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Some generalized hypergeometric d-orthogonal polynomial sets

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Abstract

In this paper, we characterize the d-orthogonal polynomial sets given by their explicit expressions in a specific basis. As application, we consider the generalized hypergeometric case to characterize d-orthogonal polynomial sets of Laguerre type, Meixner type, Meixner–Pollaczek type, Krawtchouk type, continuous dual Hahn type, and dual Hahn type. For $d = 1$, we obtain a unification of some characterization theorems in the orthogonal polynomials theory.

Keywords: Generalized hypergeometric polynomials; d-Orthogonal polynomials; Continuous dual Hahn polynomials; Dual Hahn polynomials; Meixner polynomials; Meixner–Pollaczek polynomials; Krawtchouk polynomials; Laguerre polynomials

1. Introduction

Over the past few years, there has been a growing interest in multiple orthogonal polynomials (see, for instance, [4–7,32–34]). This notion has many applications in various domains of mathematics as analytic number theory, approximation theory, special functions theory, and spectral theory of operators (see, for instance, [20]). A convenient framework to discuss explicit examples consists of considering a subclass of multiple orthogonal polynomials known as d-orthogonal polynomials, introduced by Van Iseghem [35] and completed by Maroni [29] as follows.

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n\geqslant 0}$ in $\mathcal{P}$ is called a *polynomial set* (PS, for short) if and only if $\deg P_n = n$ for all non-negative integer n. We denote by $\langle u, f\rangle$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n \geqslant 0$, the moments of u. Let $\{P_n\}_{n\geqslant 0}$ be a PS in $\mathcal{P}$. The corresponding dual sequence $(u_n)_{n\geqslant 0}$ is defined by

$$\langle u_n, P_m\rangle = \delta_{n,m}, \quad n, m = 0, 1, \ldots,$$

$\delta_{n,m}$ being the Kronecker symbol.

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Definition 1.1. (See Maroni [29] and Van Iseghem [35].) Let d be a positive integer and let $\{P_n\}_{n\geqslant 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n\geqslant 0}$ is called a d-orthogonal PS (d-OPS, for short) with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \ldots, u_{d-1})$ if it satisfies the following conditions:

$$\begin{cases} \langle u_k, P_m P_n\rangle = 0, & m > nd + k,\ n \geqslant 0, \\ \langle u_k, P_n P_{nd+k}\rangle \neq 0, & n \geqslant 0, \end{cases} \tag{1.1}$$

for each integer $k \in \{0, 1, \ldots, d-1\}$.

For the particular case $d = 1$, we meet the well-known notion of orthogonality. Maroni [29] showed that the conditions (1.1) are equivalent to the fact that the polynomials P_n, $n \geqslant 0$, satisfy a $(d+1)$-order recurrence relation of the type

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x), \tag{1.2}$$

where $\alpha_{d+1,d}(n)\alpha_{0,d}(n) \neq 0$, $n \geqslant d$, and by convention, $P_{-n} = 0$, $n \geqslant 1$.

This result, for $d = 1$, is reduced to the so-called Favard Theorem [19]. It was used to give several characterization theorems for d-OPS (see, for instance, [8–11,13–16,21–25]). This tool will be used in this paper to state other characterization theorems related to polynomials given by their explicit expansion in a specific basis. As application we solve some characterization problems associated with generalized hypergeometric polynomials.

The generalized hypergeometric functions are defined by (see, for instance, [28])

$${}_pF_q\left(\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{[\alpha_p]_m}{[\beta_q]_m}\frac{z^m}{m!}, \tag{1.3}$$

where

- p and q are positive integers or zeros,
- z is a complex variable,
- (α_p) designates the set $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$,
- $(\alpha)_k$ is the Pochhammer's symbol given by

  $$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1), \quad k = 1, 2, 3, \ldots,$$

- the numerator parameters $\alpha_1, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, β_j, $j = 1, \ldots, q$, being non-negative integers,
- $[\alpha_r]_k = \prod_{i=1}^r (\alpha_i)_k$. By convention, a product over the empty set is 1.

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates and we are led to a generalized hypergeometric polynomial. Many known orthogonal polynomials $\{P_n\}_{n\geqslant 0}$ have generalized hypergeometric representation of the form

$$P_n\big(\lambda(x); c, (\alpha_p), (\beta_q)\big) = {}_{1+s+p}F_q\left(\begin{matrix}-n, (\lambda_s(x)), (\alpha_p)\\(\beta_q)\end{matrix}; \frac{1}{c}\right), \tag{1.4}$$

where $\lambda_i(x)$, $i = 1, \ldots, s$, are polynomials of degree one not depending on n and $\lambda(x) = \prod_{i=1}^s \lambda_i(x)$.

If one of the β_j is equal to a negative integer, the corresponding sequence is finite.

The generalized hypergeometric polynomials (1.4) contain the OPSs given by Askey Scheme [27] and for which only one numerator parameter depends on n. That means: Laguerre, Meixner, Meixner–Pollaczek, continuous dual Hahn, dual Hahn and Krawtchouk polynomials. The last two ones correspond to finite sequences.

As far as we know, the only known d-OPS, $d > 1$, of type (1.4) corresponds to $s = 0$, $p = 0$ and $q = d$. That is a generalization of Laguerre polynomials, recently studied by the first author and Douak [8,10,11], Van Assche and Coussement [33] and Van Assche and Yakubovich [34].

The purpose of the paper is to state some conditions on the involved parameters in (1.4) to ensure the d-orthogonality of the corresponding PSs. This will lead us to obtain new generalized hypergeometric d-OPSs in Special Functions Theory.

The structure of the paper is as follows. In Section 2, we set and solve a d-Geronimus problem which consists to find necessary and sufficient conditions on coefficients, in a suitable basis, of a PS to be d-OPS. The obtained result is used in Section 3 to solve a characterization problem related to (1.4). We derive $2(d+1)$ classes of d-OPSs having generalized hypergeometric representation of type (1.4), that generalize in particular, Laguerre, Meixner, Charlier, Meixner–Pollaczek, continuous dual Hahn, Krawtchouk, dual Hahn families. The six 2-OPSs of type (1.4) were enumerated. The finite sequences obtained in this section seem to be the first finite ones introduced in the d-orthogonal polynomials theory. In Section 4, we discuss some questions to be treated, related to the obtained polynomials as well as to the method used in this paper.

