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On the Foundations of Combinatorial Theory. VIII. Finite Operator Calculus

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Submitted by Richard Bellman

TO SALOMON BOCHNER, MY FIRST TEACHER OF ANALYSIS, WHO TAUGHT US THAT THEORY IS THE CAPTAIN, AND COMPUTATION THE SOLDIER.

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1. Introduction

The so-called Heaviside calculus, invented by Boole and developed without interruption to our day, is the mainspring of much contemporary work in operator theory and harmonic analysis. The spectacular analytic developments in these fields in the last fifty years, coupled with current grandiose plans for unification, cannot, however, be said to have been matched by equal strides in the computational and algorithmic aspects. The algebraic aspects of the theory of special functions have not significantly changed since the nineteenth century. As a result, a deep cleavage is now apparent between the breadth of theory and the clumsiness of special cases.

In this work we reduce to a minimum the analytic apparatus of harmonic analysis on the line, by considering only polynomials. Our objective is to present a unified theory of special polynomials by exploiting to the hilt the duality between x and d/dx.

The main technique adopted here is a rigorous version—perhaps the first one—of the so-called "umbral calculus" or "symbolic calculus," widely used in the past century. This gives an effective technique for expressing a set of polynomials in terms of another. We have throughout emphasized operator methods at the expense of generating functions, which were almost exclusively used in the past. No doubt several results given later could be rephrased in terms of generating functions, but only at the expense of conceptual clarity. Umbral methods, we hope to show, are operators in disguise.

The three kinds of polynomial sequences studied are:

(a) sequences of *binomial type*, that is, sequences of polynomials $p_n(x)$ satisfying the identities

$$p_n(x+y) = \sum_{k\geqslant 0} \binom{n}{k} p_k(x)\, p_{n-k}(y).$$

These sequences were studied in the third part of the series (referred to as III), but we repeat the main results here, both in order to render this work self-contained and in order to give some results in greater generality.

(b) *Sheffer sets*, that is, sequences $s_n(x)$ of polynomials satisfying the identities

$$s_n(x+y) = \sum_{k\geqslant 0} \binom{n}{k} s_k(x)\, p_{n-k}(y),$$

where $p_n(x)$ is a given sequence of binomial type.

(c) *Cross-sequences*, namely doubly indexed sequences $p_n^{[\lambda]}(x)$ of polynomials, satisfying

$$p_n^{[\lambda+\mu]}(x+y) = \sum_{k\geqslant 0} \binom{n}{k} p_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y).$$

This last theory is only touched upon here, and remains largely undeveloped.

One of the unexpected consequences of the present algebraic approach is that the theory of eigenfunction expansions for polynomials can be rendered purely algebraic. This gives a meaning to eigenfunction expansions for Hermite polynomials of arbitrary variance and for Laguerre polynomials of arbitrary α (except a negative integer, where the gamma function is not defined).

A number of examples, each of which includes, we would like to hope, a little novelty, is given at the end, both as an illustration of the theory and to show how much of the past literature on special polynomials is the iteration of a few basic principles. We have, however, resisted the temptation of developing a theory of combinatorial identities as an application, outside of a few hints.

2. Basic Polynomials

We shall be concerned with the algebra (over a field of characteristic zero) of all polynomials $p(x)$ in one variable, to be denoted $\mathbf{P}$.

By a *polynomial sequence* we shall denote a sequence of polynomials $p_i(x)$, $i = 0, 1, 2, \ldots$, where $p_i(x)$ is exactly of degree i for all i.

A polynomial sequence is said to be of *binomial type* if it satisfies the infinite sequence of identities

$$p_n(x+y) = \sum_{k\geq 0} \binom{n}{k} p_k(x)\, p_{n-k}(y), \qquad n = 0, 1, 2,\ldots$$

The simplest sequence of binomial type is of course x^n, but we give some nontrivial examples. Other examples are found in III.

The present theory revolves around the interplay between the algebra of polynomials and another algebra, to be presently introduced and to be denoted by Σ, namely, the *algebra of shift-invariant operators*. All operators we consider are, of course, tacitly assumed to be *linear*. We denote the action of an operator T on the polynomial $p(x)$ by $Tp(x)$. This notation is not, strictly speaking, correct; a correct version is $(Tp)(x)$. However, our notational license results in greater readability.

The most important shift-invariant operators are the shift operators, written E^a, that is, $E^a p(x) = p(x+a)$. Other examples are given later.

An operator T which commutes with all shift operators is called a *shift-invariant operator*. In symbols, $TE^a = E^a T$, for all real a in the field.

We define a *delta operator*, usually denoted by the letter Q, as a shift-invariant operator for which Qx is a nonzero constant.

Delta operators possess many of the properties of the derivative operator, as we will show. In fact our first objective is to exploit the analogy between delta operators and the ordinary derivative.

PROPOSITION 1. *If Q is a delta operator, then $Qa = 0$ for every constant a.*

Proof. Since Q is shift invariant, we have

$$QE^a x = E^a Qx.$$

By the linearity of Q,

$$QE^a x = Q(x+a) = Qx + Qa = c + Qa,$$

since Qx is equal to some nonzero constant c by definition. But also

$$E^a Qx = E^a c = c$$

and so $c + Qa = c$. Hence, $Qa = 0$. Q.E.D.

PROPOSITION 2. *If $p(x)$ is a polynomial of degree n and Q is a delta operator, then $Qp(x)$ is a polynomial of degree $n-1$.*

Proof. It is sufficient to consider the special case $p(x) = x^n$. From the binomial theorem and the linearity of Q, we have

$$Q(x + a)^n = \sum_{k \geqslant 0} \binom{n}{k} a^k Q x^{n-k}.$$

Also by the shift-invariance of Q

$$Q(x + a)^n = QE^a x^n = E^a Q x^n = r(x + a)$$

say, so that

$$r(x + a) = \sum_{k \geqslant 0} \binom{n}{k} a^k Q x^{n-k}.$$

Setting $x = 0$, we have expressed the polynomial $r(x)$ as a polynomial in the parameter a,

$$r(a) = \sum_{k \geqslant 0} \binom{n}{k} a^k [Q x^{n-k}]_{x=0} \cdot$$

The coefficient of a^n is

$$[Qx^{n-n}]_{x=0} = [Q1]_{x=0} = 0$$

by Proposition 1. Further, the coefficient of a^{n-1} is

$$\binom{n}{n-1} [Qx^{n-n+1}]_{x=0} = n[Qx]_{x=0} = nc \neq 0.$$

Hence r is of degree $n - 1$. Q.E.D.

Let Q be a delta operator. A polynomial sequence $p_n(x)$ is called the sequence of *basic polynomials* for Q if:

(1) $p_0(x) = 1$;

(2) $p_n(0) = 0$ whenever $n > 0$;

(3) $Qp_n(x) = np_{n-1}(x)$.

