbol{u})+\psi\boldsymbol{u},Q_{n-1}^\star\rangle,\quad n\geqslant 1,$$

and

$$\langle \boldsymbol{u},\lambda_n B_n\rangle=\lambda_n\langle c\boldsymbol{b}_0,B_n\rangle=\lambda_n c\delta_{0n}=0,\quad n\geqslant 1.$$

Therefore, $\langle -\Theta(\phi\boldsymbol{u})+\psi\boldsymbol{u},Q_{n-1}^\star\rangle=0$ $n\geqslant 1\Leftrightarrow\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$. □

Remark 2.21. Notice that from the above proposition and Theorem 2.18 the only polynomial solutions which are orthogonal with respect to a quasi-definite q-classical functional $\boldsymbol{u}$ are the solutions of the hypergeometric-type difference equation (2.15).

The following proposition is very useful for the complete classification of the q-classical polynomials [24,25].

Proposition 2.22. *Let* ϕ, $\phi^\star$ *and* $\psi\in\mathbb{P}$, $\deg\phi\leqslant 2$, $\deg\phi^\star\leqslant 2$, $\deg\psi=1$, *such that* $\phi^\star=q^{-1}\phi+(q^{-1}-1)x\psi$, *or equivalently*, $\phi=q\phi^\star+(q-1)x\psi$. *Then, the following statements are equivalent*

(a) $\phi\Theta\Theta^\star\pi+\psi\Theta^\star\pi=\hat{\lambda}_n\pi$, $\forall\pi\in\mathbb{P}$,
(b) $q^{-1}\phi\cdot H\pi-(q^{-1}\phi+q\phi^\star)\pi+q\phi^\star\cdot H^{-1}\pi=(q-1)(1-q^{-1})x^2\lambda_n\pi$, $\forall\pi\in\mathbb{P}$,
(c) $\phi^\star\Theta^\star\Theta\pi+\psi\Theta\pi=\hat{\lambda}_n\pi$, $\forall\pi\in\mathbb{P}$.

Proof. To prove the equivalence of (a) and (b), notice that

$$\phi\Theta\Theta^\star\pi+\psi\Theta^\star\pi=\hat{\lambda}_n\pi \quad\Leftrightarrow\quad \phi\Theta\frac{H^{-1}\pi-\pi}{(q^{-1}-1)x}+\psi\frac{H^{-1}\pi-\pi}{(q^{-1}-1)x}=\hat{\lambda}_n\pi$$

$$\Leftrightarrow \phi\frac{H\pi-(1+q)\pi+qH^{-1}\pi}{(q-1)^2x^2}+\psi\frac{q(1-q)x(H^{-1}\pi-\pi)}{(q-1)^2x^2}=\hat{\lambda}_n\pi$$

$$\Leftrightarrow \phi H\pi-(\underbrace{(1+q)\phi+q(1-q)x\psi}_{\phi+q^2\phi^\star})\pi+(\underbrace{q\phi+q(1-q)x\psi}_{q^2\phi^\star})H^{-1}\pi=(q-1)^2x^2\hat{\lambda}_n\pi. \tag{2.17}$$

Multiplying the last expression by q^{-1} the equivalence (a) $\Leftrightarrow$ (b) follows. The other equivalence (c) $\Leftrightarrow$ (b) can be obtained in an analogous way. □

As an immediate consequence of Propositions 2.15 and 2.22 we have

Proposition 2.23. *Let* $\boldsymbol{u}\in\mathbb{P}^*$ *be a quasi-definite functional*, $(P_n)_{n\geqslant 0}=\operatorname{mops}\boldsymbol{u}$, ϕ, $\phi^\star$, $\psi\in\mathbb{P}$, *such that* $\phi^\star=q^{-1}\phi+(q^{-1}-1)x\psi$, $\deg\phi\leqslant 2$, $\deg\phi^\star\leqslant 2$ *and* $\deg\psi=1$. *Then, the following statements are equivalent*:

(a) $(P_n)_{n\geqslant 0}=\operatorname{mops}\boldsymbol{u}$ *is q-classical and* $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$,
(b) $(P_n)_{n\geqslant 0}=\operatorname{mops}\boldsymbol{u}$ *is* q^{-1}*-classical and* $\Theta^\star(\phi^\star\boldsymbol{u})=\psi\boldsymbol{u}$.

Remark 2.24. The above proposition means that all q-classical polynomials are also q^{-1} classical and vice versa. There also exists a very simple distributional proof of this equivalence between q and q^{-1} classical functionals and their corresponding monic OPS.

2.4. Structure relations and other characterizations

In 1972, Al-Salam and Chihara [2] proved that the relation, called structure relation (STR),

$$\phi\mathrm{D}P_n=a_nP_{n+1}+b_nP_n+c_nP_{n-1},\quad \deg\phi\leqslant 2,\ c_n\neq 0,\ n\geqslant 1,$$

characterizes the classical OPS. One remarkable consequence of this characterization is that, independently of the degree of the polynomial P_n, the product $\phi\mathrm{D}P_n$ can be represented as a linear combination of three consecutive polynomials. Later on, Marcellán et al. [22], proved that a similar relation involving three consecutive monic derivatives Q_n,

$$P_n=Q_n+e_nQ_{n-1}+h_nQ_{n-2},\quad n\geqslant 2\ .$$

also characterizes the classical MOPS. This second relation will be also considered as a structure relation. Finally, there is also a very useful characterization of classical polynomials, the so-called Cryer's characterization of the D-classical polynomials [10]. Here, we will give the q-analogue of the distributional Rodrigues formula obtained by Marcellán et al. [22].

Next, we are going to prove that the q-analogue of these two structure relations characterizes our q-classical polynomials.

Proposition 2.25. *Let* $\boldsymbol{u} \in \mathbb{P}^*$ *be a quasi-definite functional,* $(P_n)_{n\geqslant 0} = \operatorname{mops} \boldsymbol{u}$ *and* $Q_n = (1/[n+1])\Theta P_{n+1}$. *Then, the next three statements are equivalent*:

(a) *There exist two polynomials* $\phi, \psi \in \mathbb{P}$, $\deg \phi \leqslant 2$ *and* $\deg \psi = 1$ *such that* $\Theta(\phi \boldsymbol{u}) = \psi \boldsymbol{u}$.

(b) *There exist a polynomial* $\phi \in \mathbb{P}$, $\deg \phi \leqslant 2$ *and three sequences of complex numbers* a_n, b_n, c_n, $c_n \neq 0$, *such that*

$$\phi \Theta P_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}, \quad n \geqslant 1; \tag{2.18}$$

(c) *There exist complex numbers* e_n, h_n, *such that*

$$P_n = Q_n + e_n Q_{n-1} + h_n Q_{n-2}, \quad n \geqslant 2. \tag{2.19}$$

Proof. We will prove the equivalences (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (b): Since $\deg \phi \Theta P_n \leqslant n+1$, the polynomial $\phi\Theta P_n$ can be expanded in the basis $(P_n)_{n\geqslant 0}$

$$\phi \Theta P_n = \sum_{i=0}^{n+1} a_{ni} P_i, \quad a_{ni} = k_i^{-1} \langle \boldsymbol{u}, \phi \Theta P_n \cdot P_i \rangle, \quad k_i = \langle \boldsymbol{u}, P_i^2 \rangle \neq 0.$$

Furthermore,

$$a_{ni} = k_i^{-1} \langle \boldsymbol{u}, \phi \Theta P_n \cdot P_i \rangle = [n] k_i^{-1} \langle \phi \boldsymbol{u}, P_i Q_{n-1} \rangle = [n] k_i k'_{n-1} \delta_{n-1,i}, \quad k'_{n-1} = \langle \phi \boldsymbol{u}, Q_{n-1}^2 \rangle \neq 0.$$

Thus, for any $i < n-1$, $a_{ni} = 0$ while $a_{n,n-1} \neq 0$. Here we have used the fact that $(P_n)_{n\geqslant 0}$ is q-classical so $(Q_n)_{n\geqslant 0} = \operatorname{mops} \boldsymbol{u}$.

(b) $\Rightarrow$ (a): Let us represent the functional $\Theta(\phi\boldsymbol{u})$ in the dual basis $(\boldsymbol{p}_n)_{n\geqslant 0}$ of $(P_n)_{n\geqslant 0}$. Then,

$$\langle \Theta(\phi \boldsymbol{u}), P_n \rangle = -\langle \boldsymbol{u}, \phi \Theta P_n \rangle = -a_n \langle \boldsymbol{u}, P_{n+1} \rangle - b_n \langle \boldsymbol{u}, P_n \rangle - c_n \langle \boldsymbol{u}, P_{n-1} \rangle$$

$$= \begin{cases} 0 & \text{if } n-1 > 0 \Leftrightarrow n > 1, \\ \neq 0 & \text{if } n-1 = 0 \Leftrightarrow n = 1 \quad (c_n \neq 0). \end{cases}$$

Now, using $\psi = \sum_{i=0}^{1} \langle \Theta(\phi\boldsymbol{u}), P_i \rangle k_i^{-1} P_i$, $\deg \psi = 1$, we obtain

$$\Theta(\phi \boldsymbol{u}) = \sum_{n\geqslant 0} \langle \Theta(\phi \boldsymbol{u}), P_n \rangle \boldsymbol{p}_n = \sum_{i=0}^{1} \langle \Theta(\phi \boldsymbol{u}), P_i \rangle \boldsymbol{p}_i = \sum_{i=0}^{1} \langle \Theta(\phi \boldsymbol{u}), P_i \rangle k_i^{-1} P_i \boldsymbol{u} = \psi \boldsymbol{u}.$$

(a) $\Rightarrow$ (c): Let now represent the polynomials P_n in the basis $(Q_n)_{n\geqslant 0}$ which is, by hypothesis, orthogonal with respect to $\phi\boldsymbol{u}$. Since $\deg\phi \leqslant 2$, we get

$$P_n = Q_n + \sum_{i=0}^{n-1} b_{n,i} Q_i, \quad b_{n,i} = k_i'^{-1} \langle \phi \boldsymbol{u}, P_n Q_i \rangle, \quad k'_i = \langle \phi \boldsymbol{u}, Q_i^2 \rangle \neq 0,$$

and

$$\langle \phi \boldsymbol{u}, P_n Q_i \rangle = \langle \boldsymbol{u}, P_n \cdot \phi Q_i \rangle = 0, \quad \forall i = 0, 1, \ldots, n-3.$$

(c) $\Rightarrow$ (a): Finally, since $(Q_n)_{n\geqslant 0}$ is a basis, for its dual basis $(\boldsymbol{q}_n)_{n\geqslant 0}$ we get

$$\boldsymbol{q}_0 = \sum_{n\geqslant 0} \langle \boldsymbol{q}_0, P_n \rangle \boldsymbol{p}_n.$$

Therefore, using (2.19)

$$\langle \boldsymbol{q}_0, P_n \rangle = \langle \boldsymbol{q}_0, Q_n \rangle + e_n \langle \boldsymbol{q}_0, Q_{n-1} \rangle + h_n \langle \boldsymbol{q}_0, Q_{n-2} \rangle = 0, \quad n \geqslant 3,$$

and, as a consequence,

$$\boldsymbol{q}_0=\sum_{i=0}^{2}\langle \boldsymbol{q}_0,P_i\rangle \boldsymbol{p}_i=\sum_{i=0}^{2}\langle \boldsymbol{q}_0,P_i\rangle k_i^{-1}P_i\boldsymbol{u}=\phi\boldsymbol{u},\quad k_i=\langle \boldsymbol{u},P_i^2\rangle\neq 0. \tag{2.20}$$

On the other hand, taking into account Proposition 2.2, as well as $\boldsymbol{u}=u_0\boldsymbol{p}_0, \boldsymbol{v}=v_0\boldsymbol{q}_0$, we have

$$\Theta\boldsymbol{q}_0=-[1]\boldsymbol{p}_1=-k_1^{-1}P_1\boldsymbol{u}=\psi\boldsymbol{u}\ \Leftrightarrow\ \Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u},\quad \deg\psi=1.$$

$$\phi=\sum_{i=0}^{2}\langle \boldsymbol{q}_0,P_i\rangle k_i^{-1}P_i,\quad \deg\phi\leqslant 2.\qquad\square$$

Next, we will prove the q-analogue of the distributional Rodrigues formula.

