

Exceptional orthogonal polynomials and generalized Schur polynomials

Yves Grandati

LCP A2MC, Université de Lorraine, 1 Bd Arago, 57078 Metz, Cedex 3, France.

We show that the exceptional orthogonal polynomials can be viewed as confluent limits of the generalized Schur polynomials introduced by Sergeev and Veselov.

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I. INTRODUCTION

The concept of exceptional orthogonal polynomials (EOP) has been introduced five years ago by Gomez-Ullate, Kamran and Milson [1, 2]. They form families of orthogonal polynomials whose degree sequences present a finite number of gaps but which span a complete basis of their corresponding Hilbert spaces. They appear to be closely related to exactly solvable quantum systems built from translationally shape invariant potentials (TSIP) via chains of Darboux-Bäcklund transformations and this connection has been the subject of an active research during the last years. The exceptional Hermite, Laguerre and Jacobi polynomials are expressible as Wronskians of the corresponding classical orthogonal polynomials and give, up to a gauge factor the eigenstates of these extended potentials (see [3] and [4] and references therein).

In this letter we show that these Wronskians can be viewed as confluent limit of alternants which coincide with the generalized Schur polynomials defined in their seminal paper by Sergeev and Veselov [5] and recently generalized by Harnad and Lee [6]. The confluent limit of the generalized Jacobi-Trudy formula established by Sergeev and Veselov is also discussed.

II. CHAIN OF STATE-DELETING DARBOUX-BÄCKLUND TRANSFORMATIONS (SDDBT)

A. One step DBT

We consider a one dimensional Hamiltonian $\hat{H} = -d^2/dx^2 + V(x)$, $x \in I \subset \mathbb{R}$ and the associated Schrödinger equation

$$\psi''_{\mu}(x) + (E_{\mu} - V(x)) \psi_{\mu}(x) = 0, \quad (1)$$

$\psi_{\mu}(x)$ being a formal eigenfunction of \hat{H} for the eigenvalue E_{μ} . In the following we suppose that, with Dirichlet boundary conditions on I , \hat{H} admits a discrete spectrum of energies and eigenstates of the $(E_n, \psi_n)_{n \in \{0, \dots, n_{\max}\} \subseteq \mathbb{N}}$ where, without loss of generality, we can always suppose that the ground level of \hat{H} is zero: $E_0 = 0$.

Starting from a given solution $\psi_{\nu}(x)$ associated to the value $\mu = \nu$ of the spectral parameter (eigenvalue E_{μ}), we define the first order operator $\hat{A}(w_{\nu})$ by

$$\hat{A}(w_{\nu}) = d/dx + w_{\nu}(x), \quad (2)$$

where $w_{\mu}(x) = -\psi'_{\mu}(x)/\psi_{\mu}(x)$. For $\mu \neq \nu$, the function defined via the Darboux-Crum formula

$$\psi_{\mu}^{(\nu)} = \hat{A}(w_{\nu}) \psi_{\lambda}(x) = \frac{W(\psi_{\nu}, \psi_{\mu} | x)}{\psi_{\nu}(x)}, \quad (3)$$

where $W(y_1, \dots, y_m | x)$ denotes the Wronskian of the family of functions y_1, \dots, y_m

$$W(y_1, \dots, y_m | x) = \begin{vmatrix} y_1(x) & \dots & y_m(x) \\ \vdots & & \vdots \\ y_1^{(m-1)}(x) & \dots & y_m^{(m-1)}(x) \end{vmatrix}, \quad (4)$$

is then a solution of the Schrödinger equation

$$\psi_\mu^{(\nu)''}(x) + \left(E_\mu - V^{(\nu)}(x)\right) \psi_\mu^{(\nu)}(x) = 0, \quad (5)$$

with the same energy E_λ as in Eq(1) but with a modified potential

$$V^{(\nu)}(x) = V(x) + 2w'_\nu(x), \quad (6)$$

We call $V^{(\nu)}(x)$ an extension of $V(x)$ and the correspondence

$$\begin{pmatrix} V(x) \\ \psi_\mu(x) \end{pmatrix} \xrightarrow{A(w_\nu)} \begin{pmatrix} V^{(\nu)}(x) \\ \psi_\mu^{(\nu)}(x) \end{pmatrix} \quad (7)$$

is called a Darboux-Bäcklund Transformations (DBT). The eigenfunction ψ_ν is the seed function of the DBT $A(w_\nu)$.

Note that $\hat{A}(w_\nu)$ annihilates ψ_ν and consequently the formula Eq(3) allows to obtain an eigenfunction of $V^{(\nu)}$ for the eigenvalue E_μ only when $\mu \neq \nu$. Nevertheless, we can readily verify that $1/\psi_\nu(x)$ is such an eigenfunction. By extension, we then define the "image" by $A(w_\nu)$ of the seed eigenfunction ψ_ν itself as

$$\psi_\nu^{(\nu)}(x) \sim 1/\psi_\nu(x). \quad (8)$$

B. Formal chains of DBT

At the formal level, the DBT can be straightforwardly iterated and a chain of m DBT is simply described by the following scheme

$$\begin{cases} \psi_\mu \xrightarrow{A(w_{\nu_1})} \psi_\mu^{(\nu_1)} \xrightarrow{A(w_{\nu_2}^{(N_1)})} \psi_\mu^{(N_2)} \dots \xrightarrow{A(w_{\nu_m}^{(N_{m-1})})} \psi_\mu^{(N_m)} \\ V \xrightarrow{A(w_{\nu_1})} V^{(\nu_1)} \xrightarrow{A(w_{\nu_2}^{(N_1)})} V^{(N_2)} \dots \xrightarrow{A(w_{\nu_m}^{(N_{m-1})})} V^{(N_m)}, \end{cases} \quad (9)$$

where N_j denotes the j -uple (ν_1, \dots, ν_j) (with $N_1 = \nu_1$) of spectral indices which completely characterizes the chain. We note $(N_m, \nu_{m+1}, \dots, \nu_{m+k})$ the chain obtained by adding to the chain N_m the DBT associated to the successive eigenfunctions $\psi_{\nu_{m+1}}^{(N_m)}, \dots, \psi_{\nu_{m+k}}^{(N_{m+k-1})}$.