PROPOSITION 3. *Every delta operator has a unique sequence of basic polynomials.*

Proof. Inducing on n, assume that $p_k(x)$ has been defined for $k < n$ to satisfy the foregoing conditions. We show that $p_n(x)$ also exists and is unique. Indeed, a generic polynomial of degree n can be written in the form

$$p(x) = ax^n + \sum_{k=0}^{n-1} c_k p_k(x), \qquad a \neq 0.$$

Now,

$$Qp(x) = aQx^n + \sum_{k=1}^{n-1} c_k \cdot kp_{k-1}(x);$$

therefore, Qx^n being exactly of degree $n - 1$, there is a unique choice of the constants $c_1, ..., c_{n-1}$, a for which $Qp(x) = np_{n-1}(x)$. This determines $p(x)$ except for the constant term c_0, but this is in turn uniquely determined by the condition $p(0) = 0$. Q.E.D.

The typical example of a basic polynomial sequence is x^n, basic for the derivative operator D. Others are given later, or can be looked up in III.

Several properties of the polynomial sequence x^n can be generalized to an arbitrary sequence of basic polynomials. A basic property of x^n is that it is of binomial type. This turns out to be true for every sequence of basic polynomials and is one of our basic results.

THEOREM 1. (a) *If $p_n(x)$ is a basic sequence for some delta operator Q, then it is a sequence of polynomials of binomial type.*

(b) *If $p_n(x)$ is a sequence of polynomials of binomial type, then it is a basic sequence for some delta operator.*

Proof. (a) Iterating property (3) of basic polynomials, we see that

$$Q^k p_n(x) = (n)_k \, p_{n-k}(x),$$

where

$$(n)_k = n(n-1) \cdots (n-k+1).$$

And, hence, for $k = n$,

$$[Q^n p_n(x)]_{x=0} = n!,$$

while for $k < n$,

$$[Q^k p_n(x)]_{x=0} = 0.$$

Thus, we may trivially express $p_n(x)$ in the form

$$p_n(x) = \sum_{k \geqslant 0} \frac{p_k(x)}{k!} [Q^k p_n(x)]_{x=0} \, .$$

Since any polynomial $p(x)$ is a linear combination of the basic polynomials $p_n(x)$, this expression also holds for all polynomials $p(x)$, that is,

$$p(x) = \sum_{k \geqslant 0} \frac{p_k(x)}{k!} [Q^k p(x)]_{x=0} \, .$$

Now suppose $p(x)$ is the polynomial $p_n(x+y)$ for fixed y. Then

$$p_n(x+y) = \sum_{k\geqslant 0} \frac{p_k(x)}{k!} [Q^k p_n(x+y)]_{x=0}.$$

But

$$\begin{aligned}[Q^k p_n(x+y)]_{x=0} &= [Q^k E^y p_n(x)]_{x=0} \\ &= [E^y Q^k p_n(x)]_{x=0} = [E^y (n)_k\, p_{n-k}(x)]_{x=0} = (n)_k\, p_{n-k}(y),\end{aligned}$$

and so

$$p_n(x+y) = \sum_{k\geqslant 0} \binom{n}{k} p_k(x)\, p_{n-k}(y);$$

that is, the sequence $p_n(x)$ is of binomial type.

(b) Suppose now $p_n(x)$ is a sequence of binomial type. Setting $y = 0$ in the binomial identity, we obtain

$$\begin{aligned}p_n(x) &= \sum_{k\geqslant 0} \binom{n}{k} p_k(x)\, p_{n-k}(0) \\ &= p_n(x)\, p_0(0) + n p_{n-1}(x)\, p_1(0) + \cdots.\end{aligned}$$

Since each $p_i(x)$ is exactly of degree i, it follows that $p_0(0) = 1$ (and, hence, $p_0(x) = 1$) and $p_i(0) = 0$ for all other i. Thus. properties (1) and (2) of basic sequences are satisfied.

We next define a delta operator for which such a sequence $p_n(x)$ is the sequence of basic polynomials. Let Q be the operator defined by the property that $Qp_0(x) = 0$ and $Qp_n(x) = np_{n-1}(x)$ for $n \geqslant 1$. Clearly Qx must be a nonzero constant. Hence, all that remains to be shown is that Q is shift-invariant.

We may trivially write the property of being of binomial type in the form

$$p_n(x+y) = \sum_{k\geqslant 0} \frac{p_k(x)}{k!} Q^k p_n(y),$$

and, repeating the device used in (a), this may be extended to all polynomials:

$$p(x+y) = \sum_{k\geqslant 0} \frac{p_k(x)}{k!} Q^k p(y).$$

Now replace p by Qp and interchange x and y on the right—an operation which leaves the left side invariant—to get

$$(Qp)(x+y) = \sum_{k\geqslant 0} \frac{p_k(y)}{k!} Q^{k+1} p(x).$$

But

$$(Qp)(x+y) = E^y(Qp)(x) = E^yQp(x)$$

and

$$\sum_{k\geqslant 0} \frac{p_k(y)}{k!} Q^{k+1}p(x) = Q\left(\sum_{k\geqslant 0} \frac{p_k(y)}{k!} Q^k p(x)\right)$$
$$= Q(p(x+y)) = QE^y p(x).$$

Q.E.D.

3. The First Expansion Theorem

We study next the expansion of a shift-invariant operator in terms of a delta operator and its powers. The difficulties caused by convergence questions are minimal, and we refuse to discuss them in this paper (but see III).

The following theorem generalizes the Taylor expansion theorem to delta operators and their basic polynomials.

Theorem 2 (First Expansion Theorem). *Let T be a shift-invariant operator, and let Q be a delta operator with basic set $p_n(x)$. Then*

$$T = \sum_{k\geqslant 0} \frac{a_k}{k!} Q^k$$

with

$$a_k = [Tp_k(x)]_{x=0}\,.$$

Proof. Since the polynomials $p_n(x)$ are of binomial type (Theorem 1), we may write the binomial formula as in the preceding proof:

$$p_n(x+y) = \sum_{k\geqslant 0} \frac{p_k(x)}{k!} Q^k p_n(y).$$

Apply T to both sides (regarding x as the variable and y as a parameter) and get

$$Tp_n(x+y) = \sum_{k\geqslant 0} \frac{Tp_k(x)}{k!} Q^k p_n(y).$$

Once more, by linearity, this expression can be extended to all polynomials p. After doing this and setting x equal to zero, we can replace y by x and get

$$Tp(x) = \sum_{k\geqslant 0} \frac{[Tp_k(y)]_{y=0}}{k!} Q^k p(x).$$

Q.E.D.