2. A d-Geronimus problem type

A characterization problem consists to find all d-OPSs $\{P_n\}_{n\geqslant 0}$ satisfying a fixed property. When this property is related to the explicit expression of P_n in certain basis, the corresponding characterization problem will be called d-Geronimus problem type. More precisely, let us consider polynomial sequences $\{P_n\}_{n\geqslant 0}$ of the form

$$P_n(x)=\sum_{k=0}^{n}\gamma_n(k)\prod_{r=0}^{k-1}(x-x_r), \tag{2.1}$$

where the empty product is 1, $\gamma_n(k)$ is independent of x and $\{x_n=f(n)\}_{n\geqslant 0}$ is an arbitrary complex numbers sequence. A d-Geronimus problem type consists to characterize all d-OPSs $\{P_n\}_{n\geqslant 0}$ which take the form (2.1) for fixed conditions on $\{\gamma_n(k)\}_{n\geqslant 0,\,0\leqslant k\leqslant n}$ and $\{x_n\}_{n\geqslant 0}$. This appellation is justified by the fact that Geronimus [26] was the first to pose this problem for $d=1$ and $\gamma_n(k)=a_{n-k}b_k$ where $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ are two arbitrary complex sequences such that $b_n\neq 0$, $n\geqslant 0$. He gave necessary and sufficient conditions on the sequences $\{a_n\}$, $\{b_n\}$ and $\{x_n\}$ for which $\{P_n\}_{n\geqslant 0}$ is an OPS, but to identify such PS $\{P_n\}_{n\geqslant 0}$ more conditions, on the function f for instance, are required. Some special cases are known: The case $f\equiv 0$ corresponds to Brenke PS. It has been completely solved by Chihara [17,18]. The cases $f(x)=a+b\cos\pi x$ and $f(x)=aq^x$ were also treated by Al-Salam and Verma [2,3].

Recently, some d-Geronimus problem types, $d\geqslant 1$, were considered in [8,12,14,21,37], for $f\equiv 0$ and $f(x)=ax$ with various conditions on $\gamma_n(k)$. In this section, we consider the case where the function f is a polynomial and the coefficients $\gamma_n(k)$ satisfy: the quotient $\frac{\gamma_n(k+1)}{\gamma_n(k)}$ is a rational function in k of the type (2.3) below. Such conditions on f and $\gamma_n(k)$ appear if we use (1.3) to rewrite (1.4) (with an additional condition on λ_i; $1\leqslant i\leqslant s$) under the form (2.1) (see the proof of Theorem 3.3 below).

In this paper, we will show that many characterization theorems for orthogonal generalized hypergeometric polynomials may be deduced from the solution of this d-Geronimus problem for $d=1$.

2.1. Necessary and sufficient conditions

Throughout this paper, we will use the following notations:

- d being a positive integer.
- π being a polynomial defined by $\pi(x)=\sum_{k=1}^{\deg\pi}a_k x^{[k]}$, where $x^{[k]}$ denotes the falling factorial polynomials given by

$$x^{[0]}=1\quad\text{and}\quad x^{[k]}:=k!\binom{x}{k}=x(x-1)\cdots(x-k+1),\quad k=1,2,\ldots.$$

- $\mathbb{N}=\{0,1,\ldots\}$ being the set of non-negative integers.
- $\mathbb{C}$ being the set of complex numbers.
- $\mathfrak{B}=\{\mathcal{B}_k\}_{k\geqslant 0}$ designates the basis in $\mathcal{P}$ given by

$$\mathcal{B}_0(x)=1\quad\text{and}\quad\mathcal{B}_k(x)=\prod_{r=0}^{k-1}\bigl(x+\pi(r)\bigr),\quad k=1,2,\ldots.$$

Let us consider the PS $\{P_n\}_{n\geqslant 0}$ of the form

$$P_n(x)=\sum_{k=0}^{n}\gamma_n(k)\mathcal{B}_k(x), \tag{2.2}$$

where

$$\frac{\gamma_n(k+1)}{\gamma_n(k)}=\frac{(-n+k)}{(k+1)}\frac{N(k)}{cD(k)}, \qquad \gamma_n(0)=1, \quad k\leqslant n-1, \tag{2.3}$$

$D(x)=\sum_{r=0}^{\deg D} b_r x^{[r]}$ and N being two monic coprime polynomials with non-positive integer roots, and c is a non-zero complex number.

By convention, we put $\gamma_n(k)=0$, $k>n$.

These polynomials verify the following property.

Lemma 2.1. *The polynomial P_n defined by* (2.2) *satisfies the following relations*:

$$P_n(x)=\sum_{k=0}^{n}n^{[k]}A_k\mathcal{B}_k(x) \tag{2.4}$$

and

$$xP_n(x)=\sum_{k=1}^{n+1}\left(-ck\frac{D(k-1)}{N(k-1)}n^{[k-1]}-\pi(k)n^{[k]}\right)A_k\mathcal{B}_k(x), \tag{2.5}$$

where

$$A_0=1, \qquad A_k=\frac{(-1)^k}{c^k k!}\prod_{r=0}^{k-1}\frac{N(r)}{D(r)}, \quad k\geqslant 1. \tag{2.6}$$

Proof. From (2.3) and (2.6), it is easy to check that $\gamma_n(k)=n^{[k]}A_k$, $k=0,1,\ldots,n$.

Substituting this identity in (2.2), we get (2.4).

Consequently

$$xP_n(x)=\sum_{k=0}^{n}n^{[k]}A_k x\mathcal{B}_k(x).$$

Since $\mathcal{B}_{k+1}(x)=(x+\pi(k))\mathcal{B}_k(x)$, we deduce

$$xP_n(x)=\sum_{k=0}^{n}n^{[k]}A_k\mathcal{B}_{k+1}(x)-\sum_{k=0}^{n}n^{[k]}A_k\pi(k)\mathcal{B}_k(x).$$

On the other hand, we have $\pi(0)=0$ and $n^{[n+1]}=0$. Then, to derive (2.5), we write

$$\begin{aligned}xP_n(x)&=\sum_{k=1}^{n+1}n^{[k-1]}A_{k-1}\mathcal{B}_k(x)-\sum_{k=1}^{n+1}n^{[k]}A_k\pi(k)\mathcal{B}_k(x)\\&=\sum_{k=1}^{n+1}\left(-ck\frac{D(k-1)}{N(k-1)}n^{[k-1]}-\pi(k)n^{[k]}\right)A_k\mathcal{B}_k(x). \qquad \square\end{aligned}$$

A d-Geronimus type problem may be set as follows:

(P) *Find necessary and sufficient conditions on N, D, π and c ensuring the d-orthogonality of the polynomials given by* (2.2).

