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The Representation of Orthogonal Polynomials in Terms of a Differential Operator*

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1. INTRODUCTION

In this paper we will give representations for the Laguerre polynomials, Meixner polynomials, and the Poisson–Charlier polynomials in terms of a differential operator containing their generating function. An example of the type of representations we obtain is the following:

$$yL_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{t^{n-k}(1-t)^{2k}}{k!} \frac{d^k y}{dt^k} \tag{1.1}$$

where $L_n^{\alpha}(x)$ is the nth Laguerre polynomial, and

$$y := (1-t)^{-1-\alpha} \exp(-xt(1-t)^{-1}) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\, t^n.$$

Also, we will find representation for x^n and the nth Jacobi polynomials in terms of a differential operator that does not contain their respective generating function.

The Rodrigues type formulas are examples of representations of classical orthogonal polynomials in terms of a differential operator involving the weight functions. It is known that the Hermite polynomials $H_n(x)$ and the ultraspherical polynomials $C_n^{\alpha}(x)$ also have representations in terms of a differential operator containing the generating function. In fact, Poli [4] showed that for $n > 0$,

$$yH_n(x) = \sum_{k=0}^{n} \binom{n}{k} t^k y^{(n-k)}, \tag{1.2}$$

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where $y^{(k)} = d^k y/dt^k$, and

$$y := e^{xt-t^2/2} = \sum_{n=0}^{\infty} H_n(x)\, t^n/n!$$

More recently, Horadze [2] showed that

$$yC_n{}^\lambda(x) = \sum_{k=0}^{n} \binom{w}{k} \frac{t^k}{y^{(n-k)/\lambda}(n-k)!} \frac{d^{n-k}y}{dt^{n-k}}, \tag{1.3}$$

where $w = n + 2\lambda - 1$, and

$$y := (1 - 2xt + t^2)^{-\lambda} = \sum_{k=0}^{\infty} C_k{}^\lambda(x)\, t^k.$$

Representations of the type given by Eqs. (1.1), (1.2), and (1.3) are not only aesthetically pleasing but also, as pointed out by Haradze [2], they are useful in finding the linearly independent solutions of a class of homogeneous differential equations. For example, it follows directly from Eq. (1.1) that the Laguerre polynomial $L_n{}^\alpha(x)$ is the characteristic polynomial for the differential equation

$$\sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{t^{n-k}(1-t)^{2k}}{k!} \frac{d^k y}{dt^k} = 0. \tag{1.4}$$

Thus if we let x_i, $i = 1, 2,\ldots, n$ be the n distinct zeros of $L_n{}^\alpha(x)$ then $(1-t)^{-1-\alpha} \exp(-x_i t(1-t)^{-1})$ are the linearly independent solutions of Eq. (1.4).

To obtain Eqs. (1.2) and (1.3) both Poli and Haradze used the following technique. Let $y(x, t)$ be the generating functions for the polynomials $P_n(x)$, $n = 0, 1, 2,\ldots$. First, the authors obtained an expression for the kth derivative of $y(x, t)$ with respect to t in the following form:

$$y^{(k)} = f_k(y, t)\, P_k(z(x, t)). \tag{1.5}$$

Next, they obtain a formula of the form

$$P_n(x) = \sum_{k=0}^{n} g_k(x, t)\, P_k(z(x, t)). \tag{1.6}$$

Then, by using Eq. (1.5) in Eq. (1.6) the required results are obtained. The method we use in this paper is similar to that used by Poli and Haradze.

2. Representations

According to Rainville [5, p. 202] the Laguerre polynomials $L_n^\alpha(x)$ are defined by the generating function.

$$y := (1-t)^{-1-\alpha} \exp(-xt(1-t)^{-1}) = \sum_{n=0}^{\infty} L_n^\alpha(x)\, t^n. \tag{2.1}$$

From the three term recursion relation for the Laguerre polynomials and the fact that,

$$z \frac{dL_k^\alpha(z)}{dz} = kL_n^\alpha(z) - (\alpha + k) L_{k-1}^\alpha(z),$$

one may show by mathematical induction on k that

$$y^{(k)} = k!(1-t)^{-k}\, yL_k^\alpha(z), \tag{2.2}$$

where $z = x(1-t)^{-1}$. This equation is of the form given by Eq. (1.5). In order to find a formula analogues to Eq. (1.6) we note the hypergeometric representation for $L_n^\alpha(x)$ and use the fact that (see Luke [3, p. 7])

$$\begin{aligned} &{}_{p+1}F_q\left(\begin{matrix} -n, & \alpha_1, \alpha_2 \cdots \alpha_p; \\ & \beta_1, \beta_2 \cdots \beta_q; \end{matrix}\ \lambda x\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k}\, {}_{p+1}F_q\left(\begin{matrix} -k, & \alpha_1 \cdots \alpha_p; \\ & \beta_1 \cdots \beta_q; \end{matrix}\ x\right) \end{aligned} \tag{2.3}$$

and thus obtain,

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\, t^{n-k}(1-t)^k L_k^\alpha(z)}{(n-k)!\,(1+\alpha)_k} \tag{2.4}$$

where $z = x(1-t)^{-1}$. By using Eq. (2.2) in Eq. (2.4), we obtain

$$yL_n^\alpha(x) = \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{t^{n-k}(1-t)^{2k}\, y^{(k)}}{k!}, \tag{2.5}$$

where y is defined by Eq. (2.1).