The reader may apply the preceding theorem to derive several of the classical expansion formulas of numerical analysis. Our present application will be of a more theoretical nature:

THEOREM 3. *Let Q be a delta operator, and let* $\mathbf{F}$ *be the ring of formal power series in the variable t over the same field. Then there exists an isomorphism from F onto the ring Σ of shift-invariant operators, which carries*

$$f(t) = \sum_{k\geqslant 0} \frac{a_k t^k}{k!} \qquad \textit{into} \qquad \sum_{k\geqslant 0} \frac{a_k}{k!} Q^k.$$

Proof. The mapping is clearly linear, and by the first expansion theorem, it is onto. Therefore, all we have to verify is that the map preserves products. Let T be the shift-invariant operator corresponding to the formal power series $f(t)$ and let S be the shift-invariant operator corresponding to

$$g(t) = \sum_{k\geqslant 0} \frac{b_k}{k!} t^k.$$

We must verify that

$$[TSp_n(x)]_{x=0} = \sum_{k\geqslant 0} \binom{n}{k} a_k b_{n-k},$$

where $p_n(x)$ are the basic polynomials of Q. Now

$$\begin{aligned}[TSp_r(x)]_{x=0} &= \Big[\Big(\sum_{k\geqslant 0} \frac{a_k}{k!} Q^k \sum_{n\geqslant 0} \frac{b_n}{n!} Q^n\Big) p_r(x)\Big]_{x=0} \\ &= \Big[\sum_{k\geqslant 0} \sum_{n\geqslant 0} \frac{a_k b_n}{k! n!} Q^{k+n} p_r(x)\Big]_{x=0}.\end{aligned}$$

But $p_n(0) = 0$ for $n > 0$ and $p_0(x) = 1$. The only nonzero terms of the double sum occur when $n = r - k$. Thus,

$$\begin{aligned}[TSp_r(x)]_{x=0} &= \Big[\sum_{k\geqslant 0} \frac{a_k b_{r-k}}{k!(r-k)!} Q^r p_r(x)\Big]_{x=0} \\ &= \Big[\sum_{k\geqslant 0} \frac{a_k b_{r-k}}{k!(r-k)!} r! p_0(x)\Big]_{x=0} \\ &= \sum_{k\geqslant 0} \binom{r}{k} a_k b_{r-k}. \qquad \text{Q.E.D.}\end{aligned}$$

COROLLARY 1. *A shift-invariant operator T is invertible if and only if* $T1 \neq 0$.

In the following, we shall write $P = p(Q)$, where P is a shift-invariant operator and $p(t)$ is a formal power series, to indicate that the operator P corresponds to the formal power series $p(t)$ under the isomorphism of Theorem 3.

COROLLARY 2. *An operator P is a delta operator if and only if it corresponds, under the isomorphism of Theorem* 3, *to a formal power series $p(t)$ such that $p(0) = 0$ and $p'(0) \neq 0$.*

Recall that to every formal power series $p(t)$ such that $p(0) = 0$ and $p'(t) \neq 0$ there corresponds a unique inverse power series $p^{-1}(t)$. In symbols, if

$$p(t) = \sum_{k\geqslant 1} \frac{a_k}{k!} t^k,$$

then

$$p(p^{-1}(t)) = \sum_{k\geqslant 1} \frac{a_k}{k!} (p^{-1}(t))^k = t,$$

where the sum is well defined, since $p^{-1}(0) = 0$ and $(p^{-1})'(0) \neq 0$. Similarly we have $p^{-1}(p(t)) = t$.

Essentially, the problem we wish to solve in the present paper is the following: to what "operation" in the ring of shift-invariant operators corresponds the operation of composition $p(q(t))$ of power series with $q(0) = 0$, under the isomorphism theorem? Remarkably, this question does have an answer in the present context.

Next, we connect some of the preceding results with generating functions.

COROLLARY 3. *Let Q be a delta operator with basic polynomials $p_n(x)$, and let $q(D) = Q$. Let $q^{-1}(t)$ be the inverse formal power series. Then*

$$\sum_{n\geqslant 0} \frac{p_n(x)}{n!} u^n = e^{xq^{-1}(u)}.$$

Proof. Expand E^a in terms of Q by the first expansion theorem. The coefficients a_n are $p_n(a)$. Hence,

$$\sum_{n\geqslant 0} \frac{p_n(a)}{n!} Q^n = E^a,$$

a formula which can be considered as a generalization of Taylor's formula, and which specializes to several other classical expansions. Now use the isomorphism theorem with D as the delta operator. We get

$$\sum_{n\geqslant 0} \frac{p_n(a)}{n!} q(t)^n = e^{at},$$

whence the conclusion, upon setting $u = q(t)$ and $a = x$. Q.E.D.

This result will be interpreted more explicitly later (see Section 4). Finally, we note a fact that has already been implicitly used.

COROLLARY 4. *Any two shift-invariant operators commute.*

4. THE PINCHERLE DERIVATIVE

For the first time we introduce operators that are not shift-invariant. The simplest is multiplication by x. Let $p(x)$ be a polynomial. Multiplying each term of $p(x)$ by the variable x, that is, replacing each occurrence of x^n by x^{n+1}, $n \geqslant 0$, we obtain a new polynomial $xp(x)$. Call this the *multiplication operator* and we denote it by $\mathbf{x}$. Thus, $\mathbf{x}\colon p(x) \to xp(x)$. For any operator T defined on $\mathbf{P}$, the operator

$$T' = T\mathbf{x} - \mathbf{x}T,$$

will be called the *Pincherle derivative* of the operator T.

PROPOSITION 1. *If T is a shift-invariant operator, then its Pincherle derivative,*

$$T' = T\mathbf{x} - \mathbf{x}T,$$

is also a shift-invariant operator.

The proof is a straightforward verification.

As a special case of the first expansion theorem, it follows that any shift-invariant operator T can be expressed in terms of D, that is

$$T = \sum_{k\geqslant 0} \frac{a_k}{k!} D^k,$$

where $a_k = [Tx^k]_{x=0}$. Further, by the isomorphism theorem (Theorem 3) the formal power series corresponding to T is

$$\sum_{k\geqslant 0} \frac{a_k}{k!} t^k = f(t).$$

We call $f(t)$ the *indicator* of T.

PROPOSITION 2. *If T has indicator $f(t)$, then its Pincherle derivative T' has $f'(t)$ as its indicator.*

The proof is a direct verification. Similarly, from the isomorphism theorem and from the preceding proposition, we easily infer the following.

PROPOSITION 3. $(TS)' = T'S + TS'$.

And just as easily from the isomorphism theorem, we can infer Proposition 4.

PROPOSITION 4. *Q is a delta operator if and only if $Q = DP$ for some shift-invariant operator P, where the inverse operator P^{-1} exists.*

We come now to the main result of this section, which enables us to compute basic sets for a given delta operator.

THEOREM 4 (Closed forms). *If $p_n(x)$ is a sequence of basic polynomials for the delta operator $Q = DP$ (see Proposition 4), then for $n > 0$:*

(1) $p_n(x) = Q'P^{-n-1}x^n$;
(2) $p_n(x) = P^{-n}x^n - (P^{-n})'\, x^{n-1}$;
(3) $p_n(x) = xP^{-n}x^{n-1}$;
(4) (*Rodrigues formula*) $p_n(x) = x(Q')^{-1}\, p_{n-1}(x)$.

