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# Order of approximation by an operator involving biorthogonal polynomials

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## Abstract

The goal of this paper is to estimate the rate of convergence of a linear positive operator involving Konhauser polynomials to bounded variation functions on  $[0, 1]$ . To prove our main result, we have used some methods and techniques of probability theory.

**MSC:** 41A25; 41A35

**Keywords:** Konhauser polynomials; order of convergence; functions of bounded variation; Lebesgue-Stieltjes integration

## 1 Introduction

In 1965, Konhauser presented the general theory of biorthogonal polynomials [1]. Afterwards, in 1967 [2], he gave the following pair of biorthogonal polynomials:  $Y_\nu^{(n)}(x; k)$  and  $Z_\nu^{(n)}(x; k)$  ( $n > -1$ ,  $k \in \mathbb{Z}^+$ ), satisfying

$$Y_\nu^{(n)}(x; 1) = Z_\nu^{(n)}(x; 1) = L_\nu^{(n)}(x),$$

where  $L_\nu^{(n)}(x)$  are classical Laguerre polynomials and  $Y_\nu^{(n)}(x; k)$  Konhauser polynomials given by

$$L_\nu^{(n)}(x) = \sum_{j=0}^{\nu} (-1)^j \binom{\nu+n}{\nu-j} \frac{x^j}{j!},$$

and

$$Y_\nu^{(n)}(x; k) = \frac{1}{\nu!} \sum_{i=0}^{\nu} \frac{x^i}{i!} \sum_{j=0}^i (-1)^j \binom{i}{j} \left( \frac{j+n+1}{k} \right)_\nu.$$

The classical Meyer-König and Zeller (MKZ) operators are defined in 1960 [3] by

$$M_n(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right) \binom{n+k}{k} x^k, \quad x \in [0, 1),$$

$$M_n(f; 1) = f(1).$$

In order to give the monotonicity properties, Cheney and Sharma introduced the following type modification of the MKZ operators:

$$M_n^*(f; x) = (1 - x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) \binom{n+k}{k} x^k, \quad x \in [0, 1], \tag{1}$$

$$M_n^*(f; 1) = f(1).$$

In [4], they also introduced the operators for  $x \in [0, 1]$  and  $t \in (-\infty, 0]$

$$P_n(f; x) = \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t) x^k (1-x)^{n+1}, \tag{2}$$

$$P_n(f; 1) = f(1),$$

where  $L_k^{(n)}(t)$  denotes the classical Laguerre polynomials.

We consider the sequence of linear positive operators (similarly to [5]), which is another generalization of these operators including  $Y_v^{(n)}(x; k)$  Konhauser polynomials. For  $x \in [0, 1]$ ,  $t \in (-\infty, 0]$ , and  $k < n + 1$

$$(L_n f)(x, t; k) = \frac{1}{F_n(x, t)} \sum_{v=0}^{\infty} f\left(\frac{vk}{k(v-1) + n + 1}\right) Y_v^{(n)}(t; k) x^v, \tag{3}$$

where  $\{F_n(x, t)\}_{n \in \mathbb{N}}$  are the generating functions for the sequence of functions  $\{Y_v^{(n)}(x; k)\}_{v \in \mathbb{N}_0}$ , namely

$$F_n(x, t) = \sum_{v=0}^{\infty} Y_v^{(n)}(t; k) x^v, \quad n > 0, \tag{4}$$

and

$$F_n(x, t) = (1 - x)^{-\frac{n+1}{k}} \exp\{-t[(1 - x)^{-\frac{1}{k}} - 1]\}. \tag{5}$$

$Y_v^{(n)}(t; k) \geq 0$  for  $t \in (-\infty, 0]$ , due to Carlitz [6]. If we choose  $k = 1$  in (3), then the operators turn out to be (2). Similarly, if we choose  $k = 1$  and  $t = 0$  in (3), we get (1), which are called Bernstein power series by Cheney and Sharma in [4].

In view of (4) and (5), one has

$$1 = (1 - x)^{(n+1)/k} \exp\{t[(1 - x)^{-1/k} - 1]\} \sum_{v=0}^{\infty} Y_v^{(n)}(t; k) x^v, \quad n > 0.$$

For simplicity, we set

$$m_n(x, t; k) = (1 - x)^{(n+1)/k} \exp\{t[(1 - x)^{-1/k} - 1]\} Y_v^{(n)}(t; k) x^v. \tag{6}$$

In view of (6), we can write (3) as

$$(L_n f)(x, t; k) = \sum_{v=0}^{\infty} f\left(\frac{vk}{k(v-1) + n + 1}\right) m_n(x, t; k), \tag{7}$$

in which  $x \in [0, 1]$ ,  $t \in (-\infty, 0]$  and  $k < n + 1$ .