Proposition 2.26. *Let $\boldsymbol{u}\in\mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geqslant 0}=\operatorname{mops}\boldsymbol{u}$. Then, the following statements are equivalent*:

(a) *There exist two polynomials $\phi,\psi\in\mathbb{P}$, $\deg\phi\leqslant 2$, $\deg\psi=1$ such that $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$.*

(b) *There exist a polynomial $\phi\in\mathbb{P}$, $\deg\phi\leqslant 2$ and a sequence of complex numbers r_n, $r_n\neq 0$, $n\geqslant 1$ such that*

$$P_n\boldsymbol{u}=r_n\Theta^n(H^{(n)}\phi\cdot\boldsymbol{u}),\quad n\geqslant 1\quad \text{where } H^{(n)}\phi=\prod_{i=1}^{n}H^{i-1}\phi, \tag{2.21}$$

Proof. (a) $\Rightarrow$ (b): Keeping in mind that, by hypothesis

$$\prod_{i=0}^{k-1}H^i\phi\cdot\boldsymbol{u}=\boldsymbol{v}^{(k)}\quad\text{and}\quad (Q_n^{(k)})=\operatorname{mops}\boldsymbol{v}^{(k)},\quad Q_n^{(k)}=\frac{1}{[n+1]_{(k)}}\Theta^kP_{n+k},\quad n,k\geqslant 0,$$

and writing $\Theta^k\boldsymbol{v}^{(k)}$ in terms of the dual basis of $(P_n)_{n\geqslant 0}$, the coefficients of this expansion vanish up to one of them, i.e.,

$$\langle\Theta^k\boldsymbol{v}^{(k)},P_i\rangle=(-1)^k\langle\boldsymbol{v}^{(k)},\Theta^kP_i\rangle=(-1)^k[i-k+1]_{(k)}\langle\boldsymbol{v}^{(k)},Q_{i-k}^{(k)}\rangle=0\quad\text{if } i\neq k.$$

Therefore, $\Theta^k\boldsymbol{v}^{(k)}$ and $\boldsymbol{p}_k$ differ on a nonzero constant factor. From Proposition 2.1 $\boldsymbol{p}_n$ is, up to a factor, $P_k\boldsymbol{u}$ which concludes the proof.

(b) $\Rightarrow$ (a): Putting $k=1$ in (2.21) the result immediately follows. $\square$

Notice that there are other characterizations of the q-classical polynomials. The proof of the following theorem will be done in a forthcoming paper in the framework of q-semiclassical and q-Laguerre–Hahn polynomials [24].

Proposition 2.27. *Let $\boldsymbol{u}\in\mathbb{P}^*$ be a quasi-definite functional and $(P_n)_{n\geqslant 0}=\operatorname{mops}\boldsymbol{u}$. Then, the following statements are equivalent*:

(a) *There exist two polynomials $\phi,\psi\in\mathbb{P},\deg\phi\leqslant 2,\deg\psi=1$, such that $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$.*

(b) *There exist two polynomials ϕ and χ, $\deg\phi\leqslant 2$, $\deg\chi=1$, and a complex number ξ such that $\phi\Theta S_{\boldsymbol{u}}=\chi S_{\boldsymbol{u}}+\xi$, where $S_{\boldsymbol{u}}$ denotes the Stieltjes formal series corresponding to the functional $\boldsymbol{u}$, i.e., $S_{\boldsymbol{u}}(z)=-\sum_{n\geqslant 0}u_n/z^{n+1}$. Furthermore, $\chi=qH\psi-\Theta\phi$ and $\xi=u_0(q\hat{b}-\hat{a})$.*

(c) *There exist a polynomial* $\phi \in \mathbb{P}$, $\deg \phi \leqslant 2$, *two sequences of complex numbers* o_n, s_n *and two polynomial sequences* $\pi_n, \rho_n \in \mathbb{P}$, $\deg \pi_n \leqslant 1 \geqslant \deg \rho_n$, *such that*, *for all* $n \geqslant 1$,

$$\phi(P_n \Theta P_{n-1} - P_{n-1}\Theta P_n) = o_n P_n H P_n + \pi_n P_{n-1} H P_n + \rho_n P_n H P_{n-1} + s_n P_{n-1} H P_{n-1}.$$

The expression $P_n \Theta P_{n-1} - P_{n-1}\Theta P_n$ is usually called the q-Wronskian of P_n and P_{n-1}

$$W(P_n, P_{n-1}) = \det\begin{pmatrix} P_n & P_{n-1} \\ \Theta P_n & \Theta P_{n-1} \end{pmatrix} = P_n \Theta P_{n-1} - P_{n-1}\Theta P_n.$$

Notice that, if we define the rational function $f_n = -P_n/P_{n-1}$, we have

$$\Theta f_n = -\Theta \frac{P_n}{P_{n-1}} = -\frac{P_n \Theta P_{n-1} - P_{n-1}\Theta P_n}{P_{n-1} H P_{n-1}}.$$

Then, dividing the equation in Proposition 2.27 by $P_{n-1}HP_{n-1}$, we obtain

$$\phi \Theta f_n = o_n f_n H f_n + (-\pi_n) H f_n + (-\rho_n) f_n + s_n, \quad n \geqslant 1.$$

The above equation is a q-Riccati equation. Moreover, it is the same equation that the Stieltjes series $S_{\boldsymbol{u}^{(k)}}$ satisfies [24], where $\boldsymbol{u}^{(k)}$ is the functional with respect to which the associated polynomials of order k are orthogonal.

3. The main characteristics of the q-classical polynomials in terms of the coefficients of ϕ and ψ

In this section we will compute all the coefficients which appear in the characterizations of the q-classical polynomials given in the previous section in terms of the coefficients of the polynomials ϕ and ψ of the distributional equation. In fact we will give an explicit representation for the eigenvalues $\hat{\lambda}_n$ of the q-Sturm–Liouville operator (2.12) as well as for the values $\hat{\lambda}_n^{(k)}$ in the q-Sturm–Liouville equation for the derivatives (2.16). From these expressions we will obtain an extra information as well as an expression for the coefficient r_n of the distributional q-analogue of the Rodrigues formula (2.21). In fact, r_n is the *Fourier* coefficient of the functional $\Theta^n(H^{(n)}\phi \cdot \boldsymbol{u})$ in $(P_n\boldsymbol{u})_{n\geqslant 0}$, the dual basis of $(P_n)_{n\geqslant 0}$ (see Proposition 2.1).

After that, we will determine all the coefficients in the three-term recurrence relation for $(P_n)_{n\geqslant 0}$ (1.1)

$$xP_n = P_{n+1} + d_n P_n + g_n P_{n-1}, \quad P_{-1} = 0, P_0 = 1, \quad n \geqslant 0, \tag{3.1}$$

the structure relations (2.18)

$$\phi \Theta P_n = a_n P_{n+1} + b_n P_n + c_n P_{n-1}, \quad n \geqslant 1, \tag{3.2}$$

and (2.19)

$$P_n = Q_n + e_n Q_{n-1} + h_n Q_{n-2}, \quad n \geqslant 2, \tag{3.3}$$

as well as the coefficients of the three-term for the their monic derivatives $(Q_n)_{n\geqslant 0}$

$$xQ_n = Q_{n+1} + d'_n Q_n + g'_n Q_{n-1}. \tag{3.4}$$

There are two methods for finding all of them. The first one, is by comparison of the coefficients in (3.2), (3.3) and (3.4). These calculations are straightforward, but cumbersome, so it requires the use of a powerful symbolic algorithm. We have used *Mathematica* 3.0 [31] for finding them. The other method is based in several relation among all these coefficients from their Fourier coefficients with respect to an appropriate basis of $\mathbb{P}$. Sometimes this procedure is very straightforward (like for the structure relation II (3.3)), but usually it gives a lot of different relations and the method becomes dark itself. Nevertheless this method gives many interesting relations between the aforesaid coefficients.

For example, since

$$d_n = \frac{\langle \boldsymbol{u}, xP_n^2\rangle}{\langle \boldsymbol{u}, P_n^2\rangle}, \quad n \geqslant 0, \quad \text{and} \quad g_n = \frac{\langle \boldsymbol{u}, P_n^2\rangle}{\langle \boldsymbol{u}, P_{n-1}^2\rangle}, \quad n \geqslant 1, \tag{3.5}$$

using the fact that the polynomials P_n are monic, we have $g_n = k_n/k_{n-1}$, where as before $k_n = \langle \boldsymbol{u}, P_n^2\rangle$. Thus

$$k_n = \prod_{j=0}^{n} g_j, \quad g_0 := k_0 = u_0, \quad n \geqslant 1. \tag{3.6}$$

For the structure relation (3.2), the Fourier coefficients of $\phi\Theta P_n$ in the basis $(P_n)_{n\geqslant 0}$ are

$$a_n = \frac{\langle \boldsymbol{u}, \phi\Theta P_n \cdot P_{n+1}\rangle}{\langle \boldsymbol{u}, P_{n+1}^2\rangle}, \quad b_n = \frac{\langle \boldsymbol{u}, \phi\Theta P_n \cdot P_n\rangle}{\langle \boldsymbol{u}, P_n^2\rangle}, \quad c_n = \frac{\langle \boldsymbol{u}, \phi\Theta P_n \cdot P_{n-1}\rangle}{\langle \boldsymbol{u}, P_{n-1}^2\rangle}. \tag{3.7}$$

Then $a_n = \hat{a}[n]$.

In structure relation (3.3) we get

$$e_n = \frac{\langle \boldsymbol{v}, P_n \cdot Q_{n-1}\rangle}{\langle \boldsymbol{v}, Q_{n-1}^2\rangle}, \quad h_n = \frac{\langle \boldsymbol{v}, P_n \cdot Q_{n-2}\rangle}{\langle \boldsymbol{v}, Q_{n-2}^2\rangle}. \tag{3.8}$$

Finally, notice that the eigenvalue $\hat{\lambda}_n$ is the nth Fourier coefficient of the polynomial $\mathscr{S}\mathscr{L}P_n$ in the basis $(P_n)_{n\geqslant 0}$. So,

$$\hat{\lambda}_n = \frac{\langle \boldsymbol{u}, \mathscr{S}\mathscr{L}P_n \cdot P_n\rangle}{\langle \boldsymbol{u}, P_n^2\rangle} = -k_n^{-1}\langle \phi\boldsymbol{u}, \Theta P_n \Theta P_n\rangle = -k_n^{-1}[n]^2\langle \boldsymbol{v}, Q_{n-1}^2\rangle = -k_n^{-1}[n]^2 k'_{n-1}.$$

The relation

$$k_n = -[n]^2 k'_{n-1}\hat{\lambda}_n^{-1}, \quad n \geqslant 1, \tag{3.9}$$

is used in [17] to obtain g_n (see Appendix B). Also it can be used for finding k_n. In fact

$$k_n = -\frac{[n]^2}{\hat{\lambda}_n} k'_{n-1} = (-1)^n \frac{[n]^2}{\hat{\lambda}_n}\frac{[n-1]^2}{\hat{\lambda}_{n-1}^{(1)}} \cdots \frac{[1]^2}{\hat{\lambda}_1^{(n-1)}} k_0^{(n)}, \quad k_0^{(n)} = \langle \boldsymbol{v}^{(k)}, 1\rangle = \langle \boldsymbol{u}, H^{(n)}\phi\rangle, \tag{3.10}$$

which is an alternative expression for k_n (3.6).

3.1. The coefficients of the q-Sturm–Liouville equation and the q-Rodrigues formula

Here, we will provide a more careful study of the q-Sturm–Liouville equation that the q-classical polynomials satisfy. To obtain the explicit expression for $\hat{\lambda}_n$ we compare the coefficients of x^n in the q-Sturm–Liouville equation $\mathscr{S}\mathscr{L}P_n = \hat{\lambda}_n P_n$. This yields the expression

$$\hat{\lambda}_n = [n]^\star([n-1]\hat{a} + \hat{b}) \neq 0, \quad n \geqslant 1, \quad \hat{\lambda}_0 = 0. \tag{3.11}$$

Furthermore, the eigenvalues of the q-Sturm–Liouville operator for the kth derivatives Q_n of the q-classical polynomials are given by

$$\hat{\lambda}_n^{(k)} = [n]^\star([2k+n-1]\hat{a}+\hat{b}). \tag{3.12}$$

To prove this, it is sufficient to use the expression $\hat{\lambda}_n^{(k)} = [n]^\star([n-1]\hat{a}^{(k)} - \hat{b}^{(k)})$, where (see Theorem 2.12) $\hat{a}^{(k)} = q^{2k}$ and $\hat{b}^{(k)} = [2k]\hat{a}+\hat{b}$.

Remark 3.1. Theorem 2.18 gives also an alternative algorithm for finding an explicit expression of the eigenvalues $\hat{\lambda}_n$. In fact, applying the same procedure as before it is easy to show that the polynomials $P_n^{(k)} = \Theta^k P_n$ satisfy the same Eq. (2.16) but with the eigenvalues $\mu_n^{(k)}$ given by

$$\mu_n^{(k)} = q(\mu_n^{(k-1)} - \Theta\phi^{(k-1)}).$$

Therefore, $\mu_n^{(k)} = q^k\{\hat{\lambda}_n - [k]^\star([k-1]\hat{a}+\hat{b})\}$. But now, since P_n is a polynomial, then $\Theta^n P_n = \text{const}$, Eq. (2.16) yields the condition $\mu_n^{(n)} = 0$, which leads to the same expression for the eigenvalues $\hat{\lambda}_n$. The condition $\mu_n^{(n)} = 0$ is usually called the *hypergeometric condition* [26].

Proposition 3.2. *Let $\mathscr{SL} = \phi\Theta\Theta^\star + \psi\Theta$ be the q-Sturm–Liouville operator where $\phi = \hat{a}x^2 + \bar{a}x + \dot{a}$ and $\psi = \hat{b}x + \bar{b}$. If $\hat{\lambda}_n$ are the eigenvalues corresponding to a basis sequence of eigenfunctions of $\mathscr{SL}$, then*

$$[n]\hat{a}+\hat{b} \neq 0, \quad n \geqslant 0 \quad \Leftrightarrow \quad \hat{\lambda}_n \neq \hat{\lambda}_m, \quad n \neq m, \ n,m \geqslant 0.$$

In other words, the condition $\hat{\lambda}_n \neq \hat{\lambda}_m$ about the eigenvalues is equivalent to the necessary condition for the quasi-definiteness of $\boldsymbol{u}$.