Proof. We shall first show that the right sides of (1) and (2) define the same polynomial sequence. Indeed,

$$\begin{aligned} Q'P^{-n-1} &= (DP)'\, P^{-n-1} \\ &= (D'P + DP')\, P^{-n-1}. \end{aligned}$$

Now, $D' = I$. Hence,

$$\begin{aligned} Q'P^{-n-1} &= P^{-n} + P'P^{-n-1}D \\ &= P^{-n} - (1/n)\,(P^{-n})'\, D, \end{aligned}$$

whence

$$Q'P^{-n-1}x^n = P^{-n}x^n - (P^{-n})'\, x^{n-1},$$

as desired. Next, recalling the definition of the Pincherle derivative of $(P^{-n})'$, we have

$$\begin{aligned} P^{-n}x^n - (P^{-n})'\, x^{n-1} &= P^{-n}x^n - (P^{-n}\mathbf{x} - \mathbf{x}P^{-n})\, x^{n-1} \\ &= xP^{-n}x^{n-1}, \end{aligned}$$

and, thus, the right side of formula (3) equals that of formulas (2) and (1). Setting

$$q_n(x) = Q'P^{-n-1}x^n$$

and writing $Q = DP$, we get

$$Qq_n(x) = DPQ'P^{-n-1}x^n = Q'P^{-n}\, Dx^n = nq_{n-1}(x).$$

Thus, if we can show that $q_n(0) = 0$ for $n > 0$, the proof that $q_n(x)$ is the sequence of basic polynomials for Q will be complete, and it will follow that formulas (1)–(3) are equivalent. From the equivalence of Eqs. (1)–(3) we see that

$$q_n(x) = xP^{-n}x^{n-1},$$

and hence $q_n(0) = 0$ for $n \geqslant 1$. Thus, (1)–(3) have been proved, and $q_n(x) = p_n(x)$.

To prove (4), first invert formula (1),

$$x^n = (Q')^{-1}\, P^{n+1}p_n(x).$$

Note that Q' is invertible (Isomorphism Theorem and Proposition 2). Change n to $n - 1$ and insert the right side into the right side of (3):

$$\begin{aligned} p_n(x) &= xP^{-n}(Q')^{-1}\, P^n p_{n-1}(x) \\ &= x(Q')^{-1}\, p_{n-1}(x), \end{aligned}$$

which is Rodrigues' formula. Q.E.D.

The following formulas relate the basic polynomials of two different delta operators in an analogous way. Their proof is immediate.

COROLLARY. *Let $R = DS$ and $Q = DP$ be delta operators with basic polynomials $r_n(x)$ and $p_n(x)$, respectively, where S^{-1} and P^{-1} exist. Then*

(5) $p_n(x) = Q'(R')^{-1}\, P^{-n-1}S^{n+1}r_n(x), \quad n \geqslant 0;$

(6) $p_n(x) = x(SP^{-1})^n\, x^{-1}r_n(x), \quad n \geqslant 1.$

A last (and useful) characterization of basic sets is the following theorem.

THEOREM 5. *Let P be an invertible shift-invariant operator. Let $p_n(x)$ be a sequence of basic polynomials satisfying*

$$[x^{-1}p_n(x)]_{x=0} = n[P^{-1}p_{n-1}(x)]_{x=0}\,,$$

for all $n > 0$. Then $p_n(x)$ is the sequence of basic polynomials for the delta operator $Q = DP$.

Proof. Define the operator Q by setting $Q1 = 0$,

$$Qp_n(x) = np_{n-1}(x)$$

and extending by linearity. It is easily seen that Q is shift-invariant. In terms of Q, the preceding identity can be rewritten in the form

$$[x^{-1}p_n(x)]_{x=0} = [P^{-1}Qp_n(x)]_{x=0} .$$

By linearity, this extends to an identity for all polynomials $p(x)$ with $p\ (0) = 0$ —an argument we have often used. Thus, recalling that

$$[x^{-1}p(x)]_{x=0} = [Dp(x)]_{x=0}$$

whenever $p(0) = 0$, we have

$$[Dp(x)]_{x=0} = [P^{-1}Qp(x)]_{x=0}$$

for all polynomials $p(x)$, including those for which $p(0) \neq 0$, since the formula trivially holds for constants. Setting $p(x) = q(x + a)$ we obtain, using the shift-invariance of P and Q,

$$\begin{aligned} Dq(a) &= [P^{-1}QE^a q(x)]_{x=0} \\ &= [E^a P^{-1}Qq(x)]_{x=0} \\ &= P^{-1}Qq(a), \end{aligned}$$

for all constants a. But this means that $D = P^{-1}Q$, or $Q = DP$. Q.E.D.

COROLLARY 1. *Given any sequence of constants* $c_{n,1}$, $n = 1, 2,\ldots,$ *with* $c_{1,1} \neq 0$ *there exists a unique sequence of basic polynomials* $p_n(x)$ *such that*

$$[x^{-1}p_n(x)]_{x=0} = c_{n,1} ,$$

that is,

$$p_n(x) = \sum_{k\geqslant 1} c_{n,k}x^k, \qquad n = 1, 2,\ldots.$$

COROLLARY 2. *Let* $g(x)$ *be the indicator of* Q *in the preceding corollary. Then* $g = f^{-1}$, *where*

$$f(t) = \sum_{k\geqslant 1} c_{k,1} \frac{t^k}{k!} .$$

Proof. From Corollary 1

$$D = QP^{-1} = \sum_{k\geqslant 1} c_{k,1} \frac{Q^k}{k!} = f(Q),$$

and the result follows.

The preceding corollaries show that a sequence of basic polynomials is completely determined by the coefficients of their first power x. This fact can be made the starting point for a connection between the present theory and the theory of compound Poisson processes, as we hope to do elsewhere.

Note that the preceding corollary gives an explicit interpretation to the generating function of a sequence of basic polynomials, which can now be restated as

$$\sum_{n\geqslant 0} \frac{p_n(x)}{n!} t^n = \exp\left(x \sum_{k\geqslant 1} c_{k,1} \cdot t^k/k!\right),$$

a form which makes it almost evident.

5. Sheffer Polynomials

A polynomial sequence $s_n(x)$ is called a *Sheffer set* or a set of *Sheffer polynomials for the delta operator* Q if

(1) $s_0(x) = c \neq 0$,

(2) $Qs_n(x) = ns_{n-1}(x)$.

A Sheffer set for the delta operator Q is related to the set of basic polynomials of Q by the following.