Our paper concerns the rate of pointwise convergence of the operators given by (7). In particular, by means of the techniques of probability theory, we shall estimate the rate of convergence for the operators  $(L_n f)$  for functions of a bounded variation on  $[0, 1]$  at points  $x$  where  $f(x+)$  and  $f(x-)$  exist.

It is worthwhile to say that our present results extend some earlier results. In fact, if we choose  $k = 1$  in  $L_n f$ , then the operators reduce to the operators mentioned [7], and if  $k = 1$  and  $t = 0$  in  $L_n f$ , then we get the operator investigated in [8].

For some important papers on different operators related to the present study we refer the readers to Bojanic [9] and Bojanic and Vuilleumier [10] where they estimated the rate of convergence of Fourier series and Fourier-Legendre series of functions of bounded variation, respectively. Cheng [11, 12] gave two results of this type for the Bernstein operators and Szász operators. After the fundamental studies of Bojanic-Vuilleumier and Cheng, their methods have been used widely in many follow up investigations (see, for instance [7, 8, 13–15]).

The main theorem of this work reads as follows.

**Theorem 1.1** *Let  $f$  be a function of a bounded variation on  $[0, 1]$  ( $f \in BV[0, 1]$ ). Then for each  $x \in (0, 1)$ ,  $t \in (-\infty, 0]$ ,  $k < n + 1$ , and  $n$  sufficiently large, we have*

$$\begin{aligned} & \left| (L_n f)(x, t; k) - \frac{f(x+) + f(x-)}{2} \right| \\ & \leq \frac{1}{n} \left[ \frac{(3x^2 - 4x + 2)(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{x(1-x)^2} + 2 \right] \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x) \right\} \\ & \quad + \frac{C^*}{\sqrt{n}} A(x, t; k) \{ |f(x+) - f(x-)| + \varepsilon_n(x) |f(x) - f(x-)| \}, \end{aligned}$$

where

$$\begin{aligned} & A(x, t; k) \\ & = \left( \frac{1}{k^3} e^{t-3t(1-x)^{-1/k}} (1-x)^{-\frac{3(2+k)}{k}} x (2e^{2t}((1-x)^{\frac{1}{k}} - t)^3 x^2 \right. \\ & \quad + 3e^{t+(1-x)^{-1/k}} (t - (1-x)^{\frac{1}{k}})(1-x)^{\frac{1}{k}} x (k((1-x)^{\frac{1}{k}} - t)(1-x)^{\frac{1}{k}} \\ & \quad + (t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})x) \\ & \quad + e^{2t(1-x)^{-\frac{1}{k}}} (1-x)^{\frac{2}{k}} (3k(t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})(1-x)^{\frac{1}{k}} x \\ & \quad + (-t^3 + 6t^2(1-x)^{\frac{1}{k}} - 7t(1-x)^{\frac{2}{k}} + (1-x)^{\frac{3}{k}})x^2 \\ & \quad \left. + k^2(-t + (1-x)^{\frac{1}{k}})(1-x)^{\frac{2}{k}}(1+x)) \right) \\ & \quad / \left( \frac{1}{k^2} e^{t-2t(1-x)^{-1/k}} (1-x)^{-\frac{2(2+k)}{k}} x (e^{-t}(t - (1-x)^{\frac{1}{k}})^2 x \right. \\ & \quad \left. + e^{t(1-x)^{-\frac{1}{k}}} (1-x)^{\frac{1}{k}} (k((1-x)^{\frac{1}{k}} - t)(1-x)^{\frac{1}{k}} + (t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})x) \right)^{3/2}, \end{aligned}$$

$$\varepsilon_n(x) = \begin{cases} 1, & x = \frac{vk'}{k'(v-1)+n+1} \text{ for some } k' \in \{0, 1, 2, \dots, n\}, \\ 0, & x \neq \frac{vk'}{k'(v-1)+n+1} \text{ for all } k' \in \{0, 1, 2, \dots, n\}, \end{cases}$$

$C^*$  is a certain constant,  $\bigvee_a^b(g_x)$  is a total variation of  $g_x$  on  $[a, b]$  and

$$g_x(s) = \begin{cases} f(s) - f(x+), & x < s \leq 1, \\ 0, & s = x, \\ f(s) - f(x-), & 0 \leq s < x. \end{cases}$$

### 2 Some lemmas

We now mention certain results which are necessary to prove our main theorem.

