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The q-Exponential Operator

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Abstract

We define a q-exponential operator $R(bD_q)$ which turn out to be suitable for dealing with the Cauchy polynomials $P_n(x, y)$ and the homogeneous Rogers-Szegö polynomials $h_n(x, y|q)$. By using this operator, we derive Mehler's formula and Rogers formula for the polynomials $P_n(x, y)$ and $h_n(x, y|q)$.

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1. Introduction

In this paper we will follow the standard notations on q-series in [9] and we always assume that $|q| < 1$. The q-shifted factorial is defined by:

$$(a; q)_k = \begin{cases} 1, & \text{if } k = 0, \\ (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k = 1, 2, 3, \ldots. \end{cases}$$

We also define

$$(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

The generalized basic hypergeometric series is defined by

$$
{}_r\phi_s(a_1,a_2,\dots,a_r;b_1,b_2,\dots,b_s;q,x) = {}_r\phi_s\left(\begin{matrix} a_1,a_2,\dots,a_r \\ b_1,b_2,\dots,b_s\end{matrix};q,x\right)
$$

$$
=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n, \tag{1.1}
$$

where $q\neq 0$ when $r>s+1$. Note that

$$
{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_r\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_{r+1};q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_r;q)_n}x^n.
$$

The following easily verified identities will be frequently used in this paper:

$$
\begin{aligned}
(x;q)_n &= \frac{(x;q)_\infty}{(q^n x;q)_\infty},\\
(a;q)_{n+k} &= (a;q)_n(aq^n;q)_k.
\end{aligned}
$$

We shall adopt the following notation of multiple q-shifted factorials:

$$
\begin{aligned}
(a_1,a_2,\cdots,a_m;q)_n &= (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,\\
(a_1,a_2,\cdots,a_m;q)_\infty &= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.
\end{aligned}
$$

The q-binomial coefficients is defined by:

$$
\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{if } 0\leqslant k\leqslant n,\\ 0, & \text{otherwise.}\end{cases}
$$

One of the most classical identities in q-series is Cauchy identity

$$
\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1.
$$

The following is the homogeneous form of the q-shifted factorial:

$$
P_n(x,y)=(y/x;q)_n x^n=(x-y)(x-qy)(x-q^2y)\cdots(x-q^{n-1}y). \tag{1.2}
$$

Because the polynomials $P_n(x,y)$ occur so often in q-series, Chen et al. [7] proposed to call them the Cauchy polynomials because they are the coefficients in the expansion of the homogeneous version of the Cauchy identity (the generating function of $P_n(x,y)$):

$$
\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt;q)_\infty},\quad |xt|<1. \tag{1.3}
$$

Setting $y = 0$, the Cauchy identity becomes Euler's identity:

$$\sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_\infty} = \frac{1}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.4}$$

Setting $x = 0$, the Cauchy identity becomes, another, Euler's identity:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (yt)^n}{(q;q)_n} = (yt;q)_\infty. \tag{1.5}$$

In 1970, Goldman and Rota [10] have shown the q-binomial identity

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(x,z) P_{n-k}(z,y). \tag{1.6}$$

Setting $z = 0$ in (1.6), one obtains the following identity:

$$P_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} y^k x^{n-k}. \tag{1.7}$$

Note that, the Cauchy polynomials $P_n(x,y)$ naturally arise in the q-umbral calculus as studied by Andrews [1, 2], Goldman and Rota [10], Goulden and Jackson [11], Ihrig and Ismail [12], Johnson [14] and Roman [17].

The usual q-differential operator, or the q-derivative operator is defined by:

$$D_q \{f(x)\} = \frac{f(x) - f(qx)}{x}. \tag{1.8}$$

The Leibniz rule for D_q is the following identity:

$$D_q^n \{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k \{f(x)\} D_q^{n-k} \left\{g(q^k x)\right\}. \tag{1.9}$$

$D_q^0 f(x)$ is understood as the identity.

In [5], Chen and Liu developed a method for deriving hypergeometric identities by parameter augmentation, which means that a hypergeometric identity with multiple parameters may be derived from its special case obtained by reducing some parameters to zero.