As a solution to this problem, we obtain the following theorem.

Theorem 2.2. *The only d-OPSs defined by* (2.2) *arise for $N = 1$ in the following cases.*

Case I: $\deg\pi < \deg D = d$.
Case II: $\deg D < \deg\pi = d$.
Case III: $\deg D = \deg\pi = d$, $c \neq a_d$.
Case IV: $\deg D = \deg\pi = d + 1$, $c = a_{d+1}$ *and* $\frac{a_d}{c} - b_d \notin \mathbb{N}$.

Until now, we suppose that D has non-positive integer roots. If we remove this hypothesis by considering the case where D has at least one positive integer root, say n_0 the least one. The corresponding sequence is finite, that is $\{P_n\}_{n=0}^{n=n_0}$,

$$P_n(x) = \sum_{k=0}^{n} \gamma_n(k)\mathcal{B}_k(x), \quad n = 0, 1, \ldots, n_0, \tag{2.7}$$

where

$$\frac{\gamma_n(k+1)}{\gamma_n(k)} = \frac{(-n+k)}{(k+1)}\frac{N(k)}{cD(k)}, \qquad \gamma_n(0) = 1, \quad n = 0, 1, \ldots, n_0,\ k = 0, \ldots, n-1,$$
$$D(x) = (x - n_0)D_1(x), \qquad D_1(k) \neq 0, \quad k = 0, \ldots, n_0 - 1,$$

and $\max(\deg D, d + \deg N, \deg\pi + \deg N) \leqslant n_0 - 1$.

We derive the following corollary.

Corollary 2.3. *The only d-OPSs defined by* (2.7) *arise for $N = 1$ in the following cases.*

Case A: $\deg\pi < \deg D = d$.
Case B: $\deg D < \deg\pi = d$.
Case C: $\deg D = \deg\pi = d$, $c \neq a_d$.
Case D: $\deg D = \deg\pi = d + 1$, $c = a_{d+1}$ *and* $\frac{a_d}{c} - b_d \notin \{0, 1, \ldots, n_0 - d - 1\}$.

2.2. Proof of Theorem 2.2

To prove Theorem 2.2, we need the following three lemmas.

Lemma 2.4. *Let $L(x) = \sum_{r=0}^{\deg L} b'_r x^{[r]}$ be a polynomial. Put $m = \max(\deg\pi, \deg L)$ and let c be a non-zero complex number. Put*

$$\alpha'_{j,m}(n) = (-1)^{m-j} \sum_{r=m-j}^{m} \left[c\binom{r+1}{m-j+1} b'_r - \binom{r}{m-j} a_r \right] n^{[r]}, \quad j = 0, 1, \ldots, m, \tag{2.8}$$

and

$$\alpha'_{m+1,m}(n) = -cL(n), \tag{2.9}$$

with $b'_j = 0$ if $\deg L < j \leqslant m$, and $a_j = 0$ if $\deg\pi < j \leqslant m$.

Then

$$-ckL(k-1)n^{[k-1]} - \pi(k)n^{[k]} = \sum_{r=0}^{m+1} \alpha'_{r,m}(n)(n - m + r)^{[k]}, \quad k \in \mathbb{N}^*,\ n \in \mathbb{N}, \tag{2.10}$$

and

$$\sum_{r=0}^{m+1} \alpha'_{r,m}(n) = 0, \quad n \in \mathbb{N}. \tag{2.11}$$

Proof. The use of the following identities:

$$k^{[r]}x^{[k]} = x^{[r]}\Delta^r (x-r)^{[k]}, \quad r \in \mathbb{N},\ k \geqslant 1,\ x \in \mathbb{C},$$
$$\Delta^r (x-r)^{[k]} = \sum_{j=0}^{r} \binom{r}{j} (-1)^j (x-j)^{[k]},$$

where $\Delta(P)(x) = P(x+1) - P(x)$, leads to

$$\begin{aligned}
\pi(k)x^{[k]} &= \sum_{r=0}^{m} a_r k^{[r]} x^{[k]} \\
&= \sum_{r=0}^{m} a_r x^{[r]} \sum_{j=0}^{r} (-1)^j \binom{r}{j} (x-j)^{[k]} \\
&= \sum_{j=0}^{m} (-1)^j \left(\sum_{r=j}^{m} \binom{r}{j} a_r x^{[r]} \right) (x-j)^{[k]} \\
&= \sum_{j=0}^{m} (-1)^{m-j} (x-m+j)^{[k]} \sum_{r=m-j}^{m} \binom{r}{m-j} a_r x^{[r]}.
\end{aligned}$$

On the other hand, using the following transformation:

$$k(k-1)^{[r]}x^{[k-1]} = x^{[r]}\Delta^{r+1}(x-r)^{[k]}, \quad r \in \mathbb{N},\ k \geqslant 1,\ x \in \mathbb{C},$$

we obtain

$$\begin{aligned}
ckL(k-1)x^{[k-1]} &= \sum_{r=0}^{m} c b'_r k(k-1)^{[r]} x^{[k-1]} \\
&= \sum_{r=0}^{m} b'_r x^{[r]} \sum_{j=0}^{r+1} c(-1)^j \binom{r+1}{j} (x+1-j)^{[k]} \\
&= cL(x)(x+1)^{[k]} + \sum_{j=1}^{m+1} c(-1)^j (x+1-j)^{[k]} \sum_{r=j-1}^{m} \binom{r+1}{j} b'_r x^{[r]} \\
&= cL(x)(x+1)^{[k]} + \sum_{j=0}^{m} c(-1)^{m-j+1} (x-m+j)^{[k]} \sum_{r=m-j}^{m} \binom{r+1}{m+1-j} b'_r x^{[r]}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&-ckL(k-1)x^{[k-1]} - \pi(k)x^{[k]} \\
&\quad = -cL(x)(x+1)^{[k]} + \sum_{j=0}^{m} (-1)^{m-j} (x-m+j)^{[k]} \sum_{r=m-j}^{m} \left(c\binom{r+1}{m+1-j} b'_r - \binom{r}{m-j} a_r \right) x^{[r]}. \qquad (2.12)
\end{aligned}$$

Now, we put $x = n$ in (2.12) to obtain (2.10).