PROPOSITION 1. *Let Q be a delta operator with basic polynomial set $q_n(x)$. Then $s_n(x)$ is a Sheffer set relative to Q if and only if there exists an invertible shift invariant operator S such that*

$$s_n(x) = S^{-1}q_n(x).$$

Proof. Suppose first that $s_n(x) = S^{-1}q_n(x)$, where S is an invertible shift invariant operator. Then $S^{-1}Q = QS^{-1}$, and

$$\begin{aligned} Qs_n(x) &= QS^{-1}q_n(x) = S^{-1}Qq_n(x) \\ &= S^{-1}nq_{n-1}(x) = nS^{-1}q_{n-1}(x) = ns_{n-1}(x). \end{aligned}$$

Further, since S^{-1} is invertible $S^{-1}1 = c \neq 0$, by the isomorphism theorem, so that

$$s_0(x) = S^{-1}q_0(x) = S^{-1}1 = c.$$

Thus, $s_n(x)$ is a Sheffer set.

Conversely, if $s_n(x)$ is a Sheffer set for the delta operator Q, define S by setting

$$S : s_n(x) \to q_n(x),$$

and extending S by linearity, so that it is well defined on all polynomials.

Since the polynomials s_n and q_n are both of degree n, and $s_0(x) \neq 0$ S is invertible. It remains to show that S is shift-invariant. To this end, note that S commutes with Q. Indeed,

$$\begin{aligned} SQs_n(x) &= nSs_{n-1}(x) = nq_{n-1}(x) \\ &= Qq_n(x) = QSs_n(x), \end{aligned}$$

and again by the linearity argument we infer that $QS = SQ$; whence $SQ^n = Q^nS$. Finally, recall that by the first expansion theorem one has

$$E^t = \sum_{n\geqslant 0} \frac{a_n}{n!} Q^n, \qquad a_n = [E^t q_n(x)]_{x=0};$$

whence $E^tS = SE^t$ for all t. We conclude that S is shift-invariant.

Q.E.D.

Some of the properties of basic sets can be extended to Sheffer sets; one of the most important is

THEOREM 6 (Second Expansion Theorem). *Let Q be a delta operator with basic polynomials $q_n(x)$, let S be an invertible shift-invariant operator with Sheffer set $s_n(x)$. If T is any shift invariant operator, and $p(x)$ is any polynomial the following identity holds for all values of the parameter y:*

$$Tp(x+y) = \sum_{n\geqslant 0} \frac{s_n(y)}{n!} Q^n STp(x).$$

Proof. By the first expansion theorem we have

$$E^y = \sum_{n\geqslant 0} \frac{a_n}{n!} Q^n$$

with

$$a_n = [E^y q_n(x)]_{x=0} = [q_n(x+y)]_{x=0} = q_n(y);$$

that is,

$$E^y = \sum_{n\geqslant 0} \frac{q_n(y)}{n!} Q^n.$$

Applying this to $p(x)$,

$$E^y p(x) = p(x+y) = \sum_{n\geqslant 0} \frac{q_n(y)}{n!} Q^n p(x).$$

We may interchange the variables x and y in the sum without affecting the left side:

$$p(x+y) = \sum_{n \geqslant 0} \frac{q_n(x)}{n!} Q^n p(y).$$

Applying S^{-1}, regarding x as the variable and y as a parameter, this becomes

$$S^{-1}p(x+y) = \sum_{n \geqslant 0} \frac{S^{-1}q_n(x)}{n!} Q^n p(y)$$
$$= \sum_{n \geqslant 0} \frac{s_n(x)}{n!} Q^n p(y),$$

for all y. Again interchanging the variables x and y

$$S^{-1}p(x+y) = \sum_{n \geqslant 0} \frac{s_n(y)}{n!} Q^n p(x).$$

Now again regarding y as a constant and x as a variable, and applying S followed by T

$$Tp(x+y) = \sum_{n \geqslant 0} \frac{s_n(y)}{n!} Q^n STp(x). \qquad \text{Q.E.D.}$$

COROLLARY 1. *If $s_n(x)$ is a Sheffer set relative to the invertible shift invariant operator S and the delta operator Q, then*

$$S^{-1} = \sum_{n \geqslant 0} \frac{s_n(0)}{n!} Q^n.$$

Proof. In the preceding theorem, set $y = 0$ and $T = S^{-1}$. This gives

$$S^{-1}p(x) = \sum_{n \geqslant 0} \frac{s_n(0)}{n!} Q^n p(x),$$

for any polynomial $p(x)$, which by definition is the same as saying that

$$S^{-1} = \sum_{n \geqslant 0} \frac{s_n(0)}{n!} Q^n. \qquad \text{Q.E.D.}$$

The defining property of polynomial sequences of binomial type has the following analog for Sheffer polynomials.

PROPOSITION 2 (Binomial Theorem). *Let Q be a delta operator with basic* *polynomials* $q_n(x)$, *and let* $s_n(x)$ *be a Sheffer set relative to* Q *and to some invertible shift-invariant operator* S. *Then the following identity holds*

$$s_n(x+y) = \sum_{k\geqslant 0} \binom{n}{k} s_k(x)\, q_{n-k}(y).$$

Proof. Since $q_n(x)$ is of binomial type we have by definition

$$\sum_{k\geqslant 0} \binom{n}{k} q_k(x)\, q_{n-k}(y) = q_n(x+y).$$

Apply S^{-1} to both sides, where, of course, x is the variable, to obtain

$$\begin{aligned}\sum_{k\geqslant 0} \binom{n}{k} s_k(x)\, q_{n-k}(y) &= S^{-1} q_n(x+y)\\ &= S^{-1}E^y q_n(x) = E^y S^{-1} q_n(x) = E^y s_n(x)\\ &= s_n(x+y).\end{aligned}$$

Q.E.D.

We next show that $s_n(x)$ are completely determined by their constant terms:

COROLLARY 1. *Let the polynomials* $q_n(x)$ *and* $s_n(x)$ *be defined as in Proposition 2. Then*

$$s_n(x) = \sum_{k\geqslant 0} \binom{n}{k} s_k(0)\, q_{n-k}(x).$$

Proof. Immediate from Proposition 2 upon setting $x = 0$.
The following converse of the second expansion theorem is useful.

PROPOSITION 3. *Let* T *be an invertible shift-invariant operator, let* Q *be a delta operator, and let* $s_n(x)$ *be a polynomial sequence. Suppose that*

$$E^a f(x) = \sum_{n\geqslant 0} \frac{s_n(a)}{n!} Q^n T f(x)$$

for all polynomials $f(x)$ *and all constants* a. *Then the set* $s_n(x)$ *is the Sheffer set of the operator* T *relative to the delta operator* Q.

Proof. Operating with T^{-1} and then with T after permuting variables, as we have already repeatedly done, we can recast the previous identity in the form

$$E^a f(x) = \sum_{n\geqslant 0} \frac{T s_n(a)}{n!} Q^n f(x);$$

whereupon, setting $f(x) = p_i(x)$, where $p_i(x)$ is the basic set of Q, we obtain

$$p_i(x + a) = \sum_{n\geq 0} \binom{i}{n} Ts_n(a)\, p_{i-n}(x),$$

and setting $x = 0$, this yields $p_i(a) = Ts_i(a)$ for all a. Q.E.D.