**Lemma 2.1** For every  $x \in [0, 1]$ ,  $t \in (-\infty, 0]$ , and  $k < n + 1$ , we have

$$\begin{aligned} (L_n 1)(x, t; k) &= 1, \\ (L_n s)(x, t; k) &\leq x - \frac{tx}{n(1-x)^{1/k}}, \\ (L_n s^2)(x, t; k) &< x^2 - \frac{tx^2}{n(1-x)^{1/k}} - \frac{ktx}{n(1-x)^{1/k}} + \frac{kx}{n}, \end{aligned} \tag{8}$$

and

$$(L_n (s-x)^2)(x, t; k) \leq \frac{x(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{n}. \tag{9}$$

*Proof* For the proof of (8) see [16].

To prove (9), we can use the following equality:

$$\begin{aligned} (L_n (s-x)^2)(x, t; k) &= \sum_{v=0}^{\infty} \left( \frac{vk}{k(v-1) + n + 1} - x \right)^2 m_n(x, t; k) \\ &\leq \sum_{v=0}^{\infty} \left( \frac{vk}{k(v-1) + n + 1} \right)^2 m_n(x, t; k) \\ &\quad + 2x \sum_{v=0}^{\infty} \left( \frac{vk}{k(v-1) + n + 1} \right) m_n(x, t; k) + x^2 \sum_{v=0}^{\infty} m_n(x, t; k). \end{aligned}$$

From (8), one has (9). So, this completes the proof of Lemma 2.1. □

**Lemma 2.2** For  $x \in (0, 1)$ ,  $t \leq 0$ ,

$$\begin{aligned} \lambda_n(x, t, y; k) &= \int_0^y K_n(x, t, s; k) ds, \quad 0 \leq s \leq x \\ &\leq \frac{x(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{n(x-y)^2}, \end{aligned}$$

where

$$K_n(x, t, y; k) = \begin{cases} \sum_{k \leq \frac{(n+1)y}{v(1-y)+y}} m_n(x, t; k), & 0 < y < 1, \\ 0, & y = 0. \end{cases}$$

*Proof* For  $s < y < x$ ,

$$\begin{aligned} \lambda_n(x, t, y; k) &= \int_0^y K_n(x, t, s; k) ds \\ &\leq \int_0^y K_n(x, t, s; k) \left(\frac{x-s}{x-y}\right)^2 ds \\ &= \frac{1}{(x-y)^2} \int_0^y K_n(x, t, s; k)(x-s)^2 ds \\ &\leq \frac{1}{(x-y)^2} (L_n(s-x)^2)(x, t; k) \\ &\leq \frac{x(k+4nx-t(1-x)^{-1/k}(k+3x))}{n(x-y)^2}. \end{aligned} \quad \square$$

**Lemma 2.3** For  $x \in [0, 1]$ ,  $t \in (-\infty, 0]$ , and  $k < n + 1$ ,

$$\sum_{\nu=0}^{\infty} \nu Y_{\nu}^{(0)}(t; k)x^{\nu} = \frac{e^{t-t(1-x)^{-1/k}}(-t+(1-x)^{\frac{1}{k}})(1-x)^{-1-\frac{2}{k}}x}{k}, \tag{10}$$

$$\begin{aligned} \sum_{\nu=0}^{\infty} \nu^2 Y_{\nu}^{(0)}(t; k)x^{\nu} &= \frac{1}{k^2} e^{t-t(1-x)^{-1/k}}(1-x)^{-2-\frac{3}{k}}x(k((1-x)^{\frac{1}{k}}-t)(1-x)^{\frac{1}{k}} \\ &\quad + (t^2-3t(1-x)^{\frac{1}{k}}+(1-x)^{\frac{2}{k}})x), \end{aligned} \tag{11}$$

$$\begin{aligned} \sum_{\nu=0}^{\infty} \nu^3 Y_{\nu}^{(0)}(t; k)x^{\nu} &= \frac{1}{k^3} e^{t-t(1-x)^{-1/k}}(1-x)^{-3-\frac{4}{k}}x(3k(t^2-3t(1-x)^{\frac{1}{k}}(1-x)^{\frac{2}{k}})(1-x)^{\frac{1}{k}}x \\ &\quad + (-t^3+6t^2(1-x)^{\frac{1}{k}}-7t(1-x)^{\frac{2}{k}}+(1-x)^{\frac{3}{k}})x^2 \\ &\quad + k^2(-t+(1-x)^{\frac{1}{k}})(1-x)^{\frac{2}{k}}(1+x)). \end{aligned} \tag{12}$$