From (2.8), we have

$$\begin{aligned}
\sum_{j=0}^{m} \alpha'_{j,m}(n) &= \sum_{j=0}^{m} (-1)^{m-j} \sum_{r=m-j}^{m} \left[c\binom{r+1}{m-j+1} b'_r - \binom{r}{m-j} a_r \right] n^{[r]} \\
&= \sum_{j=0}^{m} (-1)^j \sum_{r=j}^{m} \left[c\binom{r+1}{j+1} b'_r - \binom{r}{j} a_r \right] n^{[r]}
\end{aligned}$$

$$= \sum_{r=0}^{m}\sum_{j=0}^{r}(-1)^j\left[c\binom{r+1}{j+1}b'_r - \binom{r}{j}a_r\right]n^{[r]}$$
$$= \sum_{r=0}^{m}\left(\sum_{j=0}^{r}(-1)^j\binom{r+1}{j+1}\right)cb'_r n^{[r]} - \sum_{r=0}^{m}\left(\sum_{j=0}^{r}(-1)^j\binom{r}{j}\right)a_r n^{[r]}.$$

That, by virtue of the following identities:

$$\sum_{j=0}^{r}(-1)^j\binom{r+1}{j+1} = 1, \qquad \sum_{j=0}^{r}(-1)^j\binom{r}{j} = 0,$$

leads to $\sum_{j=0}^{m}\alpha'_{j,m}(n) = cL(n) = -\alpha'_{m+1,m}(n)$. From which, we deduce (2.11). $\square$

Remark 2.5. From the identity (2.8), we deduce

- $\alpha'_{0,m}(n) \neq 0$, $n \geqslant m$, if $cb'_m - a_m \neq 0$,
- $\alpha'_{0,m}(n) = 0$, $n \in \mathbb{N}$, if $cb'_m - a_m = 0$. In this case we have $\alpha'_{1,m}(n) \neq 0$, $n \geqslant m-1$ iff $\frac{a_{m-1}}{c} - b'_{m-1} \notin \mathbb{N}$.

Lemma 2.6. *Let P_n be a polynomial defined by* (2.2) *with $N = 1$. Put $m = \max(\deg\pi, \deg D)$.*
Then the PS $\{P_n\}_{n\geqslant 0}$ satisfies the following recurrence relation:

$$xP_n(x) = \sum_{j=0}^{m+1}\alpha'_{j,m}(n)P_{n-m+j}(x), \tag{2.13}$$

where the coefficients $\alpha'_{j,m}(n)$ are defined by (2.8) *and* (2.9)*, and by convention $P_{-n} = 0$, $n \geqslant 1$.*

Proof. From Lemma 2.1, with $N = 1$, we have

$$xP_n(x) = \sum_{k=1}^{n+1}\big(-ckD(k-1)n^{[k-1]} - \pi(k)n^{[k]}\big)C_k\mathcal{B}_k(x),$$

where

$$C_0 = 1, \qquad C_k = \frac{(-1)^k}{c^k k!}\prod_{r=0}^{k-1}\frac{1}{D(r)}, \quad k \geqslant 1. \tag{2.14}$$

Using (2.10) with $L = D$, we obtain

$$xP_n(x) = \sum_{k=1}^{n+1}\left(\sum_{j=0}^{m+1}\alpha'_{j,m}(n)(n-m+j)^{[k]}\right)C_k\mathcal{B}_k(x)$$
$$= \sum_{j=0}^{m+1}\alpha'_{j,m}(n)\left(\sum_{k=0}^{n+1}(n-m+j)^{[k]}C_k\mathcal{B}_k(x)\right)$$
$$= \sum_{j=0}^{m+1}\alpha'_{j,m}(n)P_{n-m+j}(x). \qquad \square$$

Lemma 2.7. *Let P_n be a polynomial defined by* (2.2) *with $N = 1$. Put $m = \max(\deg\pi, \deg D)$ and $D(x) = \sum_{r=0}^{m} b_r x^{[r]}$ where $b_j = 0$ if $j > \deg D$. Then $\{P_n\}_{n\geqslant 0}$ is r-OPS with $r \in \{m, m-1\}$.*
Moreover

1. *$\{P_n\}_{n\geqslant 0}$ is m-OPS if $cb_m - a_m \neq 0$.*
2. *$\{P_n\}_{n\geqslant 0}$ is $(m-1)$-OPS if $cb_m - a_m = 0$ and $\frac{a_{m-1}}{c} - b_{m-1} \notin \mathbb{N}$, $m \geqslant 2$.*

Proof. From Lemmas 2.4–2.6, we deduce that the sequence $\{P_n\}_{n\geqslant 0}$ satisfies the following recurrence relation:

$$xP_n(x)=\sum_{j=0}^{m+1}\alpha'_{j,m}(n)P_{n-m+j}(x), \tag{2.15}$$

where

$$\begin{aligned}
&\alpha'_{0,m}(n)=(-1)^m(cb_m-a_m)n^{[m]},\\
&\alpha'_{1,m}(n)=(-1)^{m-1}\big[\big(m(cb_m-a_m)+cb_m\big)(n-m+1)+cb_{m-1}-a_{m-1}\big]n^{[m-1]},\\
&\alpha'_{m+1,m}(n)=-cD(n)\neq 0.
\end{aligned}$$

From Remark 2.5, we deduce that: if $cb_m-a_m\neq 0$, then $\alpha'_{0,m}(n)\neq 0$, for $n\geqslant m$, $\alpha'_{m+1,m}(n)\neq 0$. Moreover, the PS $\{P_n\}_{n\geqslant 0}$ satisfies the recurrence relation (2.15). Then, in view of (1.2), we conclude that $\{P_n\}_{n\geqslant 0}$ is m-OPS. For $cb_m-a_m=0$ and $\frac{a_{m-1}}{c}-b_{m-1}\notin\mathbb{N}$, the identity (2.15) leads to

$$xP_n(x)=\sum_{j=0}^{m}\alpha'_{j+1,m}(n)P_{n-m+j+1}(x),\qquad \alpha'_{1,m}(n)\alpha'_{m+1,m}(n)\neq 0,\quad n\geqslant m-1.$$

Then, using (1.2), we deduce that $\{P_n\}_{n\geqslant 0}$ is $(m-1)$-OPS. $\square$

Remark 2.8. From Lemma 2.7, it follows that if P_n is a polynomial defined by (2.2) with $N=1$ and $\{P_n\}_{n\geqslant 0}$ is d-OPS, then $\max(\deg\pi,\deg D)\in\{d,d+1\}$, moreover, if $\deg\pi\neq\deg D$, then $\max(\deg\pi,\deg D)=d$.