Proof. The proof is based on the fact that

$$\hat{\lambda}_n - \hat{\lambda}_m = \frac{[n-m]}{q^{n-1}}([n+m-1]\hat{a}+\hat{b}), \quad n,m \geqslant 0.$$

Then $\hat{\lambda}_n \neq \hat{\lambda}_m$ for all $n \neq m$ if and only if $[k]\hat{a}+\hat{b} \neq 0$, for all $k \geqslant 0$. □

Starting from the expression for the eigenvalues of the operator $\mathscr{SL}^{(k)}$ (3.12) we get the coefficients r_n in the q-distributional Rodrigues formula (2.21) for the P_n. In fact,

$$k_n = \langle P_n\boldsymbol{u}, P_n\rangle = \langle r_n\Theta^n H^{(n)}\phi\cdot\boldsymbol{u}, P_n\rangle = (-1)^n r_n\langle H^{(n)}\phi\cdot\boldsymbol{u}, \Theta^n P_n\rangle = (-1)^n r_n [n]! k_0^{(n)},$$

where $k_0^{(n)}$ is given in (3.10). Thus,

$$k_0^{(n)} = (-1)^n k_n \prod_{i=1}^{n} \frac{\hat{\lambda}_i^{(n-i)}}{[i]^2} = \frac{(-1)^n}{q^{\binom{n}{2}}[n]!} k_n \prod_{i=1}^{n}([2n-i-1]\hat{a}+\hat{b}).$$

For monic sequences we deduce

$$r_n = q^{\binom{n}{2}} \prod_{i=1}^{n}([2n-i-1]\hat{a}+\hat{b})^{-1}.$$

Remark 3.3. The previous algorithm can be used in the representation of $k_l^{(n)}$ as a function of k_{n+l}. So,

$$k_l^{(n)} = (-1)^l k_{n+l} \prod_{i=1}^{n} \frac{\hat{\lambda}_{k+i}^{(n-i)}}{[l+i]^2} = \frac{(-1)^l k_{n+l}}{q^{nl+\binom{n}{2}}[l+1]_{(n)}} \prod_{i=1}^{n}([2(n-i)+(l+i)-1]\hat{a}+\hat{b}).$$

3.2. The coefficients of the TTRR

In this section we are going to compute the coefficients of TTRR (3.1). We will use the following notation:

$$P_n = \sum_{i=0}^{n} a_{ni}x^i, \quad p_n{:=}a_{n,n} = 1, \ n\geqslant 0, \quad s_n{:=}a_{n,n-1}, \ n\geqslant 1, \ t_n{:=}a_{n,n-2}, \ n\geqslant 2.$$

First of all, comparing the coefficients in the TTRR, for the coefficient of x^{n-1} we get

$$s_n = s_{n+1} + d_n \ \Leftrightarrow \ d_n = s_n - s_{n+1}, \quad n\geqslant 1, \tag{3.13}$$

and for the coefficient of x^{n-2}

$$t_n = t_{n+1} + d_n s_n + g_n \ \Leftrightarrow \ g_n = (t_n - t_{n+1}) - s_n(s_n - s_{n+1}), \quad n\geqslant 2. \tag{3.14}$$

The above expression is also true for $n{=}1$ putting $t_1{=}0$. On the other hand, comparing the coefficients in the q-Sturm–Liouville equation we can express the second and third coefficients of P_n in terms of the coefficients of ϕ and ψ. Indeed, if

$$\hat{\lambda}_n = [n]^{\star}([n-1]\hat{a}+\hat{b}), \quad \bar{\lambda}_n = [n]^{\star}([n-1]\bar{a}+\bar{b}), \quad \dot{\lambda}_n = [n]^{\star}[n-1]\dot{a}, \quad n\geqslant 0,$$

then

$$s_n = \frac{\bar{\lambda}_n}{\hat{\lambda}_n - \hat{\lambda}_{n-1}}, \quad n\geqslant 1, \qquad t_n = \frac{\bar{\lambda}_n\bar{\lambda}_{n-1} + \dot{\lambda}_n(\hat{\lambda}_n - \hat{\lambda}_{n-1})}{(\hat{\lambda}_n - \hat{\lambda}_{n-1})(\hat{\lambda}_n - \hat{\lambda}_{n-2})}, \quad n\geqslant 2. \tag{3.15}$$

This yields the recurrence relations ($p_n = 1$)

$$(\hat{\lambda}_n - \hat{\lambda}_{n-1})s_n = \bar{\lambda}_n p_n, \quad n\geqslant 1, \qquad (\hat{\lambda}_n - \hat{\lambda}_{n-2})t_n = \bar{\lambda}_{n-1}s_n + \dot{\lambda}_n p_n, \quad n\geqslant 2.$$

Next, we substitute (3.15) in (3.13) to obtain

$$\begin{aligned} d_n &= \frac{\bar{\lambda}_n}{\hat{\lambda}_n - \hat{\lambda}_{n-1}} - \frac{\bar{\lambda}_{n+1}}{\hat{\lambda}_{n+1} - \hat{\lambda}_n} = \frac{[n]([n-1]\bar{a}+\bar{b})}{[2n-2]\hat{a}+\hat{b}} - \frac{[n+1]([n]\bar{a}+\bar{b})}{[2n]\hat{a}+\hat{b}} \\ &= \frac{[n]([n-1]\bar{a}+\bar{b})([2n]\hat{a}+\hat{b}) - [n+1]([n]\bar{a}+\bar{b})([2n-2]\hat{a}+\hat{b})}{([2n-2]\hat{a}+\hat{b})([2n]\hat{a}+\hat{b})}, \quad n\geqslant 1, \end{aligned}$$

and after some straightforward calculations,

$$d_n = -\frac{q^{n-1}[2][n]([n-1]\hat{a}+\hat{b})\bar{a} + q^n(([n-2]-q^{n-1}[n])\hat{a}+\hat{b})\bar{b}}{([2n-2]\hat{a}+\hat{b})([2n]\hat{a}+\hat{b})}, \quad n\geqslant 1. \tag{3.16}$$

Notice that $[n-2]-q^{n-1}[n] \equiv [n-1]-q^{n-2}[n+1] \to -2$ when $q\to 1$ which corresponds to the formula for d_n in the D-classical case [22]. Finally, for d_0, we first use the TTRR, $P_1 = x - d_0$, and, on the other hand, $\psi = \hat{\lambda}_1 P_1$, (it follows from the equation $\mathscr{S}\mathscr{L}P_1 = \psi = \hat{\lambda}_1 P_1$). So

$$P_1 = x + \bar{b}/\hat{b} = x - d_0, \quad d_0 = -\hat{b}/\bar{b}. \tag{3.17}$$

Now, to find g_n we substitute (3.15) in (3.14). This yields

$$g_n = \frac{\bar{\lambda}_n\bar{\lambda}_{n-1} + \dot{\lambda}_n(\hat{\lambda}_n - \hat{\lambda}_{n-1})}{(\hat{\lambda}_n - \hat{\lambda}_{n-1})(\hat{\lambda}_n - \hat{\lambda}_{n-2})} - \frac{\bar{\lambda}_{n+1}\bar{\lambda}_n + \dot{\lambda} n + 1(\hat{\lambda}_{n+1} - \hat{\lambda}_n)}{(\hat{\lambda}_{n+1} - \hat{\lambda}_n)(\hat{\lambda}_{n+1} - \hat{\lambda}_{n-1})} - \frac{\bar{\lambda}_n}{\hat{\lambda}_n - \hat{\lambda}_{n-1}} d_n, \quad n \geqslant 2.$$

A straightforward calculation (with the help of Mathematica 3.0 [31]) leads us to the expression

$$g_n = -\frac{q^{n-1}[n]([n-2]\hat{a} + \hat{b})}{([2n-1]\hat{a} + \hat{b})([2n-2]\hat{a} + \hat{b})^2([2n-3]\hat{a} + \hat{b})}$$

$$\times(q^{n-1}([n-1]\bar{a} + \bar{b})(q^{n-1}\hat{a}\bar{b} - \bar{a}([n-1]\hat{a} + \hat{b})) + \dot{a}([2n-2]\hat{a} + \hat{b})^2), \quad n \geqslant 2. \tag{3.18}$$

The above expression is also true for $n = 1$ and it gives $g_1 = -[\bar{b}(\hat{a}\bar{b} - \hat{b}\bar{a}) + \dot{a}\hat{b}^2]/\hat{b}^2(\hat{a} + \hat{b})$.

3.3. The coefficients of the STR

In this section we will determine the coefficients of the structure relations (3.2). In the following we will refer to this structure relation as SRT I.

Obviously to find the coefficients we can substitute in (3.2) the explicit expression of P_n and compare the corresponding coefficients. This leads to the following system:

$$a_n = [n]\hat{a}, \quad [n-1]\hat{a}s_n + [n]\bar{a} = a_n s_{n+1} + b_n,$$

$$[n-2]\hat{a}t_n + [n-1]\bar{a}s_n + [n]\dot{a} = a_n t_{n+1} + b_n s_{n+1} + c_n. \tag{3.19}$$

A simple calculation shows that (P_n is monic)

$$a_n = [n]\hat{a}, \quad b_n = \hat{a}([n-1]s_n - [n]s_{n+1}) + [n]\bar{a},$$

$$c_n = [n-2]\hat{a}t_n + [n-1]\bar{a}s_n + [n]\dot{a} - a_n t_{n+1} - s_{n+1}\{\hat{a}([n-1]s_n - [n]s_{n+1}) + [n]\bar{a}\}. \tag{3.20}$$

The second equation can be easily solved (e.g. using Mathematica [31])

$$b_n = -\frac{[n]([n-1]\hat{a} + \hat{b})\{\hat{a}\bar{b}q^{n-1}[2] - \bar{a}\hat{b} - \hat{a}\bar{a}[n](1 - q^{n-1})\}}{([2n]\hat{a} + \hat{b})([2n-2]\hat{a} + \hat{b})}. \tag{3.21}$$

The coefficient c_n can be derived by using Mathematica (although the computations are very cumbersome). So, from (3.20) we find

$$c_n = \frac{[n]\{q^{n-1}([n-1]\bar{a} + \bar{b})(q^{n-1}\hat{a}\bar{b} - \bar{a}([n-1]\hat{a} + \hat{b})) + \dot{a}([2n-2]\hat{a} + \hat{b})^2\}}{([n-2]\hat{a} + \hat{b})^{-1}([n-1]\hat{a} + \hat{b})^{-1}([2n-1]\hat{a} + \hat{b})([2n-2]\hat{a} + \hat{b})^2([2n-3]\hat{a} + \hat{b})}, \tag{3.22}$$

Let us obtain c_n also by the second method. We will start from (3.7). First of all, since $(P_n)_{n\geqslant 0}$ and $(Q_n)_{n\geqslant 0}$ are monic, $\langle \boldsymbol{v}, Q_n^2 \rangle = k'_n = \langle \boldsymbol{v}, P_n Q_n \rangle$, so,

$$k'_n = \langle \boldsymbol{v}, P_n Q_n \rangle = \frac{1}{[n+1]}\langle \phi\boldsymbol{u}, P_n \cdot \Theta P_{n+1} \rangle = \frac{1}{[n+1]}\langle \boldsymbol{u}, \phi\Theta P_{n+1} \cdot P_n \rangle = \frac{1}{[n+1]} c_{n+1} k_n,$$

and then,

$$k'_n = \frac{c_{n+1}}{[n+1]} k_n. \tag{3.23}$$

Now we use Eq. (3.9) $-\hat{\lambda}_n k_n = [n]^2 k'_{n-1}$, to find

$$k_n = -\frac{[n]^2}{\hat{\lambda}_n} \cdot \frac{c_n}{[n]} k_{n-1} \Leftrightarrow g_n = -[n]\frac{c_n}{\hat{\lambda}_n}, \Leftrightarrow c_n = -\frac{\hat{\lambda}_n}{[n]} g_n, \tag{3.24}$$

and then (3.22) immediately follows. Now we will obtain the coefficients of the structure relation II (3.3). We start from (3.8)

$$e_n = k'^{-1}_{n-1}\langle \phi \boldsymbol{u}, P_n Q_{n-1}\rangle = k'^{-1}_{n-1}\frac{1}{[n]}\langle \boldsymbol{u}, \phi \Theta P_n \cdot P_n\rangle \overset{(3.5)}{=} k'^{-1}_{n-1} k_n \frac{1}{[n]} \cdot b_n \overset{(3.9)}{=} -[n]\hat{\lambda}_n^{-1} b_n.$$

Then,

$$-\frac{\hat{\lambda}_n}{[n]} = \frac{b_n}{e_n} \Leftrightarrow e_n = -\frac{[n]}{\hat{\lambda}_n} b_n. \tag{3.25}$$

So,

$$e_n = \frac{q^{n-1}\{\hat{a}\bar{b}q^{n-1}[2] - \bar{a}\hat{b} - \hat{a}\bar{a}[n](1-q^{n-1})\}}{([2n]\hat{a}+\hat{b})([2n-2]\hat{a}+\hat{b})}. \tag{3.26}$$