$\psi_\mu^{(N_m)}$ is an eigenfunction associated to the eigenvalue E_μ of the potential (see Eq(6))

$$V^{(N_m)}(x) = V(x) + 2 \sum_{j=1}^m \left(w_{\nu_j}^{(N_{j-1})}(x)\right)' = V(x) - 2 \sum_{j=1}^m \left(\log \left(\psi_{\nu_j}^{(N_{j-1})}(x)\right)\right)'' . \quad (10)$$

It can be written as (cf Eq(3))

$$\psi_\mu^{(N_m)}(x) = \hat{A} \left(w_{\nu_m}^{(N_{m-1})}\right) \psi_\mu^{(N_{m-1})}(x) = \hat{A} \left(w_{\nu_m}^{(N_{m-1})}\right) \dots \hat{A} \left(w_{\nu_1}\right) \psi_\mu(x), \quad (11)$$

that is,

$$\psi_\mu^{(N_m)}(x) = \frac{W \left(\psi_{\nu_m}^{(N_{m-1})}, \psi_\mu^{(N_{m-1})} \mid x\right)}{\psi_{\nu_m}^{(N_{m-1})}(x)}. \quad (12)$$

A chain is non-degenerate if all the spectral indices ν_i of the chain N_m are distinct and is degenerate if some of them are repeated in the chain. For non-degenerate chains, Crum has established formulas which give the extended potentials and their eigenfunctions in terms of Wronskians of eigenfunctions of the initial potential [3, 7, 8].

Crum's formulas

When all the ν_j and λ are distinct, we have

$$\psi_{\mu}^{(N_m)}(x) = \frac{W^{(N_m, \mu)}(x)}{W^{(N_m)}(x)} \quad (13)$$

and

$$V^{(N_m)}(x) = V(x) - 2 \left(\log W^{(N_m)}(x) \right)'' , \quad (14)$$

where $W^{(N_m)}(x) = W(\psi_{\nu_1}, \dots, \psi_{\nu_m} | x)$.

The eigenfunctions $\psi_{\nu_1}, \dots, \psi_{\nu_m}$ of V are called the seed functions of the chain of DBT associated to the m -uple of spectral indices $N_m = (\nu_1, \dots, \nu_m)$.

C. Chains of SDDBT and partitions

In the following, we call state-deleting DBT (SDDBT) every DBT whose seed function is an eigenstate. A chain of SDDBT is then characterized by an m -uple $N_m = (n_1, \dots, n_m)$, $n_{i+1} > n_i \geq 0$, of distinct positive integers such that the corresponding seed functions ψ_{n_i} are eigenstates of the initial potential.

Consider a chain of SDDBT associated to a m -uple of spectral indices $N_m = (n_1, \dots, n_m)$ with $n_m > \dots > n_1 \geq 0$. We can equivalently characterize this chain by a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of length $l(\lambda) = m$, where $\lambda_1 > \dots > \lambda_m \geq 0$ and

$$\lambda_i = n_{m-i+1} - m + i. \quad (15)$$

λ is a partition of the integer

$$|\lambda| = \sum_{i=1}^m \lambda_i = \sum_{i=1}^m n_i - m(m-1)/2. \quad (16)$$

Note that, contrarily to the usual convention, we authorize λ to contain at its end a string of zeros (this string corresponding to a complete chain of SDDBT). The reduced form $\tilde{\lambda}$ of the partition is obtained by suppressing this last string of zeros. If λ does not contain such a chain we say that it is an irreducible partition.

To this partition is associated a Young diagram which characterizes the spectral shape of the extension. The λ_i are the lengths of the columns of the Young diagram starting from the left corner. Following [4, 9], we define the double partition λ^2 of λ as the partition of length $l(\lambda^2) = 2m$ defined as $\lambda^2 = (\lambda_1^2, \dots, \lambda_m^2)$, where we note λ_i^k for λ_i repeated k times ($\lambda_i^k = \lambda_i, \dots, \lambda_i$). We call an Adler partition a partition λ whose reduced form, $\tilde{\lambda} = (\lambda_1^{m_1}, \dots, \lambda_k^{m_k})$, is a double partition.

The qualifier state-deleting used above is somewhat abusive, since strictly speaking it have to be reserved to chains leading to regular potentials. Indeed, in this case, the action on the spectrum of the successive DBT based on eigenstates corresponds to suppress at each step the level associated to the used seed function, which then justifies the denomination "state-deleting" DBT.

Krein [10] and later Adler [11] have given a necessary and sufficient regularity condition for the final extensions of such non degenerate chains of SDDBT. The Krein-Adler theorem can be rewritten as a structural condition for the partition associated to the chain of SDDBT [4, 10, 11] as follows

Krein-Adler theorem

The final extension of the chain of SDDBT associated to the m -uple $N_m = (n_1, \dots, n_m)$, $n_{i+1} > n_i \geq 0$, or equivalently to the partition $\lambda = (\lambda_1^{m_1}, \dots, \lambda_k^{m_k}, 0^r)$, $\sum_{i=1}^k m_i + r = m$, is regular iff λ is an Adler partition, that is, iff $m_i \in 2\mathbb{N}$, $\forall i \in \{1, \dots, k\}$. The spectrum of the final extension $V^{(N_m)}(x)$ contains only even gaps (gaps constituted by an even number of consecutive missing levels).