Proof of Theorem 2.2. If $\{P_n\}_{n\geqslant 0}$ is a d-OPS of type (2.2), then it satisfies the condition (1.2). For $n\geqslant d$, replace in (1.2) xP_n and P_{n-d+k} by their explicit expressions given, respectively, by (2.5) and (2.4), to obtain

$$\sum_{k=0}^{n+1}\left(-ck\frac{D(k-1)}{N(k-1)}n^{[k-1]}-\pi(k)n^{[k]}\right)A_k\mathcal{B}_k(x)=\sum_{k=0}^{n+1}\left(\sum_{r=0}^{d+1}\alpha_{r,d}(n)(n-d+r)^{[k]}\right)A_k\mathcal{B}_k(x). \tag{2.16}$$

By identification and using the identity $(-x)^{[k]}=(-1)^k(x)_k$ we get, for $n\geqslant d$,

$$\begin{aligned}
&\sum_{r=0}^{d+1}\alpha_{r,d}(n)=0,\\
&ck\frac{D(k-1)}{N(k-1)}(-n)_{k-1}-\pi(k)(-n)_k=\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_k,\quad 1\leqslant k\leqslant n+1.
\end{aligned} \tag{2.17}$$

Substituting in (2.17) the following identities:

$$(-n)_{k-1}=\begin{cases}\frac{(-n)_d}{(-n-1+k)_{d+1-k}} & \text{if } 1\leqslant k\leqslant d+1,\\ (-n)_d(-n+d)_{k-d-1} & \text{if } d+1\leqslant k\leqslant n+1,\end{cases}$$

and

$$(-n+d-r)_k=\begin{cases}\frac{(-n-1+k)_{d+1-r}(-n+d-r)_r}{(-n-1+k)_{d+1-k}} & \text{if } 1\leqslant k\leqslant d+1,\\ (-n+d-r)_r(-n+d)_{k-1-d}(-n-1+k)_{d+1-r} & \text{if } d+1\leqslant k\leqslant n+1,\end{cases}$$

multiplying both sides of the obtained expression by

$$\varepsilon(n,d,k)=\begin{cases}(-n-1+k)_{d+1-k} & \text{if } 1\leqslant k\leqslant d+1,\\ \frac{1}{(-n+d)_{k-d-1}} & \text{if } d+1\leqslant k\leqslant n+1,\end{cases}$$

we get, for $1\leqslant k\leqslant n+1$,

$$ck(-n)_d\frac{D(k-1)}{N(k-1)}-(-n)_d\pi(k)(-n-1+k)=\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_r(-n-1+k)_{d+1-r}. \tag{2.18}$$

Put $Q_{n,d}(x)=\sum_{r=0}^{d+1}\alpha_{r,d}(n)(-n+d-r)_r(-n-1+x)_{d+1-r}$. It is easy to verify that $Q_{n,d}$ is a polynomial of degree $d+1$ and $Q_{n,d}(0)=0$.

Then the identity (2.18) can be rewritten under the form

$$cD(k-1)-N(k-1)\left[(-n-1+k)\frac{\pi(k)}{k}+\frac{Q_{n,d}(k)}{(-n)_d k}\right]=0,\quad k=1,\dots,n+1. \tag{2.19}$$

Put $R_{n,d}(x)=cD(x-1)-N(x-1)[(-n-1+x)\frac{\pi(x)}{x}-\frac{Q_{n,d}(x)}{(-n)_d x}]$. Since $\pi(0)=Q_{n,d}(0)=0$, $R_{n,d}$ is a polynomial in x of degree less or equal to $\max(\deg D,\deg\pi+\deg N,d+\deg N)$, not depending on n. Let $n\geqslant\max(d,\deg R_{n,d})$ for the general case and $n=n_0-1$ for the finite case. From (2.19), we deduce that $R_{n,d}$ has $n+1$ roots, namely $x=1,\dots,n+1$. So $R_{n,d}\equiv 0$ and consequently $\frac{D}{N}$ is a polynomial. But D and N are coprime. Then $N(x)=1$ for all $x\in\mathbb{C}$. As $\{P_n\}_{n\geqslant 0}$ is a d-OPS, by Lemma 2.7 $\max(\deg\pi,\deg D)\in\{d,d+1\}$ and $\deg D\leqslant\max(\deg\pi,d)$.

That leads us to the following three cases:

1. $\deg\pi<d$: For this case $\deg D=d$. Otherwise, according to Remark 2.8, $\{P_n\}_{n\geqslant 0}$ will not be a d-OPS.
2. $\deg\pi=d$: For this case $\deg D\leqslant d$. But if $\deg D=d$, we have, according to Lemma 2.7, the additional condition $a_d\neq c$.
3. $\deg\pi=d+1$: For this case, according to Lemma 2.7 and Remark 2.8, $\deg D=d+1$ with the additional conditions $c=a_{d+1}$ and $\frac{a_d}{c}-b_d\notin\mathbb{N}$. Otherwise $\{P_n\}_{n\geqslant 0}$ will not be a d-OPS.

From which we deduce the four cases described by Theorem 2.2.

To show the converse, we suppose $N=1$ and we consider the following cases.

Case I: $\deg\pi<\deg D=d$. In this case we have $d=\max(\deg\pi,\deg D)$, $a_d=0$ and $b_d=1$. That leads to $cb_d-a_d\neq 0$. Using Lemma 2.7 we deduce that $\{P_n\}_{n\geqslant 0}$ is a d-OPS.

Case II: $\deg D<\deg\pi=d$. In this case we have $d=\max(\deg\pi,\deg D)$, $a_d\neq 0$ and $b_d=0$. That leads to $cb_d-a_d\neq 0$. Using Lemma 2.7 we deduce that $\{P_n\}_{n\geqslant 0}$ is a d-OPS.

Case III: $\deg D=\deg\pi=d$, $c\neq a_d$. In this case we have $cb_d-a_d\neq 0$. Using Lemma 2.7 we deduce that $\{P_n\}_{n\geqslant 0}$ is a d-OPS.