Again, from (3.8)

$$h_n = k'^{-1}_{n-2}\langle \phi \boldsymbol{u}, P_n Q_{n-2}\rangle = k'^{-1}_{n-2}\frac{1}{[n-1]}\langle \boldsymbol{u}, \phi\Theta P_{n-1} \cdot P_n\rangle = k'^{-1}_{n-2}\frac{1}{[n-1]}\langle \boldsymbol{u}, [n-1]\hat{a}x^n \cdot P_n\rangle$$

$$= k'^{-1}_{n-2} k_n \hat{a} = (k'^{-1}_{n-2} k_{n-1})(k^{-1}_{n-1} k_n)\hat{a} = -\frac{[n-1]^2}{\hat{\lambda}_{n-1}} g_n \hat{a}.$$

In the last equality we have used $g_n = k_n/k_{n-1}$ and (3.9), respectively. This yields

$$h_n = -\frac{q^{n-2}[n-1]\hat{a}}{[n-2]\hat{a}+\hat{b}} g_n. \tag{3.27}$$

Thus,

$$h_n = \frac{\hat{a}q^{2n-3}[n-1][n]\{q^{n-1}([n-1]\bar{a}+\bar{b})(q^{n-1}\hat{a}\bar{b} - \bar{a}([n-1]\hat{a}+\hat{b})) + \dot{a}([2n-2]\hat{a}+\hat{b})^2\}}{([2n-1]\hat{a}+\hat{b})([2n-2]\hat{a}+\hat{b})^2([2n-3]\hat{a}+\hat{b})}. \tag{3.28}$$

3.4. The coefficients for the TTRR of the q-derivatives

Finally, we will obtain the coefficients g'_n and d'_n of the TTRR for the first q-derivatives $(Q_n)_{n\geqslant 0}$ (3.4). First of all, we use the fact that $g_n = k_n/k_{n-1}$ and $g'_{n-1} = k'_{n-1}/k'_{n-2}$. Then Eq. (3.9) gives

$$g_n = \frac{[n]^2}{[n-1]^2}\frac{\hat{\lambda}_{n-1}}{\hat{\lambda}_n} g'_{n-1} \Rightarrow g'_n = \frac{[n]^2}{[n+1]^2}\frac{\hat{\lambda}_{n+1}}{\hat{\lambda}_n} g_{n+1}, \quad n\geqslant 0,$$

and we get,

$$g'_n = -\frac{q^{n-1}[n]([n]\hat{a}+\hat{b})\{q^n([n]\bar{a}+\bar{b})(q^n\hat{a}\bar{b} - \bar{a}([n]\hat{a}+\hat{b})) + \dot{a}([2n]\hat{a}+\hat{b})^2\}}{([2n+1]\hat{a}+\hat{b})([2n]\hat{a}+\hat{b})^2([2n-1]\hat{a}+\hat{b})}. \tag{3.29}$$

For the other coefficient d'_n first we will take q-derivatives on the TTRR and then use TTRR for $(Q_n)_{n\geqslant 0}$. Thus,

$$\begin{aligned} xP_n &= P_{n+1} + d_nP_n + g_nP_{n-1} \xrightarrow{\Theta} P_n + qx\Theta P_n = \Theta P_{n+1} + d_n\Theta P_n + g_n\Theta P_{n-1} \\ &\Leftrightarrow P_n + q[n]xQ_{n-1} = [n+1]Q_n + d_n[n]Q_{n-1} + g_n[n-1]Q_{n-2} \\ &\overset{\text{TTRR}}{\Leftrightarrow} P_n = [n+1]Q_n + d_n[n]Q_{n-1} + g_n[n-1]Q_{n-2} - q[n](Q_n + d'_{n-1}Q_{n-1} + g'_{n-1}Q_{n-2}) \\ &= ([n+1]-q[n])Q_n + [n](d_n - qd'_{n-1})Q_{n-1} + ([n-1]g_nq[n]g'_{n-1})Q_{n-2}). \end{aligned}$$

Now, comparing it with the structure relation II we get

$$e_n = [n](d_n - qd'_{n-1}), \tag{3.30}$$

$$h_n = [n-1]g_n - q[n]g'_{n-1}. \tag{3.31}$$

Thus, using (3.25) and (3.30) we find

$$d_n - qd'_{n-1} = -\frac{b_n}{\hat{\lambda}_n} \quad\Leftrightarrow\quad d'_{n-1} = q^{-1}\left(d_n + \frac{b_n}{\hat{\lambda}_n}\right). \tag{3.32}$$

So,

$$d'_n = -\frac{q^n\{\bar{b}\hat{b} + \bar{b}\hat{a}(1-q^n)[n+1] + 2\bar{a}\hat{a}[n][n+1] + \bar{a}\hat{b}([n]+[n+1])\}}{(\hat{a}[2n]+\hat{b})(\hat{a}[2n+2]+\hat{b})}. \tag{3.33}$$

Notice that the second equation in (3.31) gives an alternative expression for the coefficient g'_n. In fact from (3.31) we find

$$g'_n = \frac{[n]([2n-1]\hat{a}+\hat{b})}{q[n+1]([n-2]\hat{a}+\hat{b})}g_n.$$

Remark 3.4. To conclude this section let us point out that it is possible to show that the coefficients g_n, b_n and e_n can be expressed as follows [24] (see also Appendix B):

$$g_n = -\frac{q^{n-1}[n]([n-2]\hat{a}+\hat{b})}{([2n-3]\hat{a}+\hat{b})([2n-1]\hat{a}+\hat{b})}\cdot\phi^{(n-1)}\left(-\frac{[n-1]\bar{a}+\hat{b}}{[2n-2]\bar{a}+\hat{b}}\right),$$

$$b_n = \frac{1}{1+q^{-1}}(-\psi(d_n)+(1-q^{-n})([n-1]\hat{a}+\hat{b})d_n), \quad e_n = \frac{q^n}{[2]([n-1]\hat{a}+\hat{b})}\psi(d_n) + \frac{1-q^n}{[2]}d_n.$$

In particular, the above representation of g_n leads directly to the same condition of existence of an infinite sequence or orthogonal polynomials as in Proposition A.1 (see Appendix A). In fact the conditions (b) and (d) in Proposition A.1 mean that $\phi_n(d_0^{(n)}) \neq 0$, $n\geqslant 0$, which, keeping in mind the necessary condition for the quasi-definiteness, guarantees us that $g_n \neq 0$, $n\geqslant 1$.

Remark 3.5. Notice that in all cases, when $q \to 1$, we obtain the corresponding D-classical relation [22].

4. Some examples

In this section, we will study some special families of q-polynomials and we will compute all of their principal characteristics. We will use the Proposition 2.22 to identify the families of q-classical polynomials among all the families in the so-called q-Askey Scheme [18]. In fact comparing the difference equation

$$\phi \cdot HP_n - (\phi + q^2\phi^\star)P_n + q^2\phi^\star \cdot H^{-1}P_n = (q-1)^2x^2\lambda_n P_n, \tag{4.34}$$

with those given in [18] one can easily see that the following q-polynomials are q-classical ones [24]: The Big q-Jacobi, Big q-Laguerre, Little q-Jacobi, Little q-Laguerre (Wall), q-Laguerre, Alternative q-Charlier, Al-Salam–Carlitz I, Al-Salam–Carlitz II, Stieltjes–Wigert, Discrete q-Hermite, Discrete q^{-1}-Hermite II, q-Hahn, q-Meixner, Quantum q-Kravchuk (Krawtchouk), q-Kravchuk, Affine q-Kravchuk and q-Charlier.

Eq. (4.34) gives all the information about the q-classical functional (and then about the corresponding MOPS). Moreover, it is *summarized* in the polynomials ϕ and $\phi^\star$ instead of ϕ and ψ. Furthermore, the interest of the polynomials ϕ and $\phi^\star$ is not reduced only to the aforesaid equation but also because using them one can classify all families of q-classical polynomials [24,25]. Another reason for taking into account both polynomials (and not only ϕ, like in the continuous case) is the fact that (see Proposition 2.23)) all q-classical families are q^{-1}-classical. In the following we will assume that $0 < q < 1$. In such a way, since $\phi(0) = 0$ if and only if $\phi^\star(0) = 0$, in a first step, it is natural to classify the q-classical polynomials in two wide groups: the $\emptyset$-families, i.e., the families such that $\phi(0) \neq 0$ and the 0-families, i.e., the ones with $\phi(0) = 0$.

The next step is, to classify each member in the aforesaid two wide classes in terms of the degree of the polynomials ϕ and $\phi^\star$ as well as the multiplicity of their roots in the case of 0-families. In fact, if ϕ has two simple roots, the polynomials belong to the 0-Jacobi/-family while if the roots are multiple, then they are 0-Bessel/-family. So, we have the following scheme for the q-classical OPS (for more details see [24,25]):

$$\emptyset\text{-families}\begin{cases}\emptyset\text{-Jacobi/Jacobi}\\ \emptyset\text{-Jacobi/Laguerre}\\ \emptyset\text{-Jacobi/Hermite}\\ \\ \emptyset\text{-Laguerre/Jacobi}\\ \emptyset\text{-Hermite/Jacobi}\end{cases} \qquad 0\text{-families}\begin{cases}0\text{-Bessel/Jacobi}\\ 0\text{-Bessel/Laguerre}\\ \\ 0\text{-Jacobi/Jacobi}\\ 0\text{-Jacobi/Laguerre}\\ 0\text{-Jacobi/Bessel}\\ \\ 0\text{-Laguerre/Jacobi}\\ 0\text{-Laguerre/Bessel}\end{cases}$$

Here, for example, $\emptyset$-Hermite/Jacobi means that the corresponding polynomials are such that $\phi(0) \neq 0$, where $\deg\phi = 0$ (i.e., a q-analogue of the Hermite polynomials), $\deg\phi^\star = 2$ (i.e., a q^{-1}-analogue of the Jacobi polynomials). Finally, let us point out that in all cases, except in the 0-Jacobi/Bessel and 0-Laguerre/Bessel ones there exist positive-definite families, i.e., families orthogonal with respect to a positive-definite functional.

In the following we will follow the standard notation for basic polynomials [14]:

$$ {}_r\varphi_p\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_p \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_p;q)_k} \frac{z^k}{(q;q)_k} [(-1)^k q^{k(k-1)/2}]^{p-r+1}, \tag{4.35} $$

where

$$ (a;q)_k = \prod_{m=0}^{k-1} (1 - aq^m). \tag{4.36} $$

In this section we will give the main data for some of the above families. The Big q-Jacobi polynomials $p_n(x;a,b,c;q)$ are defined by the following basic hypergeometric series [18]

$$ p_n(x;a,b,c;q) = \frac{(aq;q)_n (cq;q)_n}{(abq^{n+1};q)_n} \, {}_3\varphi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q\right). $$

Their main data are shown in Table 1. Notice that for these polynomials $\phi(0) \neq 0$, $\deg\phi = \deg\phi^{\star} = 2$. According with the aforesaid classification they constitute a $\emptyset$-Jacobi/Jacobi family.

Since the Big q-Laguerre polynomials $p_n(x;a,c;q)$ satisfy $p_n(x;a,c;q) = p_n(x;a,0,c;q)$, then

$$ \begin{aligned} p_n(x;a,c;q) &= (aq;q)_n (cq;q)_n \, {}_3\varphi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, cq \end{matrix} \middle| q; q\right) \\ &= \frac{(aq;q)_n (cq;q)_n}{(c^{-1}q^{-n};q)_n} \, {}_2\varphi_1\left(\begin{matrix} q^{-n}, aqx^{-1} \\ aq \end{matrix} \middle| q; \frac{x}{c}\right). \end{aligned} $$

So, putting $b=0$ in the main data of the Big q-Jacobi, one obtains the data for the Big q-Laguerre. So if in Table 1 we put $b=0$ we find the corresponding data for the Big q-Laguerre. Notice also that they are a $\emptyset$-Laguerre/Jacobi family.

The Little q-Jacobi polynomials $p_n(x;a,b|q)$ are defined by the following basic hypergeometric series [18]:

$$ p_n(x;a,b|q) = \frac{(-1)^n q^{\binom{n}{2}} (aq;q)_n}{(abq^{n+1};q)_n} \, {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; qx\right). $$

Notice that, since for the Little q-Jacobi polynomials $\phi(0) = 0$, $\deg\phi = \deg\phi^{\star} = 2$, then they are a 0-Jacobi/Jacobi family. If we now put $b=0$, the Little q-Jacobi polynomials become the Little q-Laguerre or Wall polynomials $p_n(x;a|q)$, i.e., $p_n(x;a|q) = p_n(x;a,0|q)$, so

$$ p_n(x;a|q) = (-1)^n q^{\binom{n}{2}} (aq;q)_n \, {}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \middle| q; qx\right). $$

Then all their characteristics can be obtained from the ones in the Table 2 just putting $b=0$. These polynomials constitute a 0-Laguerre/Jacobi family.