Such a chain of SDDBT is said of the Krein-Adler type and in the particular case where $\tilde{\lambda} = 0$, the chain is said to be **complete**.

III. JACOBI-TRUDI FORMULA FOR EOP

A. PTSIP

Consider a potential $V(x; a)$ which depends upon a (multi)parameter $a \in \mathbb{R}^N$ and which admits a (finite or infinite) bound state spectrum $(E_n, \psi_n)_{n \geq 0}$, the ground level being supposed to be zero: $E_0(a) = 0$. In the framework of SUSY QM, such a potential is said to be shape invariant (SIP) [12–14] if its SUSY partner

$$V^{(0)}(x; a) = V(x; a) + 2w'_0(x; a), \quad (17)$$

keeps the same functional form as the initial potential. Namely

$$V^{(0)}(x; a) = V(x; f(a)) + R(a), \quad (18)$$

$R(a) \in \mathbb{R}$ and $f(a) \in \mathbb{R}^N$ being two given functions of a .

In this case, it can be shown [12–14] that the complete bound state energy spectrum of $\widehat{H}(a) = -\frac{d^2}{dx^2} + V(x; a)$ is given by:

$$E_n(a) = \sum_{k=0}^{n-1} R(a_k) = \sum_{i=0}^{n-1} E_1(a_i), \quad (19)$$

where $a_k = f^{(k)}(a) = \overbrace{f \circ \dots \circ f}^{\text{k times}}(a)$.

As for the corresponding eigenstates, they can be written as

$$\psi_n(x; a) \sim \widehat{A}^+(a)\psi_{n-1}(x; a_1) \sim \widehat{A}^+(a)\dots\widehat{A}^+(a_{n-1})\psi_0(x; a_n), \quad (20)$$

where $\widehat{A}^+(a) = -\frac{d}{dx} + w_0(x; a)$.

When f is a simple translation $f(a) = a + \varepsilon$, $\varepsilon = (\varepsilon^{(1)}, \dots, \varepsilon^{(N)}) \in \mathbb{R}^N$, V is said to be translationally shape invariant and we call it a TSIP. For all the known TSIP we have $a \in \mathbb{R}$ (first category TSIP) or $a \in \mathbb{R}^2$ (second category TSIP) [12, 13, 15].

The set of TSIP contains all the potentials classically known to be exactly solvable, ie for which we know explicitly the dispersion relation and whose the eigenfunctions can be expressed in closed analytical form in terms of elementary transcendental functions: the harmonic, isotonic, Morse, Kepler-Coulomb, Eckart, Darboux-Pöschl-Teller (hyperbolic and trigonometric) and Rosen-Morse (hyperbolic and trigonometric) potentials. These potentials are primary TSIP (PTSIP) from which it is possible in some cases to build infinite towers of secondary TSIP (STSIP) which are extensions of the previous ones and which share the same translational shape invariance properties [16].

An important feature of the the PTSIP is that their eigenfunctions ψ_n are equal, up to a gauge factor, to classical orthogonal polynomials in an appropriate variable z (which can be n dependent) and we say that ψ_n is quasi-polynomial in this variable. The confining (ie diverging at both boundaries of the definition interval) PTSIP, which then possess an infinite bound state spectrum, are the harmonic, isotonic and trigonometric Darboux-Pöschl-Teller (TDPT) potentials. For these ones, the gauge factor and the adapted variable are independent of n and their (unnormalized) eigenstates can be written as

$$\psi_n(x; a) = \psi_0(x; a)\Pi_n^a(z(x)), \quad (21)$$

where $\Pi_n^a(z)$ is a monic classical orthogonal polynomial (Hermite, Laguerre and Jacobi) [17, 18].

The $\Pi_n^a(z)$ satisfy the recursion relation

$$\begin{cases} \Pi_0^a(z) = 1 \\ z\Pi_n^a(z) = \Pi_{n+1}^a(z) + p_{n,a}\Pi_n^a(z) + q_{n,a}\Pi_{n-1}^a(z) \end{cases}, \quad (22)$$

with

$$\begin{cases} p_{n,a} = 0 \\ q_{n,a} = q_n = n/2. \end{cases}, \text{ for the Hermite case,} \quad (23)$$

$$\begin{cases} p_{n,a} = p_{n,\alpha} = 2n + 1 + \alpha \\ q_{n,a} = q_{n,\alpha} = n(n + \alpha). \end{cases}, \text{ for the Laguerre case,} \quad (24)$$

and

$$\begin{cases} p_{n,a} = p_{n,\alpha,\beta} = \frac{\beta^2 - \alpha^2}{(2n+2+\alpha+\beta)(2n+\alpha+\beta)} \\ q_{n,a} = q_{n,\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+1+\alpha+\beta)(2n+\alpha+\beta)^2(2n-1+\alpha+\beta)}. \end{cases}, \text{ for the Jacobi case.} \quad (25)$$

In the following, we give a resumed description of the spectral properties of the three confining PTSIP.

1. Harmonic oscillator

The harmonic oscillator (HO) potential (with zero ground level $E_0 = 0$) is defined on the real line by

$$V(x; \omega) = \frac{\omega^2}{4}x^2 - \frac{\omega}{2}, \quad \omega \in \mathbb{R}^+. \quad (26)$$

With Dirichlet boundary conditions at infinity it has the following spectrum ($z(x) = \sqrt{\omega/2}x$)

$$\begin{cases} E_n(\omega) = n\omega \\ \psi_n(x; \omega) = \psi_0(x; \omega)\tilde{H}_n(z) \end{cases}, \quad n \geq 0, \quad (27)$$

with

$$\psi_0(x; \omega) = \exp(-z^2/2) \quad (28)$$

and

$$\tilde{H}_n(z) = \frac{1}{2^n}H_n(z), \quad (29)$$

the $H_n(z)$ being the classical Hermite polynomials.