Case IV: $\deg D=\deg\pi=d+1$, $c=a_{d+1}$ and $\frac{a_d}{c}-b_d\notin\mathbb{N}$. Using Lemma 2.7 we deduce that $\{P_n\}_{n\geqslant 0}$ is a d-OPS. □

2.3. Characterization of OPSs defined by (2.2)

Corollary 2.9. *A PS $\{P_n\}_{n\geqslant 0}$ defined by* (2.2) *is an OPS if and only if it is one of the following*:

(a) *a Laguerre PS*

$$P_n(x;\beta,c)={}_1F_1\left(\begin{matrix}-n\\ \beta\end{matrix};\frac{x}{c}\right), \tag{2.20}$$

(b) *a Charlier PS type*

$$P_n(x;w,c)=\sum_{k=0}^{n}\frac{(-n)_k}{k!c^k}\prod_{r=0}^{k-1}(x+wr),\quad w\neq 0, \tag{2.21}$$

(c) *a Meixner PS type*

$$P_n(x;\beta,w,c)=\sum_{k=0}^{n}\frac{(-n)_k}{k!c^k(\beta)_k}\prod_{r=0}^{k-1}(x+wr),\qquad w\neq 0,\ w\neq c, \tag{2.22}$$

(d) *a continuous dual Hahn PS type*

$$P_n\big(x;(\beta_2),c,a_1\big)=\sum_{k=0}^{n}\frac{(-n)_k}{k!c^k[\beta_2]_k}\prod_{r=0}^{k-1}\big(x+cr^{[2]}+a_1r\big),\qquad \frac{a_1}{c}-(1+\beta_1+\beta_2)\notin\mathbb{N}, \tag{2.23}$$

where if one of the parameters β, β_1 and β_2 is a negative integer, the corresponding sequence is finite.

Proof. According to Theorem 2.2, the only OPSs defined by (2.2) arise for $N=1$ in the following cases:

Case I: $\deg\pi<\deg D=1$: If we put $\pi=0$ and $D(x)=x+\beta$, we obtain (2.20) (see, for instance [27,31]).

Case II: $\deg D<\deg\pi=1$: If we put $\pi(x)=wx$ and $D=1$, we obtain (2.21) which, for $w=1$, leads to Charlier polynomials [19,27]

$$C_n(x;c)={}_2F_0\left(\begin{matrix}-n,x\\ -\end{matrix};\frac{1}{c}\right).$$

Case III: $\deg D=\deg\pi=1$, $c\neq a_1$: If we put $\pi(x)=wx$ and $D(x)=x+\beta$, $c\neq w$, we obtain (2.22) which, for $w=1$, leads to Meixner polynomials [19,27]

$$M_n\left(-x;\beta,\frac{c}{c-1}\right)={}_2F_1\left(\begin{matrix}-n,x\\ \beta\end{matrix};1-\frac{1}{c}\right).$$

If $\beta=-n_0$ and $w=1$, we are faced to Krawtchouk polynomials [31]

$$K_n(x;n_0,p)={}_2F_1\left(\begin{matrix}-n,x\\ -n_0\end{matrix};\frac{1}{p}\right),\qquad n\leqslant n_0.$$

Case IV: $\deg D=\deg\pi=2$, $c=a_2$, $\frac{a_1}{c}-b_1\notin\mathbb{N}$: If we put $\pi(x)=cx^{[2]}+a_1x$, $D(x)=(x+\beta_1)(x+\beta_2)$, and $\frac{a_1}{c}-(1+\beta_1+\beta_2)\notin\mathbb{N}$, we obtain (2.23), which, for $c=1$ and $a_1=2a+1$, leads to continuous dual Hahn polynomials [27,36]

$$S_n\big(x^2;\beta_1,\beta_2,a\big)={}_3F_2\left(\begin{matrix}-n,a+ix,a-ix\\ \beta_1,\beta_2\end{matrix};1\right),$$

with $2a-\beta_1-\beta_2\notin\mathbb{N}$. If one of the parameters β_1 and β_2 is a negative integer, the corresponding sequence is the dual Hahn polynomials [27]. $\square$

3. Characterization theorems for some generalized hypergeometric polynomials

3.1. *s-Separable product sets*

To state our main result, in this section we introduce the following notion.

Definition 3.1. Let $\{\lambda_1,\dots,\lambda_s\}$ be a set of s polynomials of degree one, $s\geqslant 1$. $\{\lambda_1,\dots,\lambda_s\}$ is called an s-separable product set if and only if there exists a polynomial π such that

$$\prod_{i=1}^{s}\big(\lambda_i(x)+y\big)=\left(\prod_{i=1}^{s}\lambda_i(x)\right)+\pi(y). \tag{3.1}$$

By convention, we say that the empty set is a 0-separable product set.

As immediate consequences of this definition, we note that:

(i) π is a monic polynomial of degree s having the form

$$\pi(y)=\prod_{i=1}^{s}\big(y+\lambda_i(0)\big)-\left(\prod_{i=1}^{s}\lambda_i(0)\right). \tag{3.2}$$

So $\pi(0)=0$.

(ii) Every $\{ax+b\}$ is a 1-separable product set.

For later applications, we need the following lemma which may be proved by simple calculation.

Lemma 3.2.

(i) *The only* 2*-separable product sets are of the form* $\{ax+b_1, -ax+b_2\}$.

(ii) *The only* 3*-separable product sets are of the form* $\{ax+b_1, jax+b_2, j^2ax+b_3\}$ *where* $j=e^{\frac{2i\pi}{3}}$ *and* $b_1+jb_2+j^2b_3=0$.

As an example of a general s-separable product set we mention $\{x\exp(\frac{2ki\pi}{s}),\ k=0,\dots,s-1\}$.

3.2. A characterization problem

Let us consider polynomial sequence $\{P_n\}_{n\geqslant 0}$ having generalized hypergeometric representations of the form

$$P_n\big(\lambda(x);c,(\alpha_p),(\beta_q)\big)={}_{1+s+p}F_q\left(\begin{matrix}-n,(\lambda_s(x)),(\alpha_p)\\(\beta_q)\end{matrix};\frac{1}{c}\right), \tag{3.3}$$

where $\{\lambda_i(x);\ i=1,\dots,s\}$ is an s-separable product set, $s\geqslant 0$, and $\lambda(x)=\prod_{i=1}^{s}\lambda_i(x)$.

Notice that if one of the denominator parameters $\beta_1,\dots,\beta_q$ is a negative integer, the corresponding sequence is finite.