The Al-Salam and Carlitz polynomials I and II are defined by the expressions [18]

$$ U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} \, {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{xq}{a}\right), $$

and

$$ V_n^{(a)}(x;q) = (-a)^n q^{-\binom{n}{2}} \, {}_2\varphi_0\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix} \middle| q; \frac{q^n}{a}\right), $$

Table 1
The Big q-Jacobi polynomials

P_n	$p_n(x;a,b,c;q)$
ϕ	$aq(x-1)(bx-c)$
$\phi^\star$	$q^{-2}(x-aq)(x-cq)$
ψ	$\frac{1-abq^2}{(1-q)q}x+\frac{a(bq-1)+c(aq-1)}{1-q}$
$\hat{\lambda}_n$	$q^{-n}[n]\frac{1-abq^{n+1}}{1-q}$
r_n	$\frac{q^{n(n+1)/2}(1-q)^n}{(abq^{n+1};q)_n}$
d_n	$\frac{q^{1+n}\{c+a^2bq^n((1+b+c)q^{1+n}-q-1)+a(1+c-cq^n-cq^{1+n}+b(1-q^n-cq^n-q^{1+n}-cq^{1+n}+cq^{1+2n}))\}}{(1-abq^{2n})(1-abq^{2n+2})}$
g_n	$-\frac{aq^{n+1}(1-q^n)(1-aq^n)(1-bq^n)(1-abq^n)(c-abq^n)(1-cq^n)}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$
a_n	$abq[n]$
b_n	$-\frac{aq[n](1-abq^{n+1})\{c+ab^2q^{2n+1}+b(1-cq^n-cq^{n+1}-aq^n(1+q-cq^{n+1}))\}}{(1-abq^{2n})(1-abq^{2n+2})}$
c_n	$\frac{aq[n](1-aq^n)(1-bq^n)(1-abq^n)(c-abq^n)(1-cq^n)(1-abq^{n+1})}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$
e_n	$\frac{aq^{n+1}(1-q^n)\{c+ab^2q^{2n+1}+b(1-cq^n-cq^{n+1}-aq^n(1+q-cq^{n+1}))\}}{(1-abq^{2n})(1-abq^{2n+2})}$
h_n	$\frac{a^2bq^{2n+1}(1-q^{n-1})(1-q^n)(1-aq^n)(1-bq^n)(c-abq^n)(1-cq^n)}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{1+2n})}$
d_n'	$\frac{q^{n+1}\{c+a^2bq^{n+2}((1+qb+qc)q^{n+1}-q-1)+a(1+cq-cq^{n+1}-cq^{n+2}+bq(1-q^n-q^{n+1}-cq^{n+1}+cq^{2n+2}-cq^{n+2}))\}}{(1-abq^{2n+2})(1-abq^{2n+4})}$
g_n'	$-\frac{aq^{n+1}(1-q^n)(1-aq^{n+1})(1-bq^{n+1})(c-abq^{n+1})(1-cq^{n+1})(1-abq^{n+2})}{(1-abq^{2n+2})^2(1-abq^{2n+1})(1-abq^{2n+3})}$

respectively. Notice that in the first case $\phi(0)\neq 0$, $\deg\phi=0,\deg\phi^\star=2$, so the Al-Salam & Carlitz polynomials I are a $\emptyset$-Hermite/Jacobi family. For the the Al-Salam and Carlitz II, $\phi(0)\neq 0$, $\deg\phi=2,\deg\phi^\star=0$, i.e., they constitute a $\emptyset$-Jacobi/Hermite. If we substitute $a=-1$ in $U_n^{(a)}(x;q)$ we obtain the Discrete q-Hermite polynomials I $h_n(x;q)$. So, putting $a=-1$ in Table 3 we obtain their main data. Obviously they are also a $\emptyset$-Hermite/Jacobi family.

The Stieltjes–Wigert polynomials $S_n(x;q)$ are defined by [18]

$$S_n(x;q)=(-1)^n q^{-n^2}\,{}_1\varphi_1\left(\begin{matrix}q^{-n}\\0\end{matrix}\,\middle|\,q;-xq^{n+1}\right).$$

Their main characteristics are also given in Table 3. Moreover, they are a 0-Bessel/Laguerre family, since $\phi(0)=0$, $\deg\phi=2,\deg\phi^\star=1$.

Table 2
The Little q-Jacobi polynomials and q-Charlier polynomials

P_n	$p_n(x;a,b\|q)$	$C_n(x;a;q)$
ϕ	$ax(bqx-1)$	$x(x-1)$
$\phi^\star$	$q^{-2}x(x-1)$	$q^{-2}ax$
ψ	$\frac{1-abq^2}{(1-q)q}x+\frac{aq-1}{(1-q)q}$	$-\frac{1}{1-q}x+\frac{a+q}{(1-q)q}$
$\hat{\lambda}_n$	$q^{-n}[n]\frac{1-abq^{n+1}}{1-q}$	$-\frac{[n]}{1-q}$
r_n	$\frac{q^{n(n+1)/2}(1-q)^n(ab;q)_{n+1}}{(ab;q)_{2n+1}}$	$q^{-n(n-1)}(q-1)^n$
d_n	$\frac{q^n\{1+a^2bq^{2n+1}+a(1-(1+b)q^n-(1+b)q^{n+1}+bq^{2n+1})\}}{(1-abq^{2n})(1-abq^{2n+2})}$	$-q^{-2n-1}\{(a-1)q^{n+1}-a(1+q)\}$
g_n	$\frac{aq^{2n-1}(1-q^n)(1-aq^n)(1-bq^n)(1-abq^n)}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$	$aq^{-4n+1}(1-q^n)(a+q^n)$
a_n	$abq[n]$	$[n]$
b_n	$-\frac{a[n](1-abq^{n+1})(1-bq^n(1+q-aq^{n+1}))}{(1-abq^{2n})(1-abq^{2n+2})}$	$q^{-2n-1}[n](a+aq+q^{n+1})$
c_n	$-\frac{a[n]q^{n-1}(1-aq^n)(1-bq^n)(1-abq^n)(1-abq^{n+1})}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$	$aq^{-4n+1}[n](a+q^n)$
e_n	$\frac{aq^n(1-q^n)\{1-bq^n(1+q-aq^{n+1})\}}{(1-abq^{2n})(1-abq^{2n+2})}$	$q^{-2n-1}(1-q^n)(a+aq+q^{n+1})$
h_n	$-\frac{a^2bq^{3n-1}(1-q^{n-1})(1-q^n)(1-aq^n)(1-bq^n)}{(1-abq^{2n})^2(1-abq^{2n-1})(1-abq^{2n+1})}$	$aq^{-4n+1}(1-q^n)(1-q^{n-1})(a+q^n)$
d'_n	$\frac{q^n\{1+a^2bq^{2n+4}+aq(1-(1+q)(1+bq)q^n+bq^{2n+2})\}}{(1-abq^{2n+2})(1-abq^{2n+4})}$	$q^{-2n-3}\{q^{n+2}+a(1+q-q^{n+1})\}$
g'_n	$\frac{aq^{2n}(1-q^n)(1-aq^{n+1})(1-bq^{n+1})(1-abq^{n+2})}{(1-abq^{2n+1})(1-abq^{2n+2})^2(1-abq^{2n+3})}$	$aq^{-4n-3}(1-q^n)(a+q^{n+1})$

The Discrete q-Hermite polynomials II $\tilde{h}_n(x;q)$ are related with $V_n^{(a)}(x;q)$ in the following way:

$$\tilde{h}_n(x;q)=i^{-n}V_n^{(-1)}(x;q)=x^n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},q^{-n+1}\\0\end{matrix}\middle|q^2;-\frac{q^2}{x^2}\right).$$

Their main characteristics are shown in Table 4. A simple inspection on this table gives $\phi(0)\neq 0$, $\deg\phi=2$, $\deg\phi^\star=0$, i.e., the q-Hermite polynomials $\tilde{h}_n(x;q)$ are a $\emptyset$-Jacobi/Hermite family.

The Alternative q-Charlier polynomials $K_n(x;a,q)$ are defined by [18]

$$K_n(x;a;q)=\frac{(-1)^nq^{\binom{n}{2}}}{(-aq^n;q)_n}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},-aq^n\\0\end{matrix}\middle|q;qx\right),$$

Table 3
The Al-Salam and Carlitz and Stieltjes–Wigert polynomials

P_n	$U_n^{(a)}(x;q)$	$V_n^{(a)}(x;q)$	$S_n(x;q)$
ϕ	a	$(x-1)(x-a)$	x^2
$\phi^\star$	$q^{-1}(1-x)(a-x)$	$q^{-1}a$	$q^{-2}x$
ψ	$\frac{1}{1-q}x-\frac{1+a}{1-q}$	$-\frac{1}{1-q}x+\frac{1+a}{1-q}$	$-\frac{1}{1-q}x+\frac{1}{q(1-q)}$
$\hat{\lambda}_n$	$\frac{q^{1-n}[n]}{1-q}$	$-\frac{[n]}{1-q}$	$-\frac{[n]}{1-q}$
r_n	$q^{n(n-1)/2}(1-q)^n$	$q^{-n(n-1)}(q-1)^n$	$q^{-n(n-1)}(q-1)^n$
d_n	$(1+a)q^n$	$(1+a)q^{-n}$	$q^{-2n-1}(1+q-q^{n+1})$
g_n	$aq^{n-1}(q^n-1)$	$aq^{-2n+1}(1-q^n)$	$q^{-4n+1}(1-q^n)$
a_n	0	$[n]$	$[n]$
b_n	0	$(1+a)[n]q^{-n}$	$[n]q^{-2n+1}(1+q)$
c_n	$a[n]$	$aq^{-2n+1}[n]$	$q^{-4n+1}[n]$
e_n	0	$(1+a)q^{-n}(1-q^n)$	$q^{-2n-1}(1+q)(1-q^n)$
h_n	0	$aq^{-2n+1}(1-q^n)(1-q^{n-1})$	$q^{-4n+1}(1-q^n)(1-q^{n-1})$
d'_n	$(1+a)q^n$	$(1+a)q^{-n-1}$	$q^{-2n-3}(1+q-q^{n+1})$
g'_n	$aq^{n-1}(q^n-1)$	$aq^{-2n-1}(1-q^n)$	$q^{-4n-3}(1-q^n)$

and the q-Laguerre polynomials $L_n^\alpha(x;q)\equiv L_n(x;a;q)$ are given by [18]

$$L_n(x;a;q)=(-1)^n q^{-n^2}a^{-n}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},-x\\0\end{matrix}\,\middle|\,q;aq^{n+1}\right).$$

Their main characteristics are presented in Table 4. Notice that the Alternative q-Charlier polynomials $K_n(x;a;q)$ are 0-Bessel/Jacobi family, $\phi(0)=0$, $\deg\phi=\deg\phi^\star=2$, whereas the q-Laguerre polynomials $L_n^\alpha(x;q)$ are 0-Jacobi/Laguerre: $\phi(0)=0$, $\deg\phi=2$, $\deg\phi^\star=1$.

Finally we will study the q-analogue of the classical discrete polynomials: Hahn, Meixner, Kravchuk and Charlier. In [18] such polynomials are the q-Hahn, q-Meixner, Quantum q-Kravchuk (Krawtchouk), q-Kravchuk, Affine q-Kravchuk and q-Charlier, respectively. All of them are defined as a basic terminating series and they are polynomials on q^{-x} instead of x. The main reason for such a choice is that, in the limit $q\to 1-$ they become the classical discrete ones. Here we will define them as polynomials in x. To recover the polynomials in [18] one needs to substitute just x by q^{-x}. This transforms $y(x+1)$ and $y(x-1)$ in [18] into $H^{-1}y$ and Hy, respectively, and divide by x^2.