As an application, we obtain a simpler proof of Rodrigues' formula for basic polynomials (Proposition 4):

PROPOSITION 4. *Let $p_n(x)$ be the basic set for the delta operator Q. Then*

$$p_n(x) = x(Q')^{-1}\, p_{n-1}(x),$$

where Q' is the Pincherle derivative of Q.

Proof. From the first expansion theorem we have

$$E^a = \sum_{n\geq 0} \frac{p_n(a)}{n!} Q^n,$$

and taking the Pincherle derivative of both sides,

$$aE^a = \sum_{n\geq 0} \frac{p_{n+1}(a)}{n!} Q^n Q'.$$

By the preceding proposition, the polynomial set $x^{-1}p_{n+1}(x)$, $n \geq 0$, is the Sheffer set for the invertible shift-invariant operator Q' relative to the delta operator Q, as desired.

Next, using the notion of indicator developed in Section 4, we derive the generating function for the Sheffer polynomials.

PROPOSITION 5. *Let Q be a delta operator, and let S be an invertible shift-invariant operator. Let $s(t)$ and $q(t)$ be the indicators of S and Q, and let $q^{-1}(t)$ be the formal power series inverse to $q(t)$.*

Then the generating function for the sequence $s_n(x)$ is given by

$$\frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} = \sum_{n\geq 0} \frac{s_n(x)}{n!} t^n.$$

Proof. From the proof of the first expansion theorem,

$$E^x = \sum_{n\geq 0} \frac{q_n(x)}{n!} Q^n, \qquad \text{and} \qquad S^{-1}E^x = \sum_{n\geq 0} \frac{s_n(x)}{n!} Q^n.$$

Also, since x^n is the basic set for the delta operator D, we have after a change of variable

$$E^x = \sum_{n\geqslant 0} \frac{x^n}{n!} D^n,$$

and consequently the indicator of E^x relative to D is e^{xt}. By the isomorphism theorem we may pass to indicators in the expansion for $S^{-1}E^x$ thereby obtaining

$$\frac{1}{s(t)} e^{xt} = \sum_{n\geqslant 0} \frac{s_n(x)}{n!} (q(t))^n.$$

Now set $u = q(t)$ and replace u by t to obtain the conclusion.

As a further consequence of Proposition 3, we have the following characterization of Sheffer polynomials by binomial identities.

PROPOSITION 6. *A sequence $s_n(x)$ is a Sheffer set relative to a basic set $q_n(x)$ if and only if*

$$s_n(x+y) = \sum_{k\geqslant 0} \binom{n}{k} s_k(x)\, q_{n-k}(y).$$

6. RECURRENCE FORMULAS

Given a set of polynomials $p_n(x)$, with $p_0(x) = 1$, under what conditions are they Sheffer polynomials? A simple answer is given by

PROPOSITION 1. *Let $p_n(x)$ be a polynomial sequence with $p_0(x) = 1$. If $p_n(x)$ is a Sheffer set then for every delta operator A there exists a sequence of constants s_n such that*

$$Ap_n(x) = \sum_{k\geqslant 0} \binom{n}{k} p_k(x)\, s_{n-k}, \qquad n \geqslant 0. \tag{*}$$

Also, if (*) holds for some delta operator A and some sequence s_n, then $p_n(x)$ is a Sheffer set.

Note that A need not be the delta operator associated with the set $p_n(x)$.

Proof. Assume that there exists a delta operator A and a sequence of numbers s_n so that (*) holds. We wish to show that $p_n(x)$ is a Sheffer set associated with some delta operator Q.

Define the linear operator Q by

$$Qp_n(x) = np_{n-1}(x), \qquad n > 0$$
$$Qp_0(x) = 0.$$

To prove that Q is a delta operator we need only show it is shift invariant. First note that $AQ = QA$ since

$$\begin{aligned} QAp_n(x) &= Q \sum_{k \geq 0} \binom{n}{k} p_{n-k}(x)\, s_k \\ &= \sum_{k \geq 0} \binom{n}{k} (n-k)\, p_{n-k-1}(x)\, s_k \\ &= n \sum_{k \geq 0} \binom{n-1}{k} p_{n-k-1}(x)\, s_k = nAp_{n-1}(x) = AQp_n(x), \end{aligned}$$

where we have used the identity

$$(n-k)\binom{n}{k} = n\binom{n-1}{k}.$$

The next to last equality is, by definition of the operator Q, the recurrence formula (*) with $n-1$ in the place of n. Thus, $AQp_n(x) = QAp_n(x)$ for all n; by the familiar linearity argument, this implies $AQ = QA$, whence $A^kQ = QA^k$ for all positive integers k, and finally by the First Expansion Theorem that Q is shift-invariant. Thus, $p_n(x)$ is a Sheffer set associated with the delta operator Q.

To prove the converse, let $p_n(x)$ be a Sheffer set relative to the delta operator Q with basic set $q_n(x)$, and let A be an arbitrary delta operator. By the isomorphism theorem (see also Proposition 4 of Section 4) it is easily shown that an invertible shift-invariant operator R exists with the property that $Q = AR$. From this, the proof is concluded as follows. By the binomial theorem (Proposition 6 of the preceding section) we have

$$p_n(x+y) = \sum_{k \geq 0} \binom{n}{k} p_k(x)\, q_{n-k}(y).$$

Apply $Q = AR$ to both sides, recalling that y is a parameter, and obtain

$$(ARp_n)(x+y) = \sum_{k \geq 0} \binom{n}{k} ARp_k(x)\, q_{n-k}(y).$$

Now interchange x and y, as we may since the left side is symmetric in x and y, and then operate with the operator R^{-1}. This gives

$$Ap_n(x + y) = \sum_{k \geq 0} \binom{n}{k} ARp_k(y)\, R^{-1}q_{n-k}(x).$$

Again permute x with y, and recall that $ARp_k(x) = kp_{k-1}(x)$. The right side, therefore, equals

$$\sum_{k \geq 0} \binom{n}{k} kp_{k-1}(x)\, R^{-1}q_{n-k}(y).$$

Setting $y = 0$ gives

$$Ap_n(x) = \sum_{k \geq 1} \binom{n}{k-1} p_{k-1}(x)\, [R^{-1}q_{n-k}(y)]_{y=0}\, (n - k + 1).$$

Defining

$$[R^{-1}q_{k-1}(y)]_{y=0}\, (k) = s_k \qquad \text{and} \qquad s_0 = 0,$$

we find

$$Ap_n(x) = \sum_{k \geq 0} \binom{n}{k} p_k(x)\, s_{n-k}\,.$$

Q.E.D.