Definition (3.1) contains all the generalized hypergeometric orthogonal polynomials given in Askey scheme [27] and for which only one numerator parameter depends on n. It contains also the finite sequence corresponding to Laguerre polynomials considered by Routh [30]. Such orthogonal polynomials may be distributed into four classes of polynomials defined by (3.3):

1. Laguerre class for which $p=0$, $s=0$, $q=1$ and $\frac{1}{c}=x$. It contains Laguerre polynomials ($\beta_1=\alpha+1$) and Routh polynomials ($\beta_1=-N$) [30].
2. Charlier class for which $p=0$, $s=1$, $q=0$, $\lambda_1(x)=ax+b$. It contains Charlier polynomials ($\lambda_1(x)=x$).
3. Meixner class for which $p=0$, $s=1$, $q=1$, $c\neq 1$, $\lambda_1(x)=ax+b$. It contains Meixner polynomials ($\lambda_1(x)=x$, $\beta_1=\beta$), Meixner–Pollaczek ($\lambda_1(x)=ix+\alpha$, $\beta_1=2\alpha$) and Krawtchouk polynomials ($\lambda_1(x)=x$, $\beta_1=-N$).
4. Dual Hahn class for which $p=0$, $s=2$, $q=2$, $c=1$, $\lambda_1(x)=ax+b_1$, $\lambda_2(x)=-ax+b_2$, $b_1+b_2-\beta_1-\beta_2\notin\mathbb{N}$. It contains dual Hahn polynomial ($\lambda_1(x)=-x$, $\lambda_2(x)=x+\gamma+\delta+1$, $\beta_1=\gamma+1$, $\beta_2=-N$) and continuous dual Hahn polynomials ($\lambda_1(x)=ix+a$, $\lambda_2(x)=-ix+a$, $\beta_1=a+b$, $\beta_2=a+c$, $-(b+c)\notin\mathbb{N}$).

Here, two natural questions arise:

(Q_1) *Are there other OPSs in each one of the previous four classes*?

(Q_2) *Are there other OPSs defined by* (3.3)?

To treat these questions, we consider the following more general problem:

(P_1) *Find necessary and sufficient conditions on* p, q, s *and* c *such that the PS* $\{P_n\}_{n\geqslant 0}$ *defined by* (3.3) *is a* d*-OPS.*

This problem, for $s=0$ and $d=1$, was treated by Abdul-Halim and Al-Salam [1]. The generalization to a positive integer d was solved by the first author and Douak [8,10,11]. The obtained polynomials for $d=2$ were also considered by Van Assche and Yakubovich [34].

As solution to this problem, we state the following.

Theorem 3.3. *There exist only* $2(d+1)$ *subclasses of d-OPSs defined by* (3.3) *corresponding to the following cases*:

Case 1: $p=0$; $s=0,1,\dots,d-1$; $q=d$.
Case 2: $p=0$; $s=d$; $q=0,1,\dots,d-1$.
Case 3: $p=0$; $s=q=d$; $c\neq 1$.
Case 4: $p=0$; $s=q=d+1$; $c=1$ *and* $\sum_{i=1}^{d+1}\lambda_i(0)-\sum_{i=1}^{d+1}\beta_i\notin\mathbb{N}$.

Proof. Using (3.3), (1.3) and the fact that $\{\lambda_1,\dots,\lambda_s\}$ is an s-separable product set, we have

$$\begin{aligned}
P_n\big(\lambda(x);c,(\alpha_p),(\beta_q)\big) &= {}_{1+s+p}F_q\left(\begin{matrix}-n,(\lambda_s(x)),(\alpha_p)\\(\beta_q)\end{matrix};\frac{1}{c}\right)\\
&=\sum_{k=0}^{n}\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}\prod_{i=1}^{s}\big(\lambda_i(x)\big)_k\\
&=\sum_{k=0}^{n}\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}\prod_{i=1}^{s}\prod_{r=0}^{k-1}\big(\lambda_i(x)+r\big)\\
&=\sum_{k=0}^{n}\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}\prod_{r=0}^{k-1}\prod_{i=1}^{s}\big(\lambda_i(x)+r\big)\\
&=\sum_{k=0}^{n}\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}\prod_{r=0}^{k-1}\big(\lambda(x)+\pi(r)\big),
\end{aligned}$$

where π is the polynomial of degree s given by (3.2).

It follows that

$$P_n\big(x;c,(\alpha_p),(\beta_q)\big)=\sum_{k=0}^{n}\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}\prod_{r=0}^{k-1}\big(x+\pi(r)\big). \tag{3.4}$$

Put $\gamma_n(k)=\frac{(-n)_k[\alpha_p]_k}{c^k k![\beta_q]_k}$. Then the coefficient $\gamma_n(k)$ satisfies the identity (2.3) with

$$N(x)=\begin{cases}1 & \text{if } p=0,\\ \prod_{j=1}^{p}(x+\alpha_j) & \text{if } p\geqslant 1,\end{cases}$$

and

$$D(x)=\begin{cases}1 & \text{if } q=0,\\ \prod_{i=1}^{q}(x+\beta_i) & \text{if } q\geqslant 1.\end{cases} \tag{3.5}$$

Consequently, the polynomials given by (3.4) are of type (2.2). We apply Theorem 2.2 to obtain the desired result. The different cases in Theorem 3.3 follow from the corresponding ones in Theorem 2.2. In fact, Case 1 (respectively Case 2) in Theorem 3.3 follows from Case I (respectively Case II) in Theorem 2.2, i.e. the case "$N=1$, $\deg\pi=s<\deg D=q=d$" (respectively "$N=1$, $\deg D=q<\deg\pi=s=d$") may be enumerated by "$p=0$, $s=0,1,\dots,d-1$, $q=d$" (respectively "$p=0$, $s=d$, $q=0,1,\dots,d-1$").

For Cases 3–4, we need to rewrite the polynomials D and π given respectively by (3.5) and (3.2) under the form

$$D(x)=\sum_{r=0}^{q}b_r x^{[r]}\quad\text{and}\quad \pi(x)=\sum_{k=1}^{s}a_k x^{[k]},$$

and to express a_s, a_{s-1} and b_{q-1} in terms of β_i, $1 \leqslant i \leqslant q$ and λ_j, $1 \leqslant j \leqslant s$. To this end, we recall that a monic polynomial f of degree m may be written under the two following forms:

$$\begin{aligned} f(x) &= x^m + \alpha_{m-1}x^{m-1} + Q_1(x), \quad Q_1 \text{ being polynomial of degree } < m-1 \\ &= x^{[m]} + \beta_{m-1}x^{[m-1]} + Q_2(x), \quad Q_2 \text{ being polynomial of degree } < m-1, \end{aligned}$$

and we have

$$x^m = x^{[m-1]} + \frac{m(m-1)}{2}x^{[m-1]} + Q_3(x), \qquad x^{m-1} = x^{[m-1]} + Q_4(x),$$

where Q_3 and Q_4 are two polynomials of degree $< m-1$. Then the coefficients α_{m-1} and β_{m-1} are related by

$$\beta_{m-1} = \alpha_{m-1} + \frac{m(m-1)}{2}.$$

Apply now this result to the polynomials D and π. Since

$$D(x) = x^q + \left(\sum_{i=1}^{q}\beta_i\right)x^{q-1} + D_1(x) = x^{[q]} + \left(\sum_{i=1}^{q}\beta_i + \frac{q(q-1)}{2}\right)x^{[q-1]} + D_2(x)$$

and

$$\pi(x) = x^s + \left(\sum_{i=1}^{q}\lambda_i(0)\right)x^{q-1} + \pi_1(x) = x^{[s]} + \left(\sum_{i=1}^{s}\lambda_i(0) + \frac{s(s-1)}{2}\right)x^{[s-1]} + \pi_2(x),$$

we have

$$a_s = 1, \qquad b_{q-1} = \sum_{i=1}^{q}\beta_i + \frac{q(q-1)}{2}, \qquad a_{s-1} = \sum_{i=1}^{s}\lambda_i(0) + \frac{s(s-1)}{2}.$$