In both the Poisson–Charlier polynomial case and the Meixner polynomial case the equation for the kth derivative of their corresponding generating function can be found directly by using the Leibnitz rule for the differentiation of a product. For the Poisson–Charlier polynomials $\{c_n(x; a)\}$, as defined by Szego [6, p. 35] by the generating function

$$y := e^{-t}(1 + a^{-1}t)^x = \sum_{n=0}^{\infty} c_n(x; a)\, t^n/n!, \tag{2.6}$$

we obtain

$$y^{(k)} = yc_k(x; a + t). \tag{2.7}$$

For the Meixner polynomials $\{m_n(x; \beta, c)\}$ as defined in [1, p. 225] by the generating function

$$Y := (1 - t/c)^x (1 - t)^{-x-\beta} = \sum_{n=0}^{\infty} m_n(x; \beta, c)\, t^n/n!, \tag{2.8}$$

we obtain

$$Y^{(k)} = Y(1 - t)^{-k} m_k(x; \beta, (c - t)(1 - t)^{-1}). \tag{2.9}$$

In order to obtain formulas for the Poisson–Charlier polynomial case and the Meixner polynomial case that are analogous to Eq. (1.6) we use the same technique as was used for the Laguerre polynomial case and thus obtain,

$$c_n(x; a) = \sum_{i=0}^{n} \binom{n}{i} (1 + t/a)^i (t/a)^i c_i(x; t + a) \tag{2.10}$$

and

$$m_n(x; \beta, c) = \frac{(\beta)_n}{c^n} \sum_{k=0}^{n} \binom{n}{k} \frac{t^{n-k}(c - t)^k}{(\beta)_k} m_k(x; \beta, (c - t)(1 - t)^{-1}). \tag{2.11}$$

By using Eq. (2.7) in (2.10) and Eq. (2.9) in (2.11) we obtain the required results:

$$c_n(x; a)\, y = \sum_{i=0}^{n} \binom{n}{i} (1 + t/a)^i (t/a)^{n-i} y^{(i)}, \tag{2.12}$$

where y is given by Eq. (2.6) and

$$m_n(x; \beta, c)\, Y = \frac{(\beta)_n}{c^n} \sum_{k=0}^{n} \binom{n}{k} \frac{t^{n-k}(c - t)^k (1 - t)^k}{(\beta)_k} Y^{(k)} \tag{2.13}$$

where Y is defined by Eq. (2.8).

3. Applications

Even though we were unable to obtain a representation of the Jacobi polynomials in terms of a differential operator containing its generating function, we can obtain one in terms of a differential operator containing the generating function for the Gegenbauer polynomials. It is known see Rainville [5, p. 274] that the Jacobi polynomials $P_n^{\alpha,\beta}(x)$ satisfies the relation,

$$P_n^{\alpha,\beta}(x) = \frac{(1+\alpha)_n}{(1+\alpha+\beta)_n} \sum_{k=0}^{n} A(k,n)\, P_k^{\alpha,\alpha}(x), \tag{3.1}$$

where

$$A(k,n) = \frac{(-1)^{n-k}\,(\beta-\alpha)_{n-k}\,(1+\alpha+\beta)_{n+k}\,(1+2\alpha)_k\,(1+2\alpha+2k)}{(n-k)!\,(1+2\alpha)_{n+k+1}\,(1+\alpha)_k}.$$

By using this formula along with Haradze's results (see Eq. (1.3)) we obtain

$$yP_n^{\alpha,\beta}(x) = \frac{(1+\alpha)_n}{2^{2n}n!} \sum_{k=0}^{n} F_k(t) \frac{t^k}{y^{(n-k)/(2\alpha-1)}} \frac{d^{n-k}y}{dt^{n-k}}, \tag{3.2}$$

where $y = (1-2xt+t^2)^{-\alpha-1/2}$ and

$$F_k(t) = {}_4F_3\left[\begin{matrix}-k, & \beta-\alpha, & 1-2\alpha-2n, & \alpha-n; \\ -2n-\alpha-\beta, & -2\alpha-n, & 1-\alpha-n; & \end{matrix} \; -t^{-1}\right].$$

where $y = (1-2xt+t^2)^{-\alpha-1/2}$.

By using a technique similar to the one used to obtain Eq. (3.2), it is possible to obtain representations of other polynomials in terms of a differential operator not containing their generating function.

For example it is known that (see Rainville [5, p. 207])

$$x^n = n!(1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^k L_k^{\alpha}(x)}{(n-k)!\,(1+\alpha)_k}.$$

By using this equation along with Eq. (2.5) it is easy to deduce that

$$yx^n = (1-t)^n\,(1+\alpha)_n \sum_{i=0}^{n} \frac{(-n)_i\,(1-t)^i\,y^{(i)}}{(1+\alpha)_i\, i!} \tag{3.3}$$

where y is given by Eq. (2.1). From this equation it follows that the characteristic equation for the differential equation

$$\sum_{i=0}^{n} \frac{(-n)_i\,(1-t)^i\,y^{(i)}}{(1+\alpha)_i\, i!} = 0$$

is $x^n = 0$. Thus the linearly independent solutions of this differential equation are $t^j/(1-t)^{j+1+\alpha}$, where $j = 0, 1, 2, \ldots, n-1$.

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