It is the most simple example of TSIP, with $a = \omega \in \mathbb{R}$ and $\varepsilon = 0$ (the parameter translation is of zero amplitude $a_1 = \omega$), that is

$$V^{(0)}(x; \omega) = V(x; \omega) + \omega. \quad (30)$$

2. Isotonic oscillator

The isotonic oscillator (IO) potential (with zero ground level $E_0 = 0$) is defined on the positive half line $]0, +\infty[$ by

$$V(x; \omega, \alpha) = \frac{\omega^2}{4}x^2 + \frac{(\alpha + 1/2)(\alpha - 1/2)}{x^2} - \omega(\alpha + 1), \quad \alpha > 1/2. \quad (31)$$

If we add Dirichlet boundary conditions at 0 and infinity and if we suppose $\alpha > 1/2$, it has the following spectrum ($z(x) = \omega x^2/2$)

$$\begin{cases} E_n(\omega) = 2n\omega \\ \psi_n(x; \omega, \alpha) = \psi_0(x; \omega, \alpha)\tilde{L}_n^\alpha(z) \end{cases}, \quad n \geq 0, \quad (32)$$

where

$$\psi_0(x; \omega, \alpha) = z^{(\alpha+1/2)/2} e^{-z/2} \quad (33)$$

and

$$\tilde{L}_n^\alpha(z) = (-1)^n n! L_n^\alpha(z), \quad (34)$$

the $L_n^\alpha(z)$ being the classical Laguerre polynomials.

It is a TSIP, with $a = (\omega, \alpha) \in \mathbb{R}^2$ and $\varepsilon = (0, +1)$

$$V^{(0)}(x; \omega, \alpha) = V(x; \omega, \alpha_1) + 2\omega. \quad (35)$$

3. TDPT

The trigonometric Darboux-Pöschl-Teller (TDPT) potential is defined on the interval $]0, \pi/2[$ by

$$V(x; \alpha, \beta) = \frac{(\alpha + 1/2)(\alpha - 1/2)}{\cos^2 x} + \frac{(\beta + 1/2)(\beta - 1/2)}{\sin^2 x} - (\alpha + \beta + 1)^2, \quad \alpha, \beta > 1/2. \quad (36)$$

With Dirichlet boundary conditions at 0 and $\pi/2$, it has the following spectrum

$$\left\{ \begin{array}{l} E_n(\alpha, \beta) = (\alpha_n + \beta_n + 1)^2 - (\alpha + \beta + 1)^2 = 4n(\alpha + \beta + 1 + n) \\ \psi_n(x; \alpha, \beta) = \psi_0(x; \alpha, \beta) \tilde{P}_n^{(\alpha, \beta)}(\cos 2x) \end{array} \right., \quad n \in \mathbb{N}, \quad (37)$$

where

$$\psi_0(x; \alpha, \beta) = (\sin x)^{\alpha+1/2} (\cos x)^{\beta+1/2} \quad (38)$$

and

$$\tilde{P}_n^{(\alpha, \beta)}(z) = \frac{2^n n!}{(\alpha + \beta + n + 1)_n} P_n^{(\alpha, \beta)}(z). \quad (39)$$

The $P_n^{(\alpha, \beta)}$ are the usual Jacobi polynomials, $(x)_n = x(x+1)\dots(x+n-1)$ is the Pochhammer symbol [17, 18] and $(\alpha_n, \beta_n) = (\alpha + n, \beta + n)$.

It is a TSIP, with $a = (\alpha, \beta) \in \mathbb{R}^2$ and $\varepsilon = (+1, +1)$

$$V^{(0)}(x; \alpha, \beta) = V(x; \alpha_1, \beta_1) + 4(\alpha + \beta + 2). \quad (40)$$

B. Jacobi-Trudi type formula for EOP

Due to the Crum formulas Eq(13) and Eq(14), the form of the extensions obtained from a PTSIP via chains of SDDBT as well as their eigenfunctions are determined by Wronskians of the type

$$W^{(N_m)}(x; a) = W(\psi_{n_1}(x; a), \dots, \psi_{n_m}(x; a) \mid x), \quad (41)$$

where the ψ_n are given by Eq(21). $W^{(N_m)}$ is characterized by the m-uple of spectral indices $N_m = (n_1, \dots, n_m)$ ($n_m > \dots > n_1 \geq 0$) or equivalently by the associated partition $\lambda = (\lambda_1, \dots, \lambda_m)$ (see Eq(15)). Considering the three confining PTSIP mentioned above, using the standard properties of Wronskians [22], this can be rewritten

$$W^{(N_m)}(x; a) = (\psi_0(x; a))^m W(\Pi_{n_1}^a(z), \dots, \Pi_{n_m}^a(z) | x) = (\psi_0(x; a))^m \left(\frac{dz}{dx}\right)^{m(m+1)/2} \mathcal{W}_\lambda(z), \quad (42)$$

where (see Eq(15))

$$\mathcal{W}_\lambda(z) = W(\Pi_{n_1}^a(z), \dots, \Pi_{n_m}^a(z) | z) = W(\Pi_{\lambda_m}^a(z), \dots, \Pi_{\lambda_1+m-1}^a(z) | z) \quad (43)$$

is a polynomial in z that we call in an abusive manner an exceptional orthogonal polynomial (EOP) enlarging the denomination associated to regular extensions [1]. Since, as we have seen above, the regularity is only related to a particular structure of the partition λ , we don't refer to it in the following.