We start with the q-Hahn family. The q-Hahn polynomials are defined by [18]

$$Q_n(x;a,b,N|q)=\frac{(aq;q)_n(q^{-N};q)_n}{(abq^{n+1};q)_n}\,{}_3\varphi_2\left(\begin{matrix}q^{-n},abq^{n+1},x\\aq,q^{-N}\end{matrix}\,\middle|\,q;q\right).$$

Table 4
The alternative q-Charlier, q-Laguerre polynomials and discrete q-Hermite II

P_n	$K_n(x;a,;q)$	$L_n^{\alpha}(x;q),\ a=q^{\alpha}$	$\tilde{h}_n(x;q)$
ϕ	ax^2	$ax(x+1)$	$1+x^2$
$\phi^{\star}$	$q^{-2}x(1-x)$	$q^{-2}x$	q^{-1}
ψ	$-\dfrac{1+aq}{(1-q)q}x+\dfrac{1}{(1-q)q}$	$-\dfrac{a}{(1-q)}x+\dfrac{1-aq}{(1-q)q}$	$-\dfrac{1}{1-q}x$
$\hat{\lambda}_n$	$\dfrac{(1-q^{-n})(1+aq^n)}{(1-q)^2}$	$-\dfrac{a[n]}{(1-q)}$	$-\dfrac{[n]}{(1-q)}$
r_n	$a^{-n}q^{-n(n-1)}(q-1)^n$	$a^{-n}q^{-n(n-1)}(q-1)^n$	$q^{-n(n-1)}(q-1)^n$
d_n	$\dfrac{q^n(1+aq^{n-1}+aq^n-aq^{2n})}{(1+aq^{2n-1})(1+aq^{2n+1})}$	$a^{-1}q^{-2n-1}(1+q-(1+a)q^{n+1})$	0
g_n	$\dfrac{aq^{3n-2}(1-q^n)(1+aq^{n-1})}{(1+aq^{2n})(1+aq^{2n-1})^2(1+aq^{2n-2})}$	$a^{-2}q^{-4n+1}(1-q^n)(1-aq^n)$	$q^{-2n+1}(1-q^n)$
a_n	$a[n]$	$a[n]$	$[n]$
b_n	$\dfrac{aq^{n-1}[n](1+q)(1+aq^n)}{(1+aq^{2n-1})(1+aq^{2n+1})}$	$q^{-2n-1}[n](1+q-aq^{n+1})$	0
c_n	$\dfrac{aq^{2n-2}[n](1+aq^n)(1+aq^{n-1})}{(1+aq^{2n})(1+aq^{2n-1})^2(1+aq^{2n-2})}$	$a^{-1}q^{-4n+1}[n](1-aq^n)$	$q^{-2n+1}[n]$
e_n	$\dfrac{aq^{2n-1}(1+q)(1-q^n)}{(1+aq^{2n-1})(1+aq^{2n+1})}$	$a^{-1}q^{-2n-1}(1-q^n)(1+q-aq^{n+1})$	0
h_n	$\dfrac{a^2q^{4n-3}(1-q^{n-1})(1-q^n)}{(1+aq^{2n})(1+aq^{2n-2})(1+aq^{2n-1})^2}$	$a^{-2}q^{-4n+1}(1-q^{n-1})(1-q^n)(1-aq^n)$	$q^{-2n+1}(1-q^n)(1-q^{n-1})$
d'_n	$\dfrac{q^n(1+aq^{n+1}(1+q-q^{n+1}))}{(1+aq^{2n+1})(1+aq^{2n+3})}$	$a^{-1}q^{-2n-3}(1+q-(1+aq)q^{n+1})$	0
g'_n	$\dfrac{aq^{3n}(1-q^n)(1+aq^{n+1})}{(1+aq^{2n})(1+aq^{2n+1})^2(1+aq^{2n+2})}$	$a^{-2}q^{-4n-3}(1-q^n)(1-aq^{n+1})$	$q^{-2n-1}(1-q^n)$

Just making the change $q^{-x}\to x$ in the difference equation for the q-Hahn polynomials in [18] and comparing it with Eq. (4.34) (or comparing with the definition of the Big q-Jacobi polynomials) we notice that the q-Hahn polynomials are nothing else that Big q-Jacobi polynomials with parameter $c=q^{-N-1}$ so they are a $\emptyset$-Jacobi/Jacobi family and all of their characteristics can be obtained from Table 1 just putting $c=q^{-N-1}$. Notice also that, since $g_{N+1}=0$, they are a finite family of q-classical *orthogonal* polynomials (see Remark A.2 from the appendix).

The next family is the q-Meixner one. They are defined by [18]

$$M_n(x;b,c;q)=(-c)^n(bq;q)_n q^{-n^2}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x\\ bq\end{matrix}\,\middle|\,q;-\frac{q^{n+1}}{c}\right).$$

Their main data are in Table 5. Notice that they are a finite $\emptyset$-Jacobi/Laguerre family.

The Quantum q-Kravchuck are defined by

$$K_n^{\rm qtm}(x;p,N;q)=(p)^{-n}(q^{-N};q)_n q^{-n^2}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x\\ q^{-N}\end{matrix}\,\middle|\,q;pq^{n+1}\right).$$

Table 5
The q-Meixner and q-Kravchuk polynomials

P_n	$M_n(x;b,c;q)$	$K_n(x;p,N;q)$
ϕ	$(x-1)(x+bc)$	$px(1-x)$
$\phi^\star$	$q^{-2}c(x-bq)$	$q^{-2}x(x-q^{-N})$
ψ	$-\frac{1}{1-q}x+\frac{c+q(1-bc)}{(1-q)q}$	$\frac{1+pq}{(1-q)q}x-\frac{p+q^{-N-1}}{1-q}$
$\hat{\lambda}_n$	$-\frac{[n]}{1-q}$	$-\frac{q^{-n}[n](1+pq^n)}{1-q}$
r_n	$q^{-n(n-1)}(q-1)^n$	$p^{-n}q^{-n(n-1)}(1-q)^n$
d_n	$q^{-n}+cq^{-2n-1}(1+q-(1+b)q^{n+1})$	$\frac{1-pq^N(1-q^n)+pq^{n+N}(q+pq^n)+pq^{n-1}(1+q(1-q^n))}{q^{N-n}(1+pq^{2n-1})(1+pq^{2n+1})}$
g_n	$cq^{-4n+1}(1-q^n)(c+q^n)(1-bq^n)$	$\frac{pq^{2n-2N-2}(1+pq^{n-1})(1-q^n)(q^n-q^{N+1})(1+pq^{n+N})}{(1+pq^{2n})(1+pq^{2n-2})(1+pq^{2n-1})^2}$
a_n	$[n]$	$-p[n]$
b_n	$q^{-1-2n}[n](q^{n+1}+c(1+q-bq^{n+1}))$	$-\frac{p[n](1+pq^n)\{q^n(1+q)-q^{N+1}(1-pq^{2n})\}}{q^{N+1}(1+pq^{2n-1})(1+pq^{2n+1})}$
c_n	$cq^{-4n+1}[n](c+q^n)(1-bq^n)$	$-\frac{pq^{n-2N-2}(1+pq^{n-1})(1+pq^n)(q^n-q^{N+1})(1+pq^{n+N})}{(1+pq^{2n})(1+pq^{2n-2})(1+pq^{2n-1})^2}$
e_n	$q^{-2n-1}(1-q^n)(q^{n+1}+c(1+q-bq^{n+1}))$	$\frac{pq^{n-N-1}(1-q^n)(q^n+q^{n+1}-q^{N+1}+pq^{2n+N+1})}{(1+pq^{2n-1})(1+pq^{2n+1})}$
h_n	$cq^{-4n}(1-q^n)(1-q^n)(c+q^n)(1-bq^n)$	$\frac{p^2q^{3n-2N-3}(1-q^{n-1})(1-q^n)(q^n-q^{N+1})(1+pq^{n+N})}{(1+pq^{2n})(1+pq^{2n-2})(1+pq^{2n-1})^2}$
d_n'	$q^{-2n-3}(q^{n+2}+c(1+q-q^{n+1}-bq^{n+2}))$	$\frac{1+p^2q^{2n+N+3}+pq(q^n+q^{n+1}-q^{2n+1}-q^N+q^{n+N}+q^{n+N+1})}{q^{N-n}(1+pq^{2n+1})(1+pq^{2n+3})}$
g_n'	$cq^{-4n-3}(1-q^n)(c+q^{n+1})(1-bq^{n+1})$	$\frac{pq^{2(n-N)}(1-q^n)(1+pq^{n+1})(q^n-q^N)(1+pq^{n+N+1})}{(1+pq^{2n})(1+pq^{2n+1})^2(1+pq^{2n+2})}$

Notice that they are related with the q-Meixner ones by

$$K_n^{\mathrm{qtm}}(x;p,N;q)=M_n(x;q^{-N-1},-p^{-1};q).$$

So their main characteristics can be obtained from Table 5 just putting $b=q^{-N-1}$ and $c=-p^{-1}$. Notice that they are also a finite set of the $\emptyset$-Jacobi/Laguerre family of q-classical polynomials.

The q-Kravchuk are defined by [18]

$$K_n(x;p,N;q)=\frac{(q^{-N};q)_n}{(-pq^n;q)_n}\,{}_3\varphi_2\left(\begin{matrix}q^{-n},x,-pq^n\\ q^{-N},0\end{matrix}\,\middle|\,q;q\right)$$

$$=\frac{x^n(q^{-N}x^{-1};q)_n}{(-pq^n;q)_n}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x\\ xq^{N-n+1}\end{matrix}\,\middle|\,q;\,pq^{N+n+1}\right).$$

They also constitute a finite family of the 0-Jacobi/Jacobi q-classical polynomials. Their main data are shown in Table 5.

The affine q-Kravchuck are defined by [18]

$$K_n^{\text{aff}}(x;p,N;q)=(q^{-N};q)_n(pq;q)_n\,{}_3\varphi_2\left(\begin{matrix}q^{-n},0,x\\ pq,q^{-N}\end{matrix}\,\middle|\,q;q\right)$$

$$=(-pq)^n(q^{-N};q)_nq^{n(n-1)/2}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},q^{-N}x^{-1}\\ q^{-N}\end{matrix}\,\middle|\,q;\frac{x}{p}\right).$$

Notice that they are the Big q-Laguerre polynomials with parameters $a=q^{-N-1}$ and $c=p$, or equivalently, the Big q-Jacobi polynomial with $a=q^{-N-1}$, $b=0$ and $c=p$, so they are a $\emptyset$-Laguerre/Jacobi family and their main characteristics can be obtained from Table 1 just substituting these values for the parameters a, b, and c, respectively. They, as the q-Hanh polynomials, also constitute a finite family of q-classical polynomials.

Finally, the q-Charlier polynomials are given by [18]

$$C_n(x;a;q)=(-1)^nq^{-n^2}a^n\,{}_2\varphi_1\left(\begin{matrix}q^{-n},x\\ 0\end{matrix}\,\middle|\,q;-\frac{q^{n+1}}{a}\right).$$

So they are related with the q-Laguerre polynomials $L_n^{\alpha}(x)$ by

$$C_n(x;a;q)=L_n(-x;-a^{-1};q).$$

They constitute a 0-Jacobi/Laguerre family. Obviously their main characteristics can be obtained from Table 4 making the appropriate change of parameters and signs (since the change $x\to -x$) but we will include them in Table 2.

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Appendix A. A sufficient condition for the quasi-definiteness of a q-classical functional

Up to now, we only have obtained necessary conditions on the polynomials ϕ and ψ for the quasi-definiteness of the functional $\boldsymbol{u}$ satisfying the distributional equation $\Theta(\phi\boldsymbol{u})=\psi\boldsymbol{u}$. Our aim now is to find also sufficient conditions. Here we will follow the work by Häcker [17]. In fact, we will work with the q-Sturm–Liouville equation. The main reason is that, using this equation it is easy to show that if $\hat{\lambda}_n \neq \hat{\lambda}_m$ for all $n \neq m$ then $[k]\hat{a} + \hat{b} \neq 0$, for every $k \geqslant 0$ (the necessary conditions for the quasi-definiteness), the family of eigenfunctions P_n of the q-Sturm–Liouville operator $\mathscr{S}\mathscr{L}$ are orthogonal. In fact we have

Proposition A.1. *Let $\mathscr{S}\mathscr{L} = \phi\Theta\Theta^\star + \psi\Theta^\star$ be the q-Sturm–Liouville operator. Let $(B_n)_{n\geqslant 0}$ be a basis sequence of polynomial eigenfunctions of the operator $\mathscr{S}\mathscr{L}$ and $(\boldsymbol{b}_n)_{n\geqslant 0}$ be the dual basis of $(B_n)_{n\geqslant 0}$ and $\boldsymbol{u} = c\boldsymbol{b}_0$, $c \in \mathbb{C}$. Then, $\langle \boldsymbol{u}, B_m B_n\rangle = 0$ for all $n \neq m$.*

Proof. From Proposition 2.20, $\boldsymbol{u}$ satisfies $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$. Thus, by Lemma 2.14

$$\left.\begin{array}{rl}\langle \phi\boldsymbol{u}, \Theta B_n \Theta B_n\rangle &= -\langle \boldsymbol{u}, B_m \cdot \mathscr{S}\mathscr{L} B_n\rangle = -\lambda_n \langle \boldsymbol{u}, B_m B_n\rangle \\ &= -\langle \boldsymbol{u}, B_n \cdot \mathscr{S}\mathscr{L} B_m\rangle = -\lambda_m \langle \boldsymbol{u}, B_n B_m\rangle \end{array}\right\} \Rightarrow_{n\neq m}^{\lambda_n\neq\lambda_m} \langle \boldsymbol{u}, B_n B_m\rangle = 0. \qquad \square$$

Notice that in the proof of Propositions 2.20 and A.1 the conditions $\deg\phi \leqslant 2$, $\deg\psi \leqslant 1$ are not used. We next will show a sufficient condition for the quasi-definiteness of the functional.

Lemma A.2. *Let $\mathscr{S}\mathscr{L} = \phi\Theta\Theta^\star + \psi\Theta^\star$ be the q-Sturm–Liouville operator with* $\deg\phi \leqslant 2$, $\deg\psi = 1$, *let $(B_n)_{n\geqslant 0}$ be a basis sequence of eigenfunctions of $\mathscr{S}\mathscr{L}$ and let $(\boldsymbol{b}_n)_{n\geqslant 0}$ be the dual basis of $(B_n)_{n\geqslant 0}$. Then, for $\boldsymbol{u} = c\boldsymbol{b}_0$, $c \in \mathbb{C}$, $c \neq 0$, $\langle \boldsymbol{u}, B_1^2\rangle \neq 0$, holds if and only if ϕ and ψ are coprime.*

Proof. In the following, and without loss of generality, we will assume that $(B_n)_{n\geqslant 0}$ is a monic sequence and therefore it satisfies the TTRR

$$xB_n = B_{n+1} + d_n B_n + g_n B_{n-1}, \quad B_{-1} = 0,\ B_0 = 1, \quad n \geqslant 0.$$

We will prove the following equivalent statements: There exists $a \in \mathbb{C}$ with $\phi(a) = 0 = \psi(a)$ if and only if $\langle \boldsymbol{u}, B_1^2\rangle = 0$.