7. Umbral Composition

In its most primitive form, umbral notation, or symbolic notation as it was called by invariant theorists in the past century, is an algorithmic device for treating a sequence a_1, a_2, a_3,... as a sequence of powers a, a^2, a^3,.... Computationally, the technique turned out to be very effective in the hands of Blissard (after whom the device is sometimes named), Bell, and above all Sylvester, to name only a few. Several authors attempted to set the "calculus," as it somewhat improperly came to be called, on a rigorous foundation; the last unsuccessful attempt is Bell's paper of 1941. The present author observed in 1964 (in "The Number of Partitions of a Set") that all the mystery of the umbral calculus disappears, if we only consider a sequence a_n as defined by a *linear functional* on the space of polynomials: $a_n = L(x^n)$. The description of the sequence is then condensed into the properties of the linear functional L; only a prejudice would prevent anyone from placing such a definition of a sequence a_n on a par with a definition by recurrence or by generating function. In fact, the success of the umbral notation shows that in many cases the definition by a linear functional is preferable.

If $a_n(x)$ is a polynomial sequence, then there is a unique linear operator L on $\mathbf{P}$ such that $L(x^n) = a_n(x)$. We say that L is the *umbral representation* of the sequence $a_n(x)$.

We develop the umbral device in a form leading to a general result which embodies some of the more recondite indentities satisfied by special polynomials.

An *umbral operator* is an operator T which maps some basic sequence $p_n(x)$ into another basic sequence $q_n(x)$, that is, $Tp_n(x) = q_n(x)$. Note that an umbral operator is in general not shift-invariant. To motivate this definition, we require another definition, the *umbral composition* of two polynomial sequences:

$$a_n(x) = \sum_{k=0}^{n} a_{nk}x^k$$

and $b_n(x)$. This is the sequence of polynomials $c_n(x)$ defined by

$$c_n(x) = \sum_{k=0}^{n} a_{nk}b_k(x).$$

We use for umbral composition the notation

$$c_n(x) = a_n(\mathbf{b}(x)).$$

When $a_n(x) = x^n$, we simply write

$$c_n(x) = \mathbf{b}(x)^n.$$

There is a simple (though, if we are to judge by historical standards, not obvious) connection between umbral operators and the umbral composition of basic polynomials. For if T maps x^n to $q_n(x)$, then

$$a_n(\mathbf{q}(x)) = Ta_n(x),$$

so that umbral composition of polynomials is simply the application of umbral operators, and conversely.

Umbral composition of polynomials has been widely used; our present objective is to study the umbral composition of Sheffer and basic polynomials, thereby "explaining" a great many formulas from the intricate literature on special polynomials and mechanizing the device for guessing and proving them.

A simple instance of the use of umbral notation is the definition of a polynomial sequence of binomial type, which can be umbrally stated as

$$\mathbf{p}(x + y)^n = [\mathbf{p}(x) + \mathbf{p}(y)]^n;$$

similarly, the binomial property of Sheffer polynomials becomes

$$\mathbf{s}(x+y)^n = [\mathbf{p}(x) + \mathbf{s}(y)]^n.$$

PROPOSITION 1. *Let T be an umbral operator. Then T^{-1} exists and*

(a) *the map $S \to TST^{-1}$ is an automorphism of the algebra Σ of shift-invariant operators;*

(b) *T maps every sequence of basic polynomials into a sequence of basic polynomials;*

(c) *if Q is a delta operator, then $P = TQT^{-1}$ is also a delta operator;*

(d) *T maps every Sheffer set into a Sheffer set;*

(e) *If $S = s(Q)$, where $s(t)$ is a formal power series, then $TST^{-1} = s(P)$, where P is as in* (c).

Proof. $Tp_n(x) = q_n(x)$ for two given basic sets. To prove (a) we have the string of identities:

$$TPp_n(x) = T(np_{n-1}(x)) = nTp_{n-1}(x) = nq_{n-1}(x) = Qq_n(x) = QTp_n(x)$$

and since every polynomial is a linear combination of the $p_n(x)$'s, we infer that $TPp(x) = QTp(x)$ for all polynomials $p(x)$; that is, $TP = QT$. It is clear that T is invertible, since it maps polynomials of degree n into polynomials of degree n, for all n. Hence, $TPT^{-1} = Q$; whence, $TP^nT^{-1} = Q^n$ for all $n > 0$. Let S be any shift-invariant operator and let the expansion of S in terms of P be (first expansion theorem)

$$S = \sum_{n \geqslant 0} \frac{a_n}{n!} P^n.$$

Then

$$TST^{-1} = T\left(\sum_{n\geqslant 0} \frac{a_n}{n!} P^n\right) T^{-1} = \sum_{n\geqslant 0} \frac{a_n}{n!} Q^n, \tag{I}$$

and, thus, TST^{-1} is a shift-invariant operator. Furthermore, the map $S \to TST^{-1}$ is onto since any shift-invariant operator can be expanded in terms of Q. Thus, the map is an automorphism, as claimed.

Part (c) follows upon remarking that for delta operators the constant coefficient a_0 vanishes while $a_1 \neq 0$. This also proves (e).

To prove (b), let $r_n(x)$ be a basic sequence with delta operator R.

Let $s_n(x) = Tr_n(x)$ and let $S = TRT^{-1}$. By (c), S is a delta operator. Now,

$$Ss_n(x) = TRT^{-1}s_n(x) = TRr_n(x) = nTr_{n-1}(x) = ns_{n-1}(x).$$

To complete the proof that $s_n(x)$ are the basic polynomials of S we need only show that $s_n(0) = 0$ for $n > 0$. Now we can write

$$r_n(x) = \sum_{k\geqslant 1} a_k p_k(x),$$

since $a_0 = 0$ because $r_n(0) = 0$. Hence,

$$Tr_n(x) = \sum_{k\geqslant 1} a_k q_k(x) = s_n(x)$$

so that $s_n(0) = 0$, $n > 0$, as desired.

To prove (d), let $s_n(x)$ be a Sheffer set relative to the delta operator Q, and set $t_n(x) = Ts_n(x)$ and $P = TQT^{-1}$. By (c), P is a delta operator, and trivially $Pt_n(x) = nt_{n-1}(x)$. Q.E.D.

In view of the preceding result, it follows that the umbral composition of two sequences of basic operators is again a basic sequence. A similar phenomenon holds for Sheffer sets.

PROPOSITION 2. *Let $Wr_n(x) = s_n(x)$, where both are Sheffer sets. Then $W = S^{-1}TR$, where R and S are the invertible operators of $r_n(x)$ and $s_n(x)$ and where T is the umbral operator mapping the basic set $p_n(x)$ of $r_n(x)$ to the basic set of $q_n(x)$ of $s_n(x)$.*

Proof. Obvious.

COROLLARY. *The umbral composition of two Sheffer sets is a Sheffer set.*

The next result determines the operators corresponding to umbral composition.