For the monic classical orthogonal polynomials, we have [17, 18]

$$\frac{d}{dz}\Pi_n^a(z) = n\Pi_{n-1}^{a_1}(z), \quad (44)$$

which gives

$$\mathcal{W}_\lambda(z) = \begin{vmatrix} A_{\lambda_m}^0 \Pi_{\lambda_m}^a(z) & A_{\lambda_{m-1}+1}^0 \Pi_{\lambda_{m-1}+1}^a(z) & \dots & A_{\lambda_1+m-1}^0 \Pi_{\lambda_1+m-1}^a(z) \\ A_{\lambda_m}^1 \Pi_{\lambda_{m-1}}^{a_1}(z) & A_{\lambda_{m-1}+1}^1 \Pi_{\lambda_{m-1}}^{a_1}(z) & \dots & A_{\lambda_1+m-1}^1 \Pi_{\lambda_1+m-2}^{a_1}(z) \\ \dots & \dots & \dots & \dots \\ A_{\lambda_m}^{m-1} \Pi_{\lambda_{m-m+1}}^{a_{m-1}}(z) & A_{\lambda_{m-1}+1}^{m-1} \Pi_{\lambda_{m-1}-m+2}^{a_{m-1}}(z) & \dots & A_{\lambda_1+m-1}^{m-1} \Pi_{\lambda_1}^{a_{m-1}}(z) \end{vmatrix}, \quad (45)$$

where

$$A_n^k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}. \quad (46)$$

Defining

$$g_k^{(j)}(z) = A_{k+m-1-j}^{m-1-j} \Pi_k^{a_{m-1+j}}(z) = \frac{(k-1+m-j)!}{k!} \Pi_k^{a_{m-1+j}}(z) \quad (47)$$

and reversing the order of the columns and of the lines, we arrive to

$$\mathcal{W}_\lambda(z) = \begin{vmatrix} g_{\lambda_1}^{(0)}(z) & g_{\lambda_2-1}^{(0)}(z) & \dots & g_{\lambda_m-m+1}^{(0)}(z) \\ g_{\lambda_1+1}^{(1)}(z) & g_{\lambda_2}^{(1)}(z) & \dots & g_{\lambda_m-m+2}^{(1)}(z) \\ \dots & \dots & \dots & \dots \\ g_{\lambda_1+m-1}^{(m-1)}(z) & g_{\lambda_2+m-2}^{(m-1)}(z) & \dots & g_{\lambda_m}^{(m-1)}(z) \end{vmatrix}. \quad (48)$$

Consequently, the EOP $\mathcal{W}_\lambda(z)$ is amenable of a Jacobi-Trudi type formula a la Noumi [19], analogous to the one satisfied by the ϕ -factors in Noumi-Yamada approach of the rational solutions of the Painlevé equations. It has to be noticed that the generalized Hermite and Okamoto polynomials [20, 21] appear as particular exceptional Hermite polynomials in the enlarged sense given above.

IV. WRONSKIANS AND CONFLUENT ALTERNANTS

Consider the following alternant [22]

$$\Delta(\Phi | X) = \begin{vmatrix} \phi_1(x_1) & \dots & \phi_1(x_m) \\ \dots & \dots & \dots \\ \phi_m(x_1) & \dots & \phi_m(x_m) \end{vmatrix}, \quad (49)$$

where the ϕ_i are supposed polynomials of degree n_i and where we have noted $\Phi = (\phi_1, \dots, \phi_m)$ and $X = (x_1, \dots, x_m)$. For the case $\phi_k(x) = x^{k-1}$ the preceding determinant reduces to a Vandermondean

$$\Delta(X) = \begin{vmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \dots & \dots & \dots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{vmatrix} = \prod_{1 \leq i < j \leq m} (x_i - x_j). \quad (50)$$

$\Delta(\Phi | X)$ being a polynomial antisymmetric in the exchange of two variables x_i and x_j is divisible $\Delta(X)$ and the ratio $\frac{\Delta(\Phi | X)}{\Delta(X)}$ is a symmetric polynomial $S(\Phi | X)$.

In the case where the ϕ_k are monic polynomials of respective degrees k , then by linear combinations of the columns, we obtain

$$\Delta(\Phi | X) = \Delta(X). \quad (51)$$

We are interested in the confluent limit $x_i \rightarrow x, \forall i \in \{1, \dots, m\}$. Defining the new set of variable ε_k via $x_k = x + \varepsilon_k$, we have

$$\phi_i(x + \varepsilon_j) = \sum_{k=0}^{n_i} a_{k,i}(x) \varepsilon_j^k = p_i(\varepsilon_j), \quad (52)$$

with $a_{k,i}(x) = \phi_i^{(k)}(x)/k!$ and

$$\Delta(\Phi | X) = \begin{vmatrix} p_1(\varepsilon_1) & \dots & p_1(\varepsilon_m) \\ \dots & \dots & \dots \\ p_m(\varepsilon_1) & \dots & p_m(\varepsilon_m) \end{vmatrix} = \Delta(P | \varepsilon), \quad (53)$$

where $P = (p_1, \dots, p_m)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$. Moreover

$$\Delta(X) = \prod_{1 \leq i < j \leq m} (\varepsilon_i - \varepsilon_j) = \Delta(\varepsilon), \quad (54)$$

which implies

$$S(\Phi | X) = S(P | \varepsilon) \quad (55)$$

The ratio $S(P | \varepsilon) = \Delta(P | \varepsilon)/\Delta(\varepsilon)$ is also a symmetric polynomial and is consequently a continuous function of ε on \mathbb{R}^m . It results in particular that to calculate the value $S(P | 0)$, we can successively apply the limits $\varepsilon_1 \rightarrow 0, \varepsilon_2 \rightarrow 0, \dots, \varepsilon_m \rightarrow 0$ in this order. We then have the following result

Theorem 1:

In the confluent limit $x_i \rightarrow x, \forall i \in \{1, \dots, m\}$

$$S(\Phi | X) = \frac{\Delta(\Phi | X)}{\Delta(X)} \xrightarrow{\{x_i \rightarrow x\}} \frac{W(\phi_1, \dots, \phi_m | x)}{\prod_{j=1}^{m-1} j!}. \quad (56)$$