In this case, $\phi(a) = 0 = \psi(a) \Rightarrow (\mathscr{S}\mathscr{L} B_n)(a) = 0$, $n \geqslant 0$. Since $\lambda_n \neq 0$, $n \geqslant 1$ (see Proposition 2.15), thus $B_n(a) = 0$, $n \geqslant 1$. In particular,

$$aB_1(a) = B_2(a) + d_1 B_1(a) + g_1 B_0(a) \overset{B_0=1}{\Rightarrow} g_1 = 0 \overset{u_0\neq 0}{\Rightarrow} k_1 = 0.$$

Conversely,

$$\langle \boldsymbol{u}, B_1^2\rangle = 0 \overset{u_0\neq 0}{\Rightarrow} g_1 = 0 \overset{RRTT}{\Rightarrow} B_2 = (x - d_1)B_1 \overset{\deg B_1=1}{\Rightarrow} \exists a \in \mathbb{C},\ B_2(a) = 0 = B_1(a).$$

Now, $\mathscr{S}\mathscr{L}(B_n)_{n\geqslant 0}$, for $n = 1, 2$ gives $0 + \psi = \lambda_1 B_1 \overset{\lambda_1\neq 0}{\Rightarrow} \psi(a) = 0$, and $[2]^\star\phi + \psi\Theta^\star B_2 = \lambda_2 B_2$, so $\phi(a) = 0$, respectively. This completes the proof. $\square$

Proposition A.3. *Let $\boldsymbol{u} \in \mathbb{P}^*$, $\phi, \psi \in \mathbb{P}$, $\deg\phi \leqslant 2$, $\deg\psi = 1$ such that $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$, $(B_n)_{n\geqslant 0}$ is a sequence of eigenfunctions of $\mathscr{S}\mathscr{L} = \phi\Theta\Theta^\star + \psi\Theta^\star$ and $\boldsymbol{v}^{(k)} = H^{(k)}\phi\cdot\boldsymbol{u}$. If $(Q_n^{(k)})$ is a sequence of monic kth order q-derivatives, $k \geqslant 1$, $Q_n^{(k)} = (1/[n+1]_{(k)})\Theta^k B_{n+k}$, then, $\langle \boldsymbol{v}^{(k)}, Q_m^{(k)} Q_n^{(k)}\rangle = c_{m,n}^{(k)}\langle \boldsymbol{u}, B_{m+k} B_{n+k}\rangle$, where*

$$c_{m,n}^{(k)} = \frac{(-1)^k}{[m+1]_{(k)}[n+1]_{(k)}} \lambda_{n+1}^{(k-1)} \ldots \lambda_{n+k}^{(0)}, \quad m, n \geqslant 0.$$

Proof. According to the Theorem 2.18, $(Q_n^{(k)})$ are the eigenfunctions of the q-Sturm–Liouville operator $\mathscr{S}\mathscr{L}^{(k)} = \phi^{(k)}\Theta\Theta^\star + \psi^{(k)}\Theta^\star$. Then,

$$\mathscr{S}\mathscr{L}^{(k)} Q_n^{(k)} = \hat{\lambda}_n^{(k)} Q_n^{(k)}, \quad \hat{\lambda}_n^{(k)} \in \mathbb{C}, \quad n \geqslant 0, \ k \geqslant 1,$$

where $\hat{\lambda}_n^{(k)}$ are the corresponding eigenvalues. If we use now the Lemma 2.14,

$$\begin{aligned}
\langle \boldsymbol{v}^{(k)}, Q_m^{(k)} Q_n^{(k)} \rangle &= \frac{1}{[m+1][n+1]} \langle H^{k-1}\phi \cdot \boldsymbol{v}^{(k-1)}, \Theta Q_{m+1}^{(k-1)} \Theta Q_{n+1}^{(k-1)} \rangle \\
&= \frac{1}{[m+1][n+1]} \langle \boldsymbol{v}^{(k-1)}, Q_{m+1}^{(k-1)} \cdot \mathscr{S}\mathscr{L}_{k-1} Q_{n+1}^{(k-1)} \rangle \\
&= \frac{-1}{[m+1][n+1]} \lambda_{n+1}^{(k-1)} \langle \boldsymbol{v}^{(k-1)}, Q_{m+1}^{(k-1)} Q_{n+1}^{(k-1)} \rangle \\
&\ \ \vdots \\
&= \frac{(-1)^k}{[m+1]_{(k)}[n+1]_{(k)}} \lambda_{n+1}^{(k-1)} \ldots \lambda_{n+k}^{(0)} \langle \boldsymbol{u}, B_{m+k} B_{n+k} \rangle \\
&= c_{m,n}^{(k)} \langle \boldsymbol{u}, B_{m+k} B_{n+k} \rangle. \qquad \square
\end{aligned}$$

Remark A.4. Notice that, if for $\phi\, y \psi$, with $\deg \phi = 2$ are such that

$$[n]\hat{a} + \hat{b} \neq 0, \quad n \geqslant 0 \ \Leftrightarrow \ \hat{\lambda}_n \neq 0, \quad n \geqslant 1,$$

where $\hat{\lambda}_n$ are the eigenfunctions of the corresponding q-Sturm–Liouville operator. Then, the necessary condition for the quasi-definiteness of $\boldsymbol{u}$ leads to the necessary condition for $\boldsymbol{v}^{(k)}$, and therefore we will have the orthogonality of the basis sequence $(B_n)_{n\geqslant 0}$ as well as the orthogonality of the sequence of their derivatives. So, the above proposition together with the Lemma A.2 and the condition $\hat{\lambda}_n^{(k)} \neq 0$, $k \geqslant 0$, $n \geqslant 1$, yields

$$\psi^{(k)} \text{ are coprime } \phi^{(k)} \ \Leftrightarrow \ \langle \boldsymbol{v}^{(k)}, Q_1^{(k)} Q_1^{(k)} \rangle \neq 0 \ \Leftrightarrow \ \langle \boldsymbol{u}, P_{k+1}^2 \rangle \neq 0, \quad k \geqslant 1.$$

In the next theorem we will summarize the main results of this appendix.

Theorem A.5. *Let ϕ and $\psi \in \mathbb{P}$ such that $\phi = \hat{a}x^2 + \bar{a}x + \dot{a}$, $\psi = \hat{b}x + \bar{b}$, with $\hat{b} \neq 0$, $[n]\hat{a} + \hat{b} \neq 0$, $n \geqslant 0$ and let $\boldsymbol{u} \in \mathbb{P}^*$ be the solution of the distributional equation $\Theta(\phi\boldsymbol{u}) = \psi\boldsymbol{u}$, $u_0 \neq 0$. Then, $\boldsymbol{u}$ is quasi-definite if and only if $\psi^{(k)} = \psi + \Theta\sum_{i=1}^{k-1} H^i\phi$ and $\phi^{(k)} = H^k\phi$ are coprime for all $k \geqslant 0$, and thus, $\boldsymbol{u}$ is q-classical. Furthermore, for any ϕ with* $\deg\phi \leqslant 2$, *the following different restrictions on the coefficients of ψ should be added*:

(a) $\deg\phi = 0$, *there are not restrictions on ψ*;
(b) $\deg\phi = 1$ *the coefficients of ψ should satisfy $[k]\bar{a} + \bar{b} \neq \dot{a}\bar{a}^{-1}q^{-k}\hat{b}$, $k \geqslant 0$*;
(c) $\deg\phi = 2$, *the coefficients of ψ should satisfy $(\phi = \hat{a}(x - a_1)(x - a_2))$,*

$$[k]\bar{a} + \bar{b} \neq -q^{-k}a_i([2k]\hat{a} + \hat{b}), \quad i = 1, 2, \ k \geqslant 0.$$

Proof. The first part is a simple consequence of the previous propositions and the fact that $\boldsymbol{u} \neq \mathbf{0}$.

(a) In this case ϕ is a constant, i.e., $\phi = c \in \mathbb{C}$, $c \neq 0$ and then, $\phi^{(k)} = c$, $k \geqslant 0$, so $\psi^{(k)}$ and $\phi^{(k)}$ are coprime.

(b) Here $\phi = \bar{a}x + \dot{a}$, $\bar{a} \neq 0$. Thus, $\phi^{(k)} = q^k \bar{a}x + \dot{a}$ and $a_0 = -\dot{a}/\bar{a}q^k$ is its zero. On the other hand,

$$\psi^{(k)} = ([2k]\hat{a} + \hat{b})x + ([k]\bar{a} + \bar{b}), \quad \hat{a} = 0, \ \hat{b} \neq 0,$$

and the condition $\psi^{(k)}(a_0) \neq 0$ becomes

$$-\hat{b} \cdot \frac{\dot{a}}{\bar{a}q^k} + [k]\bar{a} + \bar{b} = 0.$$

(c) Let ϕ be the polynomial $\phi = \hat{a}(x - a_1)(x - a_2) = \hat{a}x^2 - \hat{a}(a_1 + a_2)x + \hat{a}a_1a_2$. The zero a_0 of $\psi^{(k)}$ is given by

$$\psi^{(k)} = ([2k]\hat{a} + \hat{b})x - [k]\hat{a}(a_1 + a_2) + \bar{b}, \quad a_0 = \frac{[k]\hat{a}(a_1 + a_2) - \bar{b}}{[2k]\hat{a} + \hat{b}} = -\frac{[k]\bar{a} + \bar{b}}{[2k]\hat{a} + \hat{b}}.$$

Then, since $\phi^{(k)} = H^k\phi = \hat{a}(q^kx - a_1)(q^kx - a_2)$, and $\phi^{(k)}$ are coprime $\psi^{(k)}$ then $q^ka_0 \neq a_1$ and $q^ka_0 \neq a_2$. □

Remark A.6. Theorem A.5 says that if the quasi-definiteness condition holds, i.e., while there is no $n_0 \geqslant 0$ such that $\psi_{n_0}|\phi_{n_0}$, the sequence $(P_n)_{n\geqslant 0}$ will be orthogonal, but if such n_0 being $\psi_{n_0}|\phi_{n_0}$, appears, then $k_1^{(n_0)} = 0$ and therefore,

$$k_{n_0+1} = \langle \boldsymbol{u}, P_{n_0+1}^2 \rangle = 0 \Rightarrow g_{n_0+1} = 0,$$

In this case the polynomials $(P_n)_{n\geqslant 0}$ satisfy an TTRR where one of the coefficients g_{n_0+1} vanishes. This means that the sequence is orthogonal until the polynomial of degree $n_0 + 1$, i.e.,

$$P_{n_0+1} = (x - d_{n_0})P_{n_0} - g_{n_0}P_{n_0-1}, \quad g_n \neq 0, \ n_0 \geqslant n \geqslant 1.$$

Then, the condition that $\phi^{(k)}$ and $\psi^{(k)}$ are coprime, together with the necessary condition $[n]\hat{a}+\hat{b} \neq 0$, guarantees the existence of an infinite sequence of orthogonal polynomials.

Appendix B

In this appendix we will show an alternative algorithm [17] for finding the coefficients g_n in the three-term recurrence relation (3.1) and b_n in the structure relation (3.2).