*Proof* For  $n = 0$ , let  $Y_{\nu}^{(n)}(x; k)$  be the Konhauser polynomials' generating function defined by

$$\sum_{\nu=0}^{\infty} Y_{\nu}^{(0)}(t; k)x^{\nu} = (1-x)^{-\frac{1}{k}} \exp\{-t[(1-x)^{-\frac{1}{k}}-1]\}.$$

Taking derivatives of both sides with respect to  $x$  yields

$$\sum_{\nu=0}^{\infty} \nu Y_{\nu}^{(0)}(t; k)x^{\nu-1} = \frac{e^{t-t(1-x)^{-1/k}}(-t+(1-x)^{\frac{1}{k}})(1-x)^{-1-\frac{2}{k}}}{k}.$$

Editing the equation, we have

$$\sum_{\nu=0}^{\infty} \nu Y_{\nu}^{(0)}(t; k)x^{\nu} = \frac{e^{t-t(1-x)^{-1/k}}(-t+(1-x)^{\frac{1}{k}})(1-x)^{-1-\frac{2}{k}}x}{k}.$$

The proofs of (11), (12) are similar. □

**Lemma 2.4** *Let  $\zeta_1$  be the random variable with*

$$P(\zeta_1 = k) = (1 - x)^{1/k} \exp\{t[(1 - x)^{-1/k} - 1]\} Y_v^{(0)}(t; k)x^v.$$

*Then letting  $a_1 = E|\zeta_1|$ ,  $b_1^2 = E(\zeta_1 - a_1)^2$ ,*

$$\begin{aligned} E(\zeta_1 - a_1)^2 &= \frac{1}{k^2} e^{t-2t(1-x)^{-1/k}} (1-x)^{-\frac{2(2+k)}{k}} x (e^{-t} (t - (1-x)^{\frac{1}{k}})^2 x \\ &\quad + e^{t(1-x)^{-\frac{1}{k}}} (1-x)^{\frac{1}{k}} (k((1-x)^{\frac{1}{k}} - t)(1-x)^{\frac{1}{k}} \\ &\quad + (t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})x)) \end{aligned}$$

*and*

$$\begin{aligned} E|\zeta_1 - a_1|^3 &= \frac{1}{k^3} e^{t-3t(1-x)^{-1/k}} (1-x)^{-\frac{3(2+k)}{k}} x (2e^{2t} ((1-x)^{\frac{1}{k}} - t)^3 x^2 \\ &\quad + 3e^{t+t(1-x)^{-1/k}} (t - (1-x)^{\frac{1}{k}})(1-x)^{\frac{1}{k}} x (k((1-x)^{\frac{1}{k}} - t)(1-x)^{\frac{1}{k}} \\ &\quad + (t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})x) \\ &\quad + e^{2t(1-x)^{-\frac{1}{k}}} (1-x)^{\frac{2}{k}} (3k(t^2 - 3t(1-x)^{\frac{1}{k}} + (1-x)^{\frac{2}{k}})(1-x)^{\frac{1}{k}} x \\ &\quad + (-t^3 + 6t^2(1-x)^{\frac{1}{k}} - 7t(1-x)^{\frac{2}{k}} + (1-x)^{\frac{3}{k}})x^2 \\ &\quad + k^2(-t + (1-x)^{\frac{1}{k}})(1-x)^{\frac{2}{k}}(1+x)). \end{aligned}$$

*Proof* For  $n = 0$ , we have

$$a_1 = E(\zeta_1) = \sum_{v=0}^{\infty} v Y_v^{(0)}(t; k)x^v = \frac{e^{t-t(1-x)^{-1/k}} (-t + (1-x)^{\frac{1}{k}})(1-x)^{-1-\frac{2}{k}} x}{k},$$

$$b_1^2 = E(\zeta_1 - a_1)^2 = E(\zeta_1^2) - 2a_1 E(\zeta_1) + a_1^2 = E(\zeta_1^2) - (E(\zeta_1))^2,$$