In [6], Chen and Liu realized the parameter augmentation by the q-exponential operator $T(bD_q)$, which leads to considerable simplifications of some well known q-summation and transformation formulas. The q-exponential operator is defined by

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}.$$

The following operator identities were obtained:

Theorem 1.1. (Chen and Liu [6]). *Let D_q be defined as above. Then*

$$D_q^k\{x^n\} = \frac{(q;q)_n}{(q;q)_{n-k}}x^{n-k}. \tag{1.10}$$

$$D_q^k\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{t^k}{(xt;q)_\infty}. \tag{1.11}$$

The classical Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson integral [4]. Some important results on the Rogers-Szegö polynomials naturally fall into the framework of parameter augmentation such as Mehler's formula, Rogers formula and the linearization formula and its inverse [3, 6, 13, 15, 16, 18, 20]. The classical Rogers-Szegö polynomials are defined by:

$$h_n(x|q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k,$$

which has the generating function:

$$\sum_{n=0}^{\infty}h_n(x|q)\frac{t^n}{(q;q)_n} = \frac{1}{(xt,t;q)_\infty}, \quad \max\{|xt|,|t|\}<1. \tag{1.12}$$

In the same paper, Chen and Liu represented the polynomials $h_n(x|q)$ by the augmentation operator as follows:

$$T(D_q)\{x^n\} = h_n(x|q).$$

Using the above operator definition of the Rogers-Szegö polynomials and the augmentation argument, they easily derived Mehler's formula and the Rogers formula for $h_n(x|q)$.

Theorem 1.2. (Chen and Liu [6]).
The Mehler's formula for $h_n(x|q)$ is

$$\sum_{n=0}^{\infty}h_n(x|q)h_n(y|q)\frac{t^n}{(q;q)_n} = \frac{(xyt^2;q)_\infty}{(xt,t,yt,xyt;q)_\infty}, \tag{1.13}$$

where $\max\{|xt|,|t|,|yt|,|xyt|\}<1$.
The Rogers formula for $h_n(x|q)$ is

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xst;q)_\infty}{(xs,s,xt,t;q)_\infty}, \tag{1.14}$$

where $\max\{|xs|,|s|,|xt|,|t|\}<1$.

In 2006, Zhang and Wang [19] used the q-exponential operator $T(bD_q)$ to some terminating summation formulas of basic hypergeometric series and q-integrals to obtain some q-series identities and q-integrals involving ${}_3\phi_2$. The following operator identity were obtained:

Theorem 1.3. (Zhang and Wang [19]). *Let D_q be defined as above. Then*

$$D_q^k\left\{\frac{(xv;q)_\infty}{(xt;q)_\infty}\right\} = t^k(v/t;q)_k\frac{(xvq^k;q)_\infty}{(xt;q)_\infty}. \tag{1.15}$$

In 2003, Chen et al. [7], introduced the homogenous q-difference operator D_{xy}, which is suitable for the study of the Cauchy polynomials, acting on function in two variables x and y:

$$D_{xy}f(x,y)=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}.$$

Based on the homogeneous q-difference operator, they built up the homogeneous q-shift operator as the q-exponential of the homogeneous q-difference operator:

$$E(D_{xy}) = \sum_{n=0}^{\infty}\frac{D_{xy}^n}{(q;q)_n}.$$

They also introduced the homogeneous Rogers-Szegö polynomials and derive their generating function by using the homogeneous q-shift operator $E(D_{xy})$. The homogeneous Rogers-Szegö polynomials are defined by:

$$h_n(x,y|q) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}P_k(x,y),$$

which has the generating function:

$$\sum_{n=0}^{\infty}h_n(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(yt;q)_\infty}{(xt,t;q)_\infty}, \quad \max\{|xt|,|t|\}<1.$$

In 2007, Chen et al. [8] present an operator approach to derive Mehler's formula and Rogers formula for the homogeneous Rogers-Szegö polynomials $h_n(x,y|q)$. The proofs of these results are based on parameter augmentation with respect to the q-exponential operator $T(D_q)$ and the homogeneous q-shift operator $E(D_{xy})$.