Proof:

From Eq(55) and Eq(52), we have

$$S(\Phi | X) = S(P | \varepsilon) = \frac{1}{\Delta(\varepsilon)} \begin{vmatrix} \sum_{k=0}^{n_1} a_{k,1}(x) \varepsilon_1^k & \dots & \sum_{k=0}^{n_1} a_{k,1}(x) \varepsilon_m^k \\ \dots & \dots & \dots \\ \sum_{k=0}^{n_m} a_{k,m}(x) \varepsilon_1^k & \dots & \sum_{k=0}^{n_m} a_{k,m}(x) \varepsilon_m^k \end{vmatrix}, \quad (57)$$

or by subtracting the first column to the following ones

$$S(\Phi | X) = \frac{1}{\Delta(\varepsilon_1, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) + O(\varepsilon_1) & \sum_{k=1}^{n_1} a_{k,1}(x) (\varepsilon_2^k - \varepsilon_1^k) & \dots & \sum_{k=1}^{n_1} a_{k,1}(x) (\varepsilon_m^k - \varepsilon_1^k) \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) + O(\varepsilon_1) & \sum_{k=0}^{n_m} a_{k,m}(x) (\varepsilon_2^k - \varepsilon_1^k) & \dots & \sum_{k=0}^{n_m} a_{k,m}(x) (\varepsilon_m^k - \varepsilon_1^k) \end{vmatrix}, \quad (58)$$

that is,

$$S(\Phi | X) = \frac{1}{\Delta(\varepsilon_2, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) + O(\varepsilon_1) & \sum_{k=1}^{n_1} a_{k,1}(x) \varepsilon_2^{k-1} (1 + O(\varepsilon_1)) & \dots & \sum_{k=1}^{n_1} a_{k,1}(x) \varepsilon_m^{k-1} (1 + O(\varepsilon_1)) \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) + O(\varepsilon_1) & \sum_{k=1}^{n_m} a_{k,m}(x) \varepsilon_2^{k-1} (1 + O(\varepsilon_1)) & \dots & \sum_{k=1}^{n_m} a_{k,m}(x) \varepsilon_m^{k-1} (1 + O(\varepsilon_1)) \end{vmatrix}. \quad (59)$$

If we take the limit $\varepsilon_1 \rightarrow 0$, ie $x_1 \rightarrow x$, we obtain

$$\lim_{x_1 \rightarrow x} S(\Phi | X) = \frac{1}{\Delta(\varepsilon_2, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) & \sum_{k=1}^{n_1} a_{k,1}(x) \varepsilon_2^{k-1} & \dots & \sum_{k=1}^{n_1} a_{k,1}(x) \varepsilon_m^{k-1} \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) & \sum_{k=1}^{n_m} a_{k,m}(x) \varepsilon_2^{k-1} & \dots & \sum_{k=1}^{n_m} a_{k,m}(x) \varepsilon_m^{k-1} \end{vmatrix}. \quad (60)$$

Subtracting the second columns to the following ones gives

$$\begin{aligned} S(\Phi | x, x_2, \dots, x_m) &= \frac{1}{\Delta(\varepsilon_2, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) & a_{1,1}(x) + O(\varepsilon_2) & \dots & \sum_{k=2}^{n_1} a_{k,1}(x) (\varepsilon_m^{k-1} - \varepsilon_2^{k-1}) \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) & a_{1,m}(x) + O(\varepsilon_2) & \dots & \sum_{k=2}^{n_m} a_{k,m}(x) (\varepsilon_m^{k-1} - \varepsilon_2^{k-1}) \end{vmatrix} \\ &= \frac{1}{\Delta(\varepsilon_3, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) & a_{1,1}(x) + O(\varepsilon_2) & \dots & \sum_{k=2}^{n_1} a_{k,1}(x) \varepsilon_m^{k-2} (1 + O(\varepsilon_2)) \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) & a_{1,m}(x) + O(\varepsilon_2) & \dots & \sum_{k=2}^{n_m} a_{k,m}(x) \varepsilon_m^{k-2} (1 + O(\varepsilon_2)) \end{vmatrix} \end{aligned} \quad (61)$$

and

$$S(\Phi | x, x, x_3, \dots, x_m) = \lim_{x_2 \rightarrow x} S(\Phi | x, x_2, \dots, x_m) = \frac{1}{\Delta(\varepsilon_3, \dots, \varepsilon_m)} \begin{vmatrix} a_{0,1}(x) & a_{1,1}(x) & \dots & \sum_{k=2}^{n_1} a_{k,1}(x) \varepsilon_m^{k-2} \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) & a_{1,m}(x) & \dots & \sum_{k=2}^{n_m} a_{k,m}(x) \varepsilon_m^{k-2} \end{vmatrix}. \quad (62)$$

The iteration is immediate and gives the researched result

$$S(\Phi | x, \dots, x) = \begin{vmatrix} a_{0,1}(x) & a_{1,1}(x) & \dots & a_{n_1-1,1}(x) \\ \dots & \dots & \dots & \dots \\ a_{0,m}(x) & a_{1,m}(x) & \dots & a_{n_m-1,m}(x) \end{vmatrix} = \frac{W(\phi_1, \dots, \phi_m | x)}{\prod_{j=1}^{m-1} j!}. \quad (63)$$