*and*

$$E|\zeta_1 - a_1|^3 = E(\zeta_1^3) - 3a_1 E(\zeta_1^2) + 3a_1^2 E(\zeta_1) - a_1^3 = E(\zeta_1^3) - 3E(\zeta_1)E(\zeta_1^2) + 2(E(\zeta_1))^3.$$

If we use (10)-(12), the proof of the Lemma 2.4 is completed. □

Next, we recall the well-known Berry-Esséen bound for the classical central limit theorem of probability theory.

**Lemma 2.5** (Berry-Esséen) *Let  $\{\zeta_k\}_{k=1}^{\infty}$  be a sequence of independent and identically distributed random variables with finite variance such that the expectation  $E(\zeta_1) = a_1 \in \mathbb{R}$ , the variance  $\text{Var}(\zeta_1) = E(\zeta_1 - a_1)^2 = b_1^2 > 0$ . Assume  $E|\zeta_1 - a_1|^3 < \infty$ , then there exists a constant  $C$ ,  $1/\sqrt{2\pi} \leq C \leq 0.82$ , such that for all  $n = 1, 2, \dots$  and all  $t$ ,*

$$\left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^n (\zeta_k - a_1) \leq t\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du \right| < C \frac{E|\zeta_1 - a_1|^3}{b_1^3\sqrt{n}}.$$

Its proof can be found in Shirayev [17].

**Lemma 2.6** For  $x \in (0, 1)$ ,  $t \in (-\infty, 0]$  and  $k < n + 1$ , we have

$$\left| \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \frac{1}{2} \right| \leq \frac{C}{\sqrt{n}} A(x, t; k);$$

$A(x, t; k)$  is given in Theorem 1.1.

*Proof* By direct calculation, one has from Lemmas 2.4-2.5 the desired result. □

**Lemma 2.7** For all  $x \in (0, 1)$  define the function  $\text{sgn}(s - x)$  by

$$\text{sgn}(s - x) = \begin{cases} 1, & s > x, \\ 0, & s = x, \\ -1, & s < x. \end{cases}$$

We have

$$(L_n \text{sgn}(s - x))(x, t; k) = 2 \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - 1 + \varepsilon_n(x) m_n(x, t; k'),$$

where  $\varepsilon_n(x)$  is as in Theorem 1.1 and  $k' = \frac{vk}{k(v-1)+n+1}$ .

*Proof* If we apply the operator  $L_n$  to the function of  $\text{sgn}(s - x)$ , we have

$$\begin{aligned} (L_n \text{sgn}(s - x))(x, t) &= \sum_{v=0}^{\infty} \text{sgn}\left(\frac{vk}{k(v-1)+n+1} - x\right) m_n(x, t; k) \\ &= \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) + \sum_{k < \frac{(n+1)x}{v(1-x)+x}} (-1) m_n(x, t; k), \end{aligned}$$

and we can write

$$\begin{aligned} (L_n 1)(x, t) &= \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) + \sum_{k < \frac{(n+1)x}{v(1-x)+x}} m_n(x, t; k) + \varepsilon_n(x) m_n(x, t; k'), \\ \sum_{k < \frac{(n+1)x}{v(1-x)+x}} m_n(x, t; k) &= 1 - \left[ \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \varepsilon_n(x) m_n(x, t; k') \right]. \end{aligned}$$

Thus

$$\begin{aligned} (L_n \text{sgn}(s - x))(x, t; k) &= \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \left[ 1 - \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \varepsilon_n(x) m_n(x, t; k') \right] \\ &= 2 \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - 1 + \varepsilon_n(x) m_n(x, t; k'). \end{aligned}$$

This completes the proof of Lemma 2.7. □

**Lemma 2.8** *If the conditions of Theorem 1.1 hold, we have for all  $x \in (0, 1)$*

$$\begin{aligned} & \left| \frac{f(x+) - f(x-)}{2} (L_n \operatorname{sgn}(s - x))(x, t; k) + \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] (L_n \delta)(x, t; k) \right| \\ & \leq \frac{C^*}{\sqrt{n}} A(x, t; k) \{ |f(x+) - f(x-)| + \varepsilon_n(x) |f(x) - f(x-)| \}, \end{aligned} \tag{13}$$

where  $\delta_x(s)$  is the Dirac delta function,  $C^*$  is a certain constant, and  $A(x, t; k)$  is given in Theorem 1.1.