In this paper, we introduce a new q-exponential operator $R(bD_q)$. We present an operator proof for Mehler's formula and Rogers formula for both the Cauchy polynomials $P_n(x,y)$ and the homogeneous Rogers-Szegö polynomials $h_n(x,y|q)$.

2. The q-exponential operator $R(bD_q)$

Let D_q be defined as in (1.8). We define a q-exponential operator $R(bD_q)$ as follows:

$$R(bD_q)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} b^k}{(q;q)_k}D_q^k. \tag{2.1}$$

The operator proof needs operator identities, so we derive some identities for the q-exponential operator $R(bD_q)$. We use R_a for the operator R acting on the variable a. The following theorem for the exponential operator $R_a(bD_q)$ is easy to verify.

Theorem 2.1. *We have*

$$R_a(bD_q)\left\{\frac{1}{(at;q)_\infty}\right\} = \frac{(bt;q)_\infty}{(at;q)_\infty}. \tag{2.2}$$

$$R_a(bD_q)\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\} = \frac{(av;q)_\infty}{(at;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} v/t \\ av\end{matrix};q,bt\right). \tag{2.3}$$

Theorem 2.2. *We have*

$$R_a(bD_q)\left\{\frac{(av;q)_\infty}{(at,as;q)_\infty}\right\}=\frac{(bs;q)_\infty}{(as;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} v/t, b/a \\ bs\end{matrix};q,at\right). \tag{2.4}$$

Proof. By using (1.9), we get

$$\begin{aligned}
&R_a(bD_q)\left\{\frac{(av;q)_\infty}{(at,as;q)_\infty}\right\}\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix} q^{k(k-n)}D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\}D_q^{n-k}\left\{\frac{1}{(asq^k;q)_\infty}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} b^k}{(q;q)_k}D_q^k\left\{\frac{(av;q)_\infty}{(at;q)_\infty}\right\}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n}D_q^n\left\{\frac{1}{(asq^k;q)_\infty}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} b^k}{(q;q)_k}t^k(v/t;q)_k\frac{(avq^k;q)_\infty}{(at;q)_\infty}R_a(bD_q)\left\{\frac{1}{(asq^k;q)_\infty}\right\} \quad \text{(by using (1.15))}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} b^k}{(q;q)_k}t^k(v/t;q)_k\frac{(avq^k;q)_\infty}{(at;q)_\infty}\frac{(bsq^k;q)_\infty}{(asq^k;q)_\infty} \quad \text{(by using (2.2))}\\
&= \frac{(av,bs;q)_\infty}{(at,as;q)_\infty}\sum_{k=0}^{\infty}\frac{(v/t,as;q)_k}{(q,av,bs;q)_k}(-1)^k q^{\binom{k}{2}}(bt)^k\\
&= \frac{(av,bs;q)_\infty}{(at,as;q)_\infty}\,{}_2\phi_2\left(\begin{matrix} v/t, as \\ av, bs\end{matrix};q,bt\right).
\end{aligned}$$

By Jackson's transformation [9, Appendix III, equation (III.4)], we get the required result.

∎

Theorem 2.3. *We have*

$$R_a(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(bt;q)_\infty}{(as,at;q)_\infty}\ {}_1\phi_1\left(\begin{matrix} at \\ bt \end{matrix};q,bs\right). \tag{2.5}$$

Proof. From (1.9), we get

$$\begin{aligned}
&R_a(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} D_q^k\left\{\frac{1}{(as;q)_\infty}\right\} D_q^{n-k}\left\{\frac{1}{(atq^k;q)_\infty}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}} b^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-n)} \frac{s^k}{(as;q)_\infty}\frac{(tq^k)^{n-k}}{(atq^k;q)_\infty} \qquad \text{(by using (1.11))} \\
&= \frac{1}{(as,at;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(bs)^k}{(q;q)_k}(at;q)_k \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(btq^k)^n}{(q;q)_n} \\
&= \frac{1}{(as,at;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(bs)^k}{(q;q)_k}(at;q)_k(btq^k;q)_\infty \qquad \text{(by using (1.5))} \\
&= \frac{(bt;q)_\infty}{(as,at;q)_\infty}\sum_{k=0}^{\infty}\frac{(at;q)_k}{(q,bt;q)_k}(-1)^k q^{\binom{k}{2}}(bs)^k \\
&= \frac{(bt;q)_\infty}{(as,at;q)_\infty}\ {}_1\phi_1\left(\begin{matrix} at \\ bt \end{matrix};q,bs\right).
\end{aligned}$$