V. CONFLUENT LIMITS OF THE GENERALIZED SCHUR POLYNOMIALS AND OF THE GENERALIZED JACOBI-TRUDI FORMULA

A. EOP and generalized Schur polynomials

Sergeev and Veselov [5, 6] have defined the generalized Schur polynomials associated to the family of polynomials (Π_n^a) as

$$S_\lambda(Z) = \frac{\Delta(\Pi_\lambda | Z)}{\Delta(\Pi_0 | Z)}, \quad \lambda = (\lambda_1, \dots, \lambda_m), \quad (64)$$

where

$$\Delta(\Pi_\lambda | Z) = \begin{vmatrix} \Pi_{\lambda_1+m-1}^\alpha(z_1) & \dots & \Pi_{\lambda_1+m-1}^\alpha(z_m) \\ \dots & & \dots \\ \Pi_{\lambda_m}^\alpha(z_1) & \dots & \Pi_{\lambda_m}^\alpha(z_m) \end{vmatrix}. \quad (65)$$

Note that (see Eq(51))

$$\Delta(\Pi_0 | Z) = \Delta(Z). \quad (66)$$

From theorem 1, we then deduce immediately

$$S_\lambda(z) = \lim_{\{x_i \rightarrow x\}} S_\lambda(Z) = \frac{1}{m-1} \mathcal{W}_\lambda(z) \prod_{j=1}^{m-1} j! \quad (67)$$

and the EOP considered above appear to be the confluent limits of generalized Schur polynomials.

B. Confluent limit of the generalized Jacobi-Trudi formula

Sergeev and Veselov [5, 6], have shown that the generalized Schur polynomials satisfy a generalized Jacobi-Trudi formula of the form

$$S_\lambda(Z) = \begin{vmatrix} S_{\lambda_1}^{(0,m)}(Z) & S_{\lambda_1}^{(1,m)}(Z) & \dots & S_{\lambda_1}^{(m-1,m)}(Z) \\ S_{\lambda_2-1}^{(0,m)}(Z) & S_{\lambda_2-1}^{(1,m)}(Z) & \dots & S_{\lambda_2-1}^{(m-1,m)}(Z) \\ \dots & \dots & \dots & \dots \\ S_{\lambda_m-m+1}^{(0,m)}(Z) & S_{\lambda_m-m+1}^{(1,m)}(Z) & \dots & S_{\lambda_m-m+1}^{(m-1,m)}(Z) \end{vmatrix}, \quad (68)$$

where the multivariable polynomials $S_k^{(i,m)}(Z)$ verify the recursion relations

$$\begin{cases} S_k^{(i+1,m)}(Z) = S_{k+1}^{(i,m)}(Z) + p_{k+m-1,a} S_k^{(i,m)}(Z) + q_{k+m-1,a} S_{k-1}^{(i,m)}(Z) \\ S_k^{(i+1,m)}(Z) = z_1 S_k^{(i,m)}(Z) + S_{k+1}^{(i,m-1)}(Z) \end{cases}, \quad (69)$$

the coefficients p_k and q_k being given by Eq(23), Eq(24) and Eq(25).

$S_{k \geq 0}^{(0,m)}(Z)$ is the generalized Schur polynomial associated to a column Young diagram of height k , that is,

$$S_k^{(0,m)}(Z) = S_{(k,0,\dots,0)}(Z) = \frac{1}{\Delta(Z)} \begin{vmatrix} \Pi_{k+m-1}^a(z_1) & \dots & \Pi_{k+m-1}^a(z_m) \\ \Pi_{m-2}^a(z_1) & \dots & \Pi_{m-2}^a(z_m) \\ \dots & & \dots \\ \Pi_0^a(z_1) & \dots & \Pi_0^a(z_m) \end{vmatrix}, \quad (70)$$

which is extended to negative k by setting $S_{k<0}^{(0,m)}(Z) = 0$.

The confluent limit of the generalized JT formula Eq(68)

$$S_\lambda(z) = \begin{vmatrix} S_{\lambda_1}^{(0,m)}(z) & S_{\lambda_1}^{(1,m)}(z) & \dots & S_{\lambda_1}^{(m-1,m)}(z) \\ S_{\lambda_2-1}^{(0,m)}(z) & S_{\lambda_2-1}^{(1,m)}(z) & \dots & S_{\lambda_2-1}^{(m-1,m)}(z) \\ \dots & \dots & \dots & \dots \\ S_{\lambda_m-m+1}^{(0,m)}(z) & S_{\lambda_m-m+1}^{(1,m)}(z) & \dots & S_{\lambda_m-m+1}^{(m-1,m)}(z) \end{vmatrix}, \quad (71)$$

is a priori different in nature from the Jacobi-Trudi type formula given in Eq(48). This last does not appear as the confluent limit of Sergeev -Veselov's generalized Jacobi-Trudi formula but simply as a rewriting of Eq(67) on the basis of the explicit derivation formula Eq(44). More precisely, in the confluent limit, we deduce immediately from theorem 1

$$\begin{aligned} S_k^{(0,m)}(z) &= \frac{1}{m-1} W(\Pi_{k+m-1}^a(z), \Pi_{m-2}^a(z), \dots, \Pi_0^a(z) | z) \\ &= \frac{1}{\prod_{j=0}^{m-1} j!} \begin{vmatrix} \Pi_{k+m-1}^a(z) & \Pi_{m-2}^a(z) & \dots & \Pi_1^a(z) & 1 \\ (k+m-1)\Pi_{k+m-2}^{a_1}(z) & \dots & \dots & 1! & 0 \\ \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 0 \\ (k+m-1)\dots(k+1)\Pi_k^{a_{m-1}}(z) & (m-2)! & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & 0 \end{vmatrix}, \end{aligned} \quad (72)$$

that is, using Eq(44)

$$S_k^{(0,l)}(z) = \binom{k+l-1}{k} \Pi_k^{a_{l-1}}(z) = \frac{1}{(l-1)!} \frac{d^{l-1}}{dz^{l-1}} (\Pi_{k+l-1}^a(z)). \quad (73)$$

The $g_k^{(j)}(z)$ functions appearing in Eq(48) identify then to the $S_k^{(0,l)}(z)$ rather than to the $S_k^{(j,l)}(z)$.