A.1. The coefficient g_n

To obtain g_n we will follow [17]. First of all, since (3.9), the quantities $g_n = k_n/k_{n-1}$ and $g'_{n-1} = k'_{n-1}/k'_{n-2}$ satisfy Eq. (3.29),

$$g_n = \frac{[n]^2}{[n-1]^2} \frac{\hat{\lambda}_{n-1}}{\hat{\lambda}_n} g'_{n-1}.$$

Applying the previous result to g'_{n-1}, we get

$$g_n = [n]^2 \frac{\hat{\lambda}_{n-1}}{\hat{\lambda}_n} \frac{\hat{\lambda}^{(1)}_{n-2}}{\hat{\lambda}^{(1)}_{n-1}} \cdots \frac{\hat{\lambda}^{(n-2)}_1}{\hat{\lambda}^{(n-2)}_2} g'^{(n-1)}_1. \tag{B.1}$$

Next, since $\hat{\lambda}^{(k)}_i = [i]^\star([2k+i-1]\hat{a}+\hat{b})$, then almost all factors in (B.1) cancel out

$$\frac{\hat{\lambda}_{n-1}}{\hat{\lambda}_n} \cdot \frac{\hat{\lambda}^{(1)}_{n-2}}{\hat{\lambda}^{(1)}_{n-1}} \cdots \frac{\hat{\lambda}^{(n-3)}_2}{\hat{\lambda}^{(n-3)}_3} \frac{\hat{\lambda}^{(n-2)}_1}{\hat{\lambda}^{(n-2)}_2} = \frac{\hat{\lambda}_{n-1}[n-2]^\star \cdots [2]^\star [1]^\star}{[n]^\star \cdots [3]^\star \hat{\lambda}^{(n-2)}_2} = \frac{[n-2]\hat{a}+\hat{b}}{[n]^\star([2n-3]\hat{a}+\hat{b})},$$

and therefore the following relation holds:

$$g_n = \frac{q^{n-1}[n]([n-2]\hat{a}+\hat{b})}{[2n-3]\hat{a}+\hat{b}} g^{(n-1)}_1, \quad n \geqslant 2. \tag{B.2}$$

Now we determine the coefficient $g^{(n-1)}_1$. First of all, we will obtain the coefficient $g_1 = k_1/k_0$, $k_0 = \langle \boldsymbol{u}, 1\rangle = u_0$ and

$$k_1 = \langle \boldsymbol{u}, P^2_1\rangle = \langle \boldsymbol{u}, x^2 - 2d_0 x + d^2_0\rangle = u_2 - 2d_0 u_1 + d^2_0 u_0. \tag{B.3}$$

Using the difference equation (2.4) for the moments u_n we have

$$u_1 = -\frac{\bar{b}}{\hat{b}} u_0 = d_0 u_0, \quad u_2 = \frac{-1}{\hat{a}+\hat{b}}((\bar{a}+\bar{b})u_1 + \dot{a}u_0) = \frac{-1}{\hat{a}+\hat{b}}((\bar{a}+\bar{b})d_0 + \dot{a})u_0.$$

Therefore, substituting in (B.3), we find

$$k_1 = \frac{-1}{\hat{a}+\hat{b}}((\bar{a}+\bar{b})d_0 + \dot{a})u_0 - 2d^2_0 u_0 + d^2_0 u_0 = \frac{-u_0}{\hat{a}+\hat{b}}(\phi(d_0) + d_0 \underbrace{\psi(d_0)}_{=0}) = \frac{-u_0}{\hat{a}+\hat{b}}\phi(d_0).$$

The above gives $g_1 = -(1/(\hat{a}+\hat{b}))\phi(d_0)$. Next, we apply this result to $g^{(n-1)}_1$

$$g^{(n-1)}_1 = \frac{-1}{\hat{a}^{(n-1)} + \hat{b}^{(n-1)}} \cdot \phi^{(n-1)}(d^{(n-1)}_0), \quad d^{(n-1)}_0 = -\frac{\bar{b}^{(n-1)}}{\hat{b}^{(n-1)}}. \tag{B.4}$$

Finally using the explicit expressions for the coefficients $\hat{a}^{(n-1)}$, $\bar{a}^{(n-1)}\hat{b}^{(n-1)}$ and $\bar{b}^{(n-1)}$ (see Theorem 2.12) we find

$$g^{(n-1)}_1 \overset{(B.4)}{=} \underbrace{\frac{-1}{q^{2(n-1)}\hat{a} + [2(n-1)]\hat{a} + \hat{b}}}_{[2n-1]\hat{a}+\hat{b}} \cdot \underbrace{H^{n-1}\phi\left(-\frac{[n-1]\bar{a}+\bar{b}}{[2n-2]\hat{a}+\hat{b}}\right)}_{\phi^{(n-1)}(d^{(n-1)}_0)}, \tag{B.5}$$

which gives

$$g_n = -\frac{q^{n-1}[n]([n-2]\hat{a}+\hat{b})}{([2n-3]\hat{a}+\hat{b})([2n-1]\hat{a}+\hat{b})} \cdot \phi_{n-1}(d^{(n-1)}_0).$$

Finally, from

$$\phi^{(n-1)}(d^{(n-1)}_0) = H^{n-1}\phi\left(-\frac{[n-1]\bar{a}+\bar{b}}{[2n-2]\hat{a}+\hat{b}}\right)$$

and doing some straightforward calculations, an explicit representation for g_n in terms of the coefficients of ϕ and ψ, equivalent to (3.18), follows.

A.2. The coefficient b_n

To obtain b_n we start from the expression (3.7).

$$b_n = k_n^{-1}\langle \boldsymbol{u}, \phi\Theta P_n \cdot P_n\rangle = k_n^{-1}\langle \phi\boldsymbol{u}, \Theta P_n \cdot P_n\rangle = k_n^{-1}\langle \phi\boldsymbol{u}, \Theta P_n^2 - HP_n \cdot \Theta P_n\rangle$$

$$= -\underbrace{k_n^{-1}\langle \Theta(\phi\boldsymbol{u}), P_n^2\rangle}_{(1)} - \underbrace{k_n^{-1}\langle \boldsymbol{u}, \phi\Theta P_n \cdot HP_n\rangle}_{(2)}.$$

The first term in the above sum is

$$(1) = k_n^{-1}\langle \psi\boldsymbol{u}, P_n^2\rangle = k_n^{-1}\langle \boldsymbol{u}, \psi P_n \cdot P_n\rangle = k_n^{-1}(\hat{b}\langle \boldsymbol{u}, xP_n \cdot P_n\rangle + \bar{b}\langle \boldsymbol{u}, x^n P_n\rangle) = \hat{b}d_n + \bar{b} = \psi(d_n).$$

For the second term, since there is a dilation, the calculations are more *complicated*. To avoid this we will eliminate it by using the identity $HP_n = \Theta(xP_n) - x\Theta P_n$. Then,

$$(2) = k_n^{-1}\langle \phi\boldsymbol{u}, \Theta(xP_n) \cdot \Theta P_n\rangle - k_n^{-1}\langle \phi\boldsymbol{u}, x\Theta P_n \cdot \Theta P_n\rangle$$

$$= \underbrace{[n]k_n^{-1}\langle \boldsymbol{v}, \Theta(xP_n)Q_{n-1}\rangle}_{(3)} - \underbrace{[n]^2 k_n^{-1}\langle \boldsymbol{v}, xQ_n^2\rangle}_{(4)},$$

where the orthogonality of (Q_n) with respect to $\boldsymbol{v} = \phi\boldsymbol{u}$ has been used. From the TTRR (3) becomes

$$[n]k_n^{-1}\langle \boldsymbol{v}, \Theta(P_{n+1} + d_nP_n + g_nP_{n-1})Q_{n-1}\rangle = [n]k_n^{-1} \cdot [n]d_n\langle \boldsymbol{v}, Q_{n-1}^2\rangle = [n]^2k_n^{-1}d_nk'_{n-1}.$$

For (4), a straightforward calculation yields

$$(4) = [n]^2k_n^{-1}d'_{n-1}k'_{n-1}.$$

Substituting (3) and (4) in (2) we finally obtain

$$(2) = [n]^2k_n^{-1}k'_{n-1}(d_n - d'_{n-1}) \overset{(3.9)}{=} -\hat{\lambda}_n(d_n - d'_{n-1}),$$

and therefore the following representation for b_n:

$$b_n = -\psi(d_n) + \hat{\lambda}_n(d_n - d'_{n-1}),$$

holds. As before, d'_n denotes the coefficient of Q_n in the TTRR (3.4). Now we substitute the expression (3.33) for d'_n in (3.32) to find

$$b_n = -\psi(d_n) + (d_n - d'_{n-1})\hat{\lambda}_n = -\psi(d_n) + \left(d_n - q^{-1}\left(d_n + \frac{b_n}{\hat{\lambda}_n}\right)\right)\hat{\lambda}_n$$

$$= -\psi(d_n) + (d_n - q^{-1}d_n)\hat{\lambda}_n - q^{-1}b_n \quad\Leftrightarrow\quad [2]^{\star}b_n = -\psi(d_n) + (1 - q^{-1})\hat{\lambda}_nd_n.$$

If we now substitute the explicit expression for $\hat{\lambda}_n$ and use the identity $(1 - q^{-1})[n]^{\star} = q - q^{-n}$, we finally obtain a very closed form for the coefficient b_n in (3.2)

$$[2]^{\star}b_n = -\psi(d_n) + (1 - q^{-n})([n-1]\hat{a} + \hat{b})d_n = -\bar{b} + ([n-1]\hat{a} - q^{-n}([n-1]\hat{a} + \hat{b}))d_n.$$

References

[1] W.A. Al-Salam, Characterization theorems for orthogonal polynomials, in: P. Nevai (Ed.), Orthogonal Polynomials: Theory and Practice, NATO ASI Series C, Vol. 294, Kluwer Academic Publishers, Dordrecht, 1990, pp. 1–24.
[2] W.A. Al-Salam, T.S. Chihara, Another characterization of the classical orthogonal polynomials, SIAM J. Math. Anal. 3 (1972) 65–70.
[3] R. Álvarez-Nodarse, Polinomios hipergeométricos y q-polinomios, Monografías de la Academia de Ciencias de Zaragoza, Spain, (in Spanish), to appear.
[4] R. Álvarez-Nodarse, J. Arvesú, On the q-polynomials in the exponential lattice $x(s) = c_1 q^s + c_3$, Integral Transform. Special Funct. 8 (1999) 299–324.
[5] G.E. Andrews, q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra. Conference Series in Mathematics, Vol. 66, American Mathematical Society, Providence, RI, 1986.
[6] G.E. Andrews, R. Askey, Classical orthogonal polynomials, in: C. Brezinski, et al., (Eds.), Polynômes Orthogonaux et Applications, Lecture Notes in Mathematics, Vol. 1171, Springer, Berlín, 1985, pp. 36–62.
[7] N.M. Atakishiyev, M. Rahman, S.K. Suslov, On classical orthogonal polynomials, Constr. Approx. 11 (1995) 181–226.
[8] S. Bochner, Über Sturm–Liouvillesche polynomsysteme, Math. Zeit. 29 (1929) 730–736.
[9] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach Science Publishers, New York, 1978.
[10] C.W. Cryer, Rodrigues' formula and the classical orthogonal polynomials, Boll. Un. Mat. Ital. 25 (3) (1970) 1–11.
[11] J.A. Favard, Sur les polynômes de Tchebicheff, C.R. Acad. Sci. Paris. 200 (1935) 2052–2053.
[12] N.J. Fine, Basic Hypergeometric Series and Applications, Mathematical Surveys and Monographs, Vol. 27, American Mathematical Society, Providence, RI 1988.
[13] A.G. García, F. Marcellán, L. Salto, A distributional study of discrete classical orthogonal polynomials, J. Comput. Appl. Math. 57 (1995) 147–162.
[14] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.
[15] W. Hahn, Über höhere Ableitungen von Orthogonalpolynome, Math. Zeit. 43 (1937) 101.
[16] W. Hahn, Über orthogonalpolynomen die q-differentialgleichungen genügen, Math. Nachr. 2 (1949) 4–34.
[17] S. Häcker, Polinomiale Eigenwertprobleme zweiter Ordnung mit Hahnschen q-Operatoren, Doctoral Dissertation, Universität Stuttgart, Stuttgart, 1993 (in German).
[18] R. Koekoek, R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analog, Reports of the Faculty of Technical Mathematics and Informatics No. 98-17, Delft University of Technology, Delft, 1998.
[19] T.H. Koornwinder, Orthogonal polynomials in connection with quantum groups, in: P. Nevai (Ed.), Orthogonal Polynomials, Theory and Practice, Vol. 294, Kluwer Academic Publishers, Dordrecht, 1990, pp. 257–292.
[20] T.H. Koornwinder, Compact quantum groups and q-special functions, in: V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups, Pitman Research Notes in Mathematics Series, Vol. 311, Longman Scientific & Technical, New York, 1994, pp. 46–128.
[21] P. Lesky, Über Polynomsysteme, die Sturm–Liouvilleschen differenzengleichungen genügen, Math. Zeit. 78 (1962) 439–445.
[22] F. Marcellán, A. Branquinho, J. Petronilho, Classical orthogonal polynomials: a functional approach, Acta Appl. Math. 34 (1994) 203–283.
[23] P. Maroni, Une théorie algébrique des polynômes orthogonaux semiclassiques, in: C. Brezinski et al., (Eds.), Othogonal Polynomials and their Applications, IMACS Annals on Computing and Applied Mathematics, Vol. 9, Baltzer, Basel, 1991, pp. 95–130.
[24] J.C. Medem, Polinomios q-semiclásicos, Doctoral Dissertation, Universidad Politécnica, Madrid, 1996 (in Spanish).
[25] J.C. Medem, F. Marcellán, q-Classical polynomials: a very classical approach, Electron. Trans. Numer. Annal. 9 (1999) 112–127.
[26] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal Polynomials of a Discrete Variable, Springer Series in Computational Physics, Springer, Berlin, 1991.
[27] A.F. Nikiforov, V.B. Uvarov, Classical orthogonal polynomials in a discrete variable on nonuniform lattices, Preprint Inst. Prikl. Mat. Im. M. V. Keldysha Akad. Nauk SSSR, Moscú, Vol. 17, 1983 (in Russian).

[28] A.F. Nikiforov, V.B. Uvarov, Polynomial Solutions of hypergeometric type difference Equations and their classification, Integral Transform. Spec. Funct. 1 (1993) 223–249.
[29] S.K. Suslov, The theory of difference analogs of special functions of hypergeometric type, Uspekhi Mat. Nauk. 44:2 (1989) 185–226 (English translation Russian Math. Surveys 44(2) (1989) 227–278.
[30] N.Ja. Vilenkin, A.U. Klimyk, Representations of Lie Groups and Special Functions, Vol. I, II, III, Kluwer Academic Publishers, Dordrecht, 1992.
[31] S. Wolfram, The MATHEMATICA Book, 3rd Edition, Wolfram Media & Cambridge University Press, Cambridge, 1996.