∎

Theorem 2.4. *We have*

$$R_x(yD_q)\left\{\frac{x^n}{(xt;q)_\infty}\right\} = \frac{(yt;q)_\infty P_n(x,y)}{(xt;q)_\infty(yt;q)_n}. \tag{2.6}$$

Proof. By using (2.1) and (1.9), we get

$$\begin{aligned}
&R_x(yD_q)\left\{\frac{x^n}{(xt;q)_\infty}\right\} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} y^k}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-k)} D_q^j\left\{\frac{1}{(xt;q)_\infty}\right\} D_q^{k-j}\left\{(xq^j)^n\right\} \\
&= \frac{1}{(xt;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{\binom{j}{2}}(ytq^n)^j}{(q;q)_j}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} y^k x^{n-k} \qquad \text{(by using (1.10))} \\
&= \frac{(yt;q)_\infty P_n(x,y)}{(xt;q)_\infty(yt;q)_n}. \qquad \text{(by using (1.5) and (1.7))}
\end{aligned}$$

∎

∎

3. Mehler's formula and Rogers formula for $P_n(x,y)$

By using (1.7), the Cauchy polynomials $P_n(x,y)$ can easily be represented by the augmentation operator as follows:

$$R_x(yD_q)\{x^n\} = P_n(x,y). \tag{3.1}$$

Using the operator definition (3.1) of the Cauchy polynomials $P_n(x,y)$, it is easy to give a simple derivation for Mehler's formula and Rogers formula for $P_n(x,y)$.

Theorem 3.1. (Mehler's formula for $P_n(x,y)$). *We have*

$$\sum_{n=0}^{\infty} P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n} = \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} w/z \\ xwt \end{matrix}; q, yzt\right), \quad |zxt| < 1.$$

Proof. From (3.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(x,y)P_n(z,w)\frac{t^n}{(q;q)_n} &= R_x(yD_q)\left\{\sum_{k=0}^{\infty} P_n(z,w)\frac{(xt)^n}{(q;q)_n}\right\} \\
&= R_x(yD_q)\left\{\frac{(xwt;q)_\infty}{(xzt;q)_\infty}\right\} && \text{(by using (1.3))} \\
&= \frac{(xwt;q)_\infty}{(xzt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} w/z \\ xwt \end{matrix}; q, yzt\right). && \text{(by using (2.3))}
\end{aligned}$$

∎

Theorem 3.2. (Rogers formula for $P_n(x,y)$). *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix}; q, ys\right),$$

where $\max\{|xt|,|xs|\} < 1$.

Proof. From (3.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} \\
&= R_x(yD_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\} \\
&= R_x(yD_q)\left\{\frac{1}{(xs,xt;q)_\infty}\right\} && \text{(by using (1.4))} \\
&= \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\, {}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix};q,ys\right). && \text{(by using (2.5))}
\end{aligned}$$

∎

4. Mehler's formula and Rogers formula for $h_n(x,y|q)$

The homogeneous Rogers-Szegö polynomials $h_n(x,y|q)$ can easily be represented by the augmentation operator as follows:

$$R_x(yD_q)\{h_n(x|q)\} = h_n(x,y|q). \tag{4.1}$$

Using the operator definition (4.1) of the homogeneous Rogers-Szegö polynomials $h_n(x,y|q)$, it is easy to give a simple derivation of the Mehler's formula and Rogers formula for $h_n(x,y|q)$.