The connection between the two Jacobi-Trudy type formulas becomes clearer if we return to the proof of Eq(71) in the specific confluent case. If we note

$$\mathbf{S}_\lambda^{(i,m)}(z) = \begin{pmatrix} S_{\lambda_1}^{(i,m)}(z) \\ \dots \\ S_{\lambda_m-m+1}^{(i,m)}(z) \end{pmatrix}, \quad (74)$$

we have

$$S_\lambda(z) = \det [\mathbf{S}_\lambda^{(0,m)}(z), \mathbf{S}_\lambda^{(1,m)}(z), \dots, \mathbf{S}_\lambda^{(m-1,m)}(z)]. \quad (75)$$

Eq(69) implies then

$$\mathbf{S}_\lambda^{(i+1,m)}(z) = z\mathbf{S}_\lambda^{(i,m)}(z) + \mathbf{S}_{\lambda+1}^{(i,m-1)}(z), \quad (76)$$

where $\lambda+1 = (\lambda_1+1, \dots, \lambda_m+1)$. By successively using Eq(76) and combining the columns from the last one to the second one, we obtain

$$\begin{aligned} S_\lambda(z) &= \det [\mathbf{S}_\lambda^{(0,m)}(z), \dots, \mathbf{S}_\lambda^{(m-3,m)}(z), \mathbf{S}_\lambda^{(m-2,m)}(z), z\mathbf{S}_\lambda^{(m-2,m)}(z) + \mathbf{S}_{\lambda+1}^{(m-2,m-1)}(z)] \\ &= \det [\mathbf{S}_\lambda^{(0,m)}(z), \dots, \mathbf{S}_\lambda^{(m-3,m)}(z), \mathbf{S}_\lambda^{(m-2,m)}(z), \mathbf{S}_{\lambda+1}^{(m-2,m-1)}(z)] \\ &= \det [\mathbf{S}_\lambda^{(0,m)}(z), \dots, \mathbf{S}_\lambda^{(m-3,m)}(z), z\mathbf{S}_\lambda^{(m-3,m)}(z) + \mathbf{S}_{\lambda+1}^{(m-3,m-1)}(z), \mathbf{S}_{\lambda+1}^{(m-2,m-1)}(z)] \\ &= \det [\mathbf{S}_\lambda^{(0,m)}(z), \dots, \mathbf{S}_\lambda^{(m-3,m)}(z), \mathbf{S}_{\lambda+1}^{(m-3,m-1)}(z), \mathbf{S}_{\lambda+1}^{(m-2,m-1)}(z)] \\ &= \dots \\ &= \det [\mathbf{S}_\lambda^{(0,m)}(z), \mathbf{S}_{\lambda+1}^{(0,m-1)}(z), \mathbf{S}_{\lambda+1}^{(1,m-1)}(z), \dots, \mathbf{S}_{\lambda+1}^{(m-2,m-1)}(z)]. \end{aligned} \quad (77)$$

Repeating the same procedure on the columns from the last one to the third one, gives

$$S_\lambda(z) = \det \left[\mathbf{S}_\lambda^{(0,m)}(z), \mathbf{S}_{\lambda+1}^{(0,m-1)}(z), \mathbf{S}_{\lambda+2}^{(0,m-2)}(z), \mathbf{S}_{\lambda+2}^{(1,m-2)}(z), \dots, \mathbf{S}_{\lambda+2}^{(m-3,m-2)}(z) \right]. \quad (78)$$

The recursion is immediate and we obtain

$$S_\lambda(z) = \det \left[\mathbf{S}_\lambda^{(0,m)}(z), \mathbf{S}_{\lambda+1}^{(0,m-1)}(z), \mathbf{S}_{\lambda+2}^{(0,m-2)}(z), \dots, \mathbf{S}_{\lambda+m-1}^{(0,1)}(z) \right] \quad (79)$$

$$= \begin{vmatrix} S_{\lambda_1}^{(0,m)}(z) & S_{\lambda_1+1}^{(0,m-1)}(z) & \dots & S_{\lambda_1+m-1}^{(0,1)}(z) \\ S_{\lambda_2-1}^{(0,m)}(z) & S_{\lambda_2}^{(0,m-1)}(z) & \dots & S_{\lambda_2+m-2}^{(0,1)}(z) \\ \dots & \dots & \dots & \dots \\ S_{\lambda_m-m+1}^{(0,m)}(z) & S_{\lambda_m-m+2}^{(0,m-1)}(z) & \dots & S_{\lambda_m}^{(0,1)}(z) \end{vmatrix},$$

that is,

$$S_\lambda(z) = \begin{vmatrix} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (\Pi_{\lambda_1+m-1}^a(z)) & \frac{1}{(m-2)!} \frac{d^{m-2}}{dz^{m-2}} (\Pi_{\lambda_1+m-1}^a(z)) & \dots & \Pi_{\lambda_1+m-1}^a(z) \\ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (\Pi_{\lambda_2+m-2}^a(z)) & \frac{1}{(m-2)!} \frac{d^{m-2}}{dz^{m-2}} (\Pi_{\lambda_2+m-2}^a(z)) & \dots & \Pi_{\lambda_2+m-2}^a(z) \\ \dots & \dots & \dots & \dots \\ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (\Pi_{\lambda_m}^a(z)) & \frac{1}{(m-2)!} \frac{d^{m-2}}{dz^{m-2}} (\Pi_{\lambda_m}^a(z)) & \dots & \Pi_{\lambda_m}^a(z) \end{vmatrix}. \quad (80)$$

We finally recover Eq(67) and consequently Eq(48).

VI. ACKNOWLEDGMENTS

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