*Proof* By direct calculation, we get

$$(L_n \delta)(x, t; k) = \varepsilon_n(x) m_n(x, t; k').$$

One has

$$\begin{aligned} & \left| \frac{f(x+) - f(x-)}{2} (L_n \operatorname{sgn}(s - x))(x, t; k) + \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] (L_n \delta)(x, t; k) \right| \\ & \leq \left| \frac{f(x+) - f(x-)}{2} (L_n \operatorname{sgn}(s - x))(x, t; k) \right| + \left| \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] (L_n \delta)(x, t; k) \right| \\ & \leq \left| \frac{f(x+) - f(x-)}{2} \left[ 2 \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - 1 + \varepsilon_n(x) m_n(x, t; k') \right] \right| \\ & \quad + \left| \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] \varepsilon_n(x) m_n(x, t; k') \right| \\ & \leq |f(x+) - f(x-)| \left| \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \frac{1}{2} \right| + |f(x) - f(x-)| \varepsilon_n(x) m_n(x, t; k'). \end{aligned}$$

According to Lemma 2.6, one has

$$\left| \sum_{\frac{(n+1)x}{v(1-x)+x} < k} m_n(x, t; k) - \frac{1}{2} \right| \leq \frac{C}{\sqrt{n}} A(x, t; k), \tag{14}$$

and using the method of proof of Lemma 2.6 and evaluations which are similar to the work in [13], we have

$$m_n(x, t; k') =: \frac{C_1}{\sqrt{n}} A(x, t; k). \tag{15}$$

Set  $C^* = \max\{C, C_1\}$ . Consequently from (14) and (15) we get (13).

This completes the proof of Lemma 2.8. □

**Lemma 2.9** *For  $n$  sufficiently large, we have*

$$\begin{aligned} & |(L_n g_x)(x, t; k)| \\ & \leq \frac{1}{n} \left[ \frac{(3x^2 - 4x + 2)(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{x(1-x)^2} + 2 \right] \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x) \right\}, \end{aligned}$$

$g_x(s)$  is given in Theorem 1.1.



*Proof* We recall the Lebesgue-Stieltjes integral representations

$$(L_n f)(x, t; k) = \int_0^1 f(y) d_y \lambda_n(x, t, y; k), \tag{16}$$

where  $\lambda_n(x, t, y; k) = \int_0^y K_n(x, t, s; k) ds$ ,  $0 \leq s \leq x$ . From (16), we can rewrite  $(L_n g_x)(x, t; k)$  as follows:

$$\begin{aligned} |(L_n g_x)(x, t; k)| &= \left| \sum_{v=0}^{\infty} g_x \left( \frac{vk}{k(v-1) + n + 1} \right) m_n(x, t; k) \right| \\ &= \left| \int_0^1 g_x(y) d_y \lambda_n(x, t, y; k) \right|. \end{aligned} \tag{17}$$

To estimate (17), we decompose it into three parts as follows:

$$\int_0^1 g_x(y) d_y \lambda_n(x, t, y; k) = (I_1 g_x)(x, t; k) + (I_2 g_x)(x, t; k) + (I_3 g_x)(x, t; k), \tag{18}$$

where

$$\begin{aligned} (I_1 g_x)(x, t) &= \int_0^{x-x/\sqrt{n}} g_x(y) d_y \lambda_n(x, t, y; k), \\ (I_2 g_x)(x, t) &= \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(y) d_y \lambda_n(x, t, y; k), \\ (I_3 g_x)(x, t) &= \int_{x+(1-x)/\sqrt{n}}^1 g_x(y) d_y \lambda_n(x, t, y; k). \end{aligned}$$

We shall evaluate  $(I_1 g_x)(x, t)$ ,  $(I_2 g_x)(x, t)$ , and  $(I_3 g_x)(x, t)$ .

First we estimate  $(I_2 g_x)(x, t)$ , for  $y \in [x - x/\sqrt{n}, x + (1 - x)/\sqrt{n}]$ .