From which we deduce the conditions given in Cases 3–4. □

Theorem 3.3 and Lemma 3.2, for $d = 1$, provide negative answers to questions (Q_1) and (Q_2). For $d = 2$, that allows us to enumerate the six classes of generalized hypergeometric 2-OPSs defined by (3.3) as follows.

Corollary 3.4. *A PS $\{P_n\}_{n\geqslant 0}$ defined by* (3.3) *is a* 2*-OPS iff it is one of the following*:

$${}_1F_2\left(\begin{matrix}-n\\ \beta_1,\beta_2\end{matrix};x\right),$$

$${}_2F_2\left(\begin{matrix}-n, ax+b\\ \beta_1,\beta_2\end{matrix};\frac{1}{c}\right), \quad a \neq 0,$$

$${}_3F_q\left(\begin{matrix}-n, ax+b_1, -ax+b_2\\ (\beta_q)\end{matrix};\frac{1}{c}\right), \quad q = 0,1,2,\ a \neq 0\ (c \neq 1 \text{ if } q = 2),$$

$${}_4F_3\left(\begin{matrix}-n, ax+b_1, jax+b_2, j^2ax+b_3\\ \beta_1,\beta_2,\beta_3\end{matrix};1\right), \quad a \neq 0,\ (b_1+b_2+b_3) - (\beta_1+\beta_2+\beta_3) \notin \mathbb{N},$$

where $j = e^{\frac{2i\pi}{3}}$ and $b_1 + jb_2 + j^2b_3 = 0$.

An equivalent version of Theorem 3.3 is given by the following

Theorem 3.5. *For fixed d and s, the only d-OPSs defined by* (3.3) *are the following*

$$P_n\big(\lambda(x); c, (\beta_q)\big) = {}_{1+s}F_q\left(\begin{matrix}-n, (\lambda_s(x))\\ (\beta_q)\end{matrix};\frac{1}{c}\right),$$

where

$$q = \begin{cases} 0, 1, \dots, d-1 & \textit{if } s = d, \\ d & \textit{if } s = d \textit{ and } c \neq 1 \textit{ or } s \in \{0, 1, \dots, d-1\}, \\ d+1 & \textit{if } s = d+1 \textit{ and } c = 1, \end{cases}$$

where for the last case the (β_{d+1}) *are chosen such that* $\sum_{i=1}^{d+1} \lambda_i(0) - \sum_{i=1}^{d+1} \beta_i \notin \mathbb{N}$.

For $s = 0$, we deduce the following main result obtained in [11] and characterizing d-OPSs of Laguerre type.

Corollary 3.6. *(See Ben Cheikh and Douak [11].) The only d-OPS of type*

$${}_{p+1}F_q \left(\begin{matrix} -n, (\alpha_p) \\ (\beta_q) \end{matrix}; x \right)$$

arises for $p = 0$, $q = d$.

Similar characterization theorems related to d-OPSs of Meixner type, Meixner–Pollaczek type, Krawtchouk type, continuous dual Hahn type, and dual Hahn type may be deduced from Theorem 3.5 for $s > 0$.

4. Concluding remarks

4.1. Limit relations

The limit relations between generalized hypergeometric orthogonal polynomials in Askey scheme are well known [27]. It is possible to state analogue ones for the polynomials obtained in this paper by the use of the following relations [28]:

$$\begin{aligned}
&{}_pF_q \left(\begin{matrix} (a_{p-1}), \mu \\ (b_{q-1}), \mu \end{matrix}; x \right) = {}_{p-1}F_{q-1} \left(\begin{matrix} (a_{p-1}) \\ (b_{q-1}) \end{matrix}; x \right), \\
&\lim_{\lambda \to +\infty} {}_pF_q \left(\begin{matrix} (a_{p-1}), \lambda a_p \\ (b_q) \end{matrix}; \frac{x}{\lambda} \right) = {}_{p-1}F_q \left(\begin{matrix} (a_{p-1}) \\ (b_q) \end{matrix}; a_p x \right), \\
&\lim_{\lambda \to +\infty} {}_pF_q \left(\begin{matrix} (a_p) \\ (b_{q-1}), \lambda b_q \end{matrix} \lambda x \right) = {}_pF_{q-1} \left(\begin{matrix} (a_p) \\ (b_{q-1}) \end{matrix}; \frac{x}{b_q} \right), \\
&\lim_{\lambda \to +\infty} {}_pF_q \left(\begin{matrix} (a_{p-1}), \lambda a_p \\ (b_{q-1}), \lambda b_q \end{matrix} \lambda x \right) = {}_{p-1}F_{q-1} \left(\begin{matrix} (a_{p-1}) \\ (b_{q-1}) \end{matrix}; \frac{a_p x}{b_q} \right).
\end{aligned}$$

4.2. Further d-OPSs

In this paper, we solved a d-Geronimus problem type to characterize generalized hypergeometric d-OPSs for which only one numerator depends on n. The method may be extended to investigate d-OPSs having generalized hypergeometric representations with more than one numerator parameters, depending on n and their q-analogues.

4.3. Properties of the obtained polynomials

The polynomials corresponding to $s = 0$ and $q = d$ obtained in Theorem 3.3 were deeply investigated by the first author and Douak [8,10,11]. They state some of the properties generalizing in a natural way the Laguerre polynomials: a differential equation of order $d + 1$, a generating functions, a differential formulas, links with known polynomial families, an explicit expression of the corresponding d-dimensional function vector. It is then significant to investigate analogue properties for some other d-OPSs given by Theorem 3.3, especially the finite sequences.

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