Theorem 4.1. (Mehler's formula for $h_n(x,y|q)$). *We have*

$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} = \frac{(yt,xvt;q)_\infty}{(xt,t,xut;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} y,xt,v/u \\ yt,xvt \end{matrix};q,ut\right),$$

where $\max\{|xt|,|t|,|ut|,|xut|\}<1$.

Proof. From (4.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n} \\
&= R_x(yD_q)R_u(vD_q)\left\{\sum_{n=0}^{\infty} h_n(x|q)h_n(u|q)\frac{t^n}{(q;q)_n}\right\} \\
&= R_x(yD_q)\left\{\frac{1}{(xt,t;q)_\infty}R_u(vD_q)\left\{\frac{(uxt^2;q)_\infty}{(ut,uxt;q)_\infty}\right\}\right\} && \text{(by using (1.13))} \\
&= R_x(yD_q)\left\{\frac{(vxt;q)_\infty}{(xt,t,uxt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} xt,v/u \\ vxt \end{matrix};q,ut\right)\right\}. && \text{(by using (2.4))}
\end{aligned}$$

By Heine's transformation ${}_2\phi_1$ series [9, Appendix III, equation (III.2)], we get

$$ {}_2\phi_1\left(\begin{matrix} xt, v/u \\ vxt \end{matrix}; q, ut\right) = \frac{(xut, vt; q)_\infty}{(vxt, ut; q)_\infty}\ {}_2\phi_1\left(\begin{matrix} t, v/u \\ vt \end{matrix}; q, xut\right). $$

Hence

$$ \begin{aligned} &\sum_{n=0}^{\infty} h_n(x, y|q) h_n(u, v|q) \frac{t^n}{(q;q)_n} \\ &= \frac{(vt;q)_\infty}{(t, ut; q)_\infty} \sum_{n=0}^{\infty} \frac{(t, v/u; q)_n (ut)^n}{(q, vt; q)_n} R_x(yD_q)\left\{\frac{x^n}{(xt;q)_\infty}\right\} \\ &= \frac{(vt, yt; q)_\infty}{(xt, t, ut; q)_\infty} \sum_{n=0}^{\infty} \frac{(t, v/u; q)_n (ut)^n}{(q, vt; q)_n} \frac{P_n(x, y)}{(yt;q)_n} \qquad \text{(by using (2.6))} \\ &= \frac{(vt, yt; q)_\infty}{(xt, t, ut; q)_\infty}\ {}_3\phi_2\left(\begin{matrix} t, v/u, y/x \\ vt, yt \end{matrix}; q, xut\right). \qquad \text{(by using (1.1) and (1.2))} \end{aligned} $$

By transformation ${}_3\phi_2$ series [9, Appendix III, equation (III.9)], we get the required result. ∎

Theorem 4.2. (Rogers formula for $h_n(x, y|q)$). *We have*

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(xs, s, xt; q)_\infty}\ {}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix}; q, t\right), $$

where $\max\{|xs|, |s|, |xt|, |t|\} < 1$.

Proof. From (4.1), we get

$$ \begin{aligned} &\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x, y|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\ &= R_x(yD_q)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}\right\} \\ &= \frac{1}{(s, t; q)_\infty} R_x(yD_q)\left\{\frac{(xst;q)_\infty}{(xs, xt; q)_\infty}\right\} \qquad \text{(by using (1.14))} \\ &= \frac{1}{(s, t; q)_\infty} \frac{(ys;q)_\infty}{(xs;q)_\infty}\ {}_2\phi_1\left(\begin{matrix} s, y/x \\ ys \end{matrix}; q, xt\right). \qquad \text{(by using (2.4))} \end{aligned} $$

By Heine's transformation ${}_2\phi_1$ series [9, Appendix III, equation (III.3)], we get desired result.

∎

Our derivation for Mehler's formula and Rogers formula for $h_n(x, y|q)$ seems shorter and simpler than the one given in [8], because we only use the q-exponential operator $R(bD_q)$, while their proofs are based on parameter augmentation with respect to the q-exponential operator $T(D_q)$ and the homogeneous q-shift operator $E(D_{xy})$.

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