Note that  $g_x(x) = 0$  and  $|\int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} d_y \lambda_n(x, t, y; k)| \leq 1$ , so we have

$$\begin{aligned} |(I_2 g_x)(x, t; k)| &= \left| \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x(y) - g_x(x)) d_y \lambda_n(x, t, y; k) \right| \\ &\leq \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} |g_x(y) - g_x(x)| d_y \lambda_n(x, t, y; k) \\ &\leq \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x) \leq \frac{1}{n-1} \sum_{l=2}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x). \end{aligned} \tag{19}$$

Next we estimate  $(I_1 g_x)(x, t)$ . Using partial Lebesgue-Stieltjes integrations, we obtain

$$\begin{aligned} (I_1 g_x)(x, t; k) &= \int_0^{x-x/\sqrt{n}} g_x(y) d_y \lambda_n(x, t, y; k) \\ &= g_x(x - x/\sqrt{n}) \lambda_n(x, t, x - x/\sqrt{n}; k) - g_x(0) \lambda_n(x, t, 0; k) \\ &\quad - \int_0^{x-x/\sqrt{n}} \lambda_n(x, t, y; k) d_y g_x(y). \end{aligned}$$

Since  $|g_x(x - x/\sqrt{n})| = |g_x(x - x/\sqrt{n}) - g_x(x)| \leq \sqrt[ x - x/\sqrt{n} ]{g_x}$ , it follows that

$$|(I_1 g_x)(x, t; k)| \leq \int_{x-x/\sqrt{n}}^x (g_x) \lambda_n(x, t, x - x/\sqrt{n}; k) + \int_0^{x-x/\sqrt{n}} \lambda_n(x, t, y; k) d_y \left( -\sqrt[ x ]{g_x} \right).$$

From Lemma 2.2, it is clear that

$$\lambda_n(x, t, x - x/\sqrt{n}; k) \leq \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n(\frac{x}{\sqrt{n}})^2}.$$

It follows that

$$\begin{aligned} |(I_1 g_x)(x, t; k)| &\leq \int_{x-x/\sqrt{n}}^x (g_x) \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n(\frac{x}{\sqrt{n}})^2} \\ &\quad + \int_0^{x-x/\sqrt{n}} \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n(x - y)^2} d_y \left( -\sqrt[ x ]{g_x} \right) \\ &= \int_{x-x/\sqrt{n}}^x (g_x) \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n(\frac{x}{\sqrt{n}})^2} \\ &\quad + \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n} \int_0^{x-x/\sqrt{n}} \frac{1}{(x - y)^2} d_y \left( -\sqrt[ x ]{g_x} \right). \end{aligned}$$

Furthermore, since

$$\begin{aligned} &\int_0^{x-x/\sqrt{n}} \frac{1}{(x - y)^2} d_y \left( -\sqrt[ x ]{g_x} \right) \\ &= -\frac{1}{(x - y)^2} \sqrt[ x ]{g_x} \Big|_0^{x-x/\sqrt{n}} + \int_0^{x-x/\sqrt{n}} \frac{2}{(x - y)^3} \sqrt[ x ]{g_x} dy \\ &= -\frac{1}{(\frac{x}{\sqrt{n}})^2} \sqrt[ x ]{g_x} + \frac{1}{x^2} \sqrt[ x ]{g_x} + \int_0^{x-x/\sqrt{n}} \frac{2}{(x - y)^3} \sqrt[ x ]{g_x} dy, \end{aligned}$$

and putting  $y = x - x/\sqrt{u}$  in the last integral, we get

$$\int_0^{x-x/\sqrt{n}} \frac{2}{(x - y)^3} \sqrt[ x ]{g_x} dy = \frac{1}{x^2} \int_1^n \sqrt[ x ]{g_x} du = \frac{1}{x^2} \sum_{l=1}^n \sqrt[ x ]{g_x}.$$

Consequently,

$$\begin{aligned} |(I_1 g_x)(x, t)| &\leq \int_{x-x/\sqrt{n}}^x (g_x) \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n(\frac{x}{\sqrt{n}})^2} \\ &\quad + \frac{x(k + 4nx - t(1 - x)^{-1/k}(k + 3x))}{n} \left\{ -\frac{1}{(\frac{x}{\sqrt{n}})^2} \sqrt[ x ]{g_x} \right. \end{aligned}$$

$$\begin{aligned}
 & \left. + \frac{1}{x^2} \bigvee_0^x(g_x) + \frac{1}{x^2} \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^x(g_x) \right\} \\
 &= \frac{x(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{n} \left\{ \frac{1}{x^2} \bigvee_0^x(g_x) + \frac{1}{x^2} \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^x(g_x) \right\} \\
 &= \frac{k + 4nx - t(1-x)^{-1/k}(k + 3x)}{nx} \left\{ \bigvee_0^x(g_x) + \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^x(g_x) \right\}.
 \end{aligned}$$

Moreover,

$$\bigvee_0^x(g_x) \leq \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^x(g_x) \leq \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}}(g_x)$$

yields

$$|(I_1g_x)(x, t)| \leq \frac{2(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{nx} \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}}(g_x) \right\}. \tag{20}$$

Using a similar method for estimating  $|(I_3g_x)(x, t)|$ , we get

$$|(I_3g_x)(x, t)| \leq \frac{x(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{n(1-x)^2} \left\{ \sum_{l=1}^n \bigvee_x^{x+(1-x)/\sqrt{l}}(g_x) \right\}. \tag{21}$$

Putting (19)-(21) in (18), we get

$$\begin{aligned}
 |(L_n g_x)(x, t)| &\leq |(I_1g_x)(x, t)| + |(I_2g_x)(x, t)| + |(I_3g_x)(x, t)| \\
 &\leq \frac{2(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{nx} \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}}(g_x) \right\} \\
 &\quad + \frac{1}{n-1} \sum_{l=2}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}}(g_x) \\
 &\quad + \frac{x(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{n(1-x)^2} \left\{ \sum_{l=1}^n \bigvee_x^{x+(1-x)/\sqrt{l}}(g_x) \right\}. \tag{22}
 \end{aligned}$$

Obviously,

$$\frac{2}{x} + \frac{x}{(1-x)^2} = \frac{3x^2 - 4x + 2}{x(1-x)^2}$$

and

$$\bigvee_x^{x+(1-x)/\sqrt{l}}(g_x) \leq \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}}(g_x).$$

Hence, we obtain from (22)

$$\begin{aligned}
 |(L_n g_x)(x, t)| &\leq \left(\frac{2}{x} + \frac{x}{(1-x)^2}\right) \left(\frac{k + 4nx - t(1-x)^{-1/k}(k + 3x)}{n}\right) \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x) \right\} \\
 &\quad + \frac{1}{n-1} \sum_{l=2}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x) \\
 &\leq \frac{1}{n} \left[ \frac{(3x^2 - 4x + 2)(k + 4nx - t(1-x)^{-1/k}(k + 3x))}{x(1-x)^2} + 2 \right] \\
 &\quad \times \left\{ \sum_{l=1}^n \bigvee_{x-x/\sqrt{l}}^{x+(1-x)/\sqrt{l}} (g_x) \right\}.
 \end{aligned}$$

This inequality is equivalent to the one to be proved. □

### 3 Proof of the main theorem

Now we are ready to establish our main theorem.

*Proof* For any  $f \in BV[0, 1]$ , we decompose  $f$  into four parts on  $[0, 1]$  for sufficiently large  $n$ ,

$$\begin{aligned}
 f(s) &= \frac{f(x+) + f(x-)}{2} + g_x(s) + \frac{f(x+) - f(x-)}{2} \operatorname{sgn}(s - x) \\
 &\quad + \delta_x(s) \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right], \tag{23}
 \end{aligned}$$

where

$$\delta_x(s) = \begin{cases} 1, & x = s, \\ 0, & x \neq s. \end{cases}$$

If we apply the operator  $L_n$  to the both sides of (23), then we have

$$\begin{aligned}
 (L_n f)(x, t; k) &= \frac{f(x+) + f(x-)}{2} (L_n 1)(x, t; k) + (L_n g_x)(x, t; k) \\
 &\quad + \frac{f(x+) - f(x-)}{2} (L_n \operatorname{sgn}(s - x))(x, t; k) \\
 &\quad + \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] (L_n \delta_x)(x, t; k).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &\left| (L_n f)(x, t; k) - \frac{f(x+) + f(x-)}{2} \right| \\
 &\leq |(L_n g_x)(x, t; k)| + \left| \frac{f(x+) - f(x-)}{2} (L_n \operatorname{sgn}(s - x))(x, t; k) \right| \\
 &\quad + \left| \left[ f(x) - \frac{f(x+) + f(x-)}{2} \right] (L_n \delta_x)(x, t; k) \right|.
 \end{aligned}$$

By Lemmas 2.8-2.9, we get the required result. Thus the proof is completed. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

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