THEOREM 7 (Umbral Composition). *Let $s_n(x)$ and $t_n(x)$ be Sheffer sets relative to the delta operators Q and P, and to the invertible shift-invariant operators S and T, respectively. Let $q_n(x)$ and $p_n(x)$ be the basic sets for Q and P, and let the indicators of S, Q, and P be*

$$S = s(D), \qquad Q = q(D), \qquad P = p(D),$$

where $s(t)$, $q(t)$ and $p(t)$ are formal power series. Define $r_n(x)$ to be the umbral composition of $s_n(x)$ and $t_n(x)$, in symbols

$$r_n(x) = s_n(\mathbf{t}(x)).$$

Then $r_n(x)$ is a Sheffer set relative to the shift-invariant operator

$$Ts(P) = t(D)\, s(p(D))$$

and the delta operator

$$q(p(D)),$$

having as basic set the sequence

$$q_n(\mathbf{p}(x)).$$

Proof. We begin by establishing the special case where S and T are the identity operators, so that we wish to find the delta operator of the sequence $u_n(x) = q_n(\mathbf{p}(x))$, which we know to be a basic sequence by Proposition 1. Thus, let $V: x^n \to p_n(x)$ be an umbral operator. Then $u_n(x) = Vq_n(x)$, and by (c) of Proposition 1 the delta operator VQV^{-1} of $u_n(x)$ is of the form $q(P) = q(p(D))$ as desired. Next, suppose that T is the identity operator, but not S. We study the sequence $s_n(\mathbf{p}(x))$. But

$$s_n(\mathbf{p}(x)) = Vs_n(x) = VS^{-1}q_n(x), \tag{*}$$

and from $Vq_n(x) = q_n(\mathbf{p}(x))$ we infer that $q_n(x) = V^{-1}q_n(\mathbf{p}(x))$, so that, substituting in (*), we obtain

$$s_n(\mathbf{p}(x)) = VS^{-1}V^{-1}q_n(\mathbf{p}(x)) = VS^{-1}V^{-1}u_n(x).$$

This proves that it is a Sheffer sequence relative to the basic set $u_n(x)$ and the shift-invariant operator VSV^{-1}; and $VQV^{-1} = q(p(D))$, $VSV^{-1} = s(p(D))$, as follows from part (e) of Proposition 1.

Now to the general case, S and T arbitrary. By definition we have

$$t_n(x) = T^{-1}p_n(x), \qquad \text{and} \qquad r_n(x) = T^{-1}s_n(\mathbf{p}(x));$$

thus, we are reduced to the previous case, and the proof is complete.

Several special cases of the preceding theorems are worth stating. A Sheffer set relative to the delta operator D, namely, ordinary differentiation, is called an *Appell set*. The theory of Appell sets is quite old, in fact classical enough to be included in Bourbaki.

Corollary 1. *If $p_n(x)$ and $q_n(x)$ are basic sets with delta operators $P = p(D)$ and $Q = q(D)$, then $p_n(\mathbf{q}(x))$ is a basic set with delta operator $p(q(D))$.*

Corollary 2. *If $s_n(x)$ and $t_n(x)$ are Appell sets, then $s_n(\mathbf{t}(x))$ is an Appell set with operator ST; in particular, $s_n(t(x)) = t_n(\mathbf{s}(x))$.*

Corollary 3. *If $r_n(x)$ is a Sheffer set, then there is a unique Sheffer set $s_n(x)$, called the inverse set, such that $r_n(\mathbf{s}(x)) = x^n$. If $p_n(x)$ and $q_n(x)$ are the corresponding basic sequences, then the basic sequence of $r_n(\mathbf{s}(x))$ is $p_n(\mathbf{q}(x))$.*

The following result gives the solution of the so-called "problem of the connection constants."

Corollary 4. *Given Sheffer sets, $u_n(x)$ relative to the delta operator $U = u(D)$ and the invertible operator $W = w(D)$, and $t_n(x)$ as in Theorem 7, the constants s_{nk} such that*

$$\sum_{k=0}^{n} s_{nk} t_k(x) = u_n(x), \qquad n = 0, 1, \ldots$$

are uniquely determined as follows. The polynomial sequence,

$$s_n(x) = \sum_{k=0}^{n} s_{nk} x^k,$$

is the Sheffer set with delta operator $u(p^{-1}(D))$ and invertible operator $w(p^{-1}(D))/t(p^{-1}(D))$.

The following result gives one of several closed-formula expressions for the coefficients of the Sheffer polynomials.

Corollary 5. *Let $s_n(x)$ be Sheffer polynomials as in Theorem 7 and let V be an umbral operator such that $Vs_n(x) = u_n(x)$ and $V^{-1}s_n(x) = v_n(x)$. Then*

$$s_n(x) = \sum_{k=0}^{n} \frac{v_k(x)}{k!} [SQ^k u_n(x)]_{x=0} \,.$$

Proof. By the second expansion theorem we have

$$Vs_n(x+y) = \sum_{k \geqslant 0} \frac{s_k(x)}{k!} [SQ^k Vs_n(y)];$$

setting $y = 0$ and applying the operator V^{-1} to both sides the result follows.

The following special case is useful.

Corollary 6. *Suppose $p_n(x)$ and $q_n(x)$ are the basic sequences for the delta operators P and Q, respectively. If $q_n(x)$ is inverse to $p_n(x)$, then*

$$p_n(x) = \sum_{k \geqslant 0} \frac{x^k}{k!} [Q^k x^n]_{x=0} \,.$$

Conversely, if the foregoing identity holds for a given delta operator Q, then the $p_n(x)$ are the basic sets for the inverse operator.

COROLLARY 7 (Summation Formula). *Let $f(x)$ be any polynomial. Then, in the notation of the preceding corollary.*

$$f(\mathbf{p}(x)) = \sum_{k\geq 0} \frac{x^k}{k!} [Q^k f(x)]_{x=0} .$$

The prototype of this formula is the classical formula of Dobinsky for the exponential polynomials (see III).

PROPOSITION 3. *Let $W: p_n(x) \to x^n$ be an umbral operator, and let Q be the delta operator of $p_n(x)$. Then*

$$Wxp(x) = xWQ'p(x)$$

for all polynomials $p(x)$, or $W' = xW(Q' - I)$.

Proof. Set $r_n(x) = (Q')^{-1} p_n(x)$, so that $xr_n(x) = p_{n+1}(x)$ by Theorem 4. Now, $Wxr_n(x) = x^{n+1} = xWp_n(x)$, so that

$$Wx(Q')^{-1} p_n(x) = xWp_n(x).$$

By linearity, this holds for all polynomials $p(x)$;

$$Wx(Q')^{-1} p(x) = xWp(x),$$

replacing $p(x)$ by $Q'p(x)$ the result follows.

It would be of interest to develop a theory of operator differential equations in the Pincherle derivative strong enough to give an explicit solution to the previous "differential equation" for the umbral operator W. An example of umbral operator is $Wp_k(x) = a^k p_k(x)$, which is a Sheffer set whenever $p_k(x)$ is. If Q is the delta operator of $p_k(x)$, then $a^{-1}Q$ is the delta operator of $a^k p_k(x)$. Similarly, $p_k(ax)$ is a Sheffer set, and if $Q = f(D)$, then the delta operator for $p_k(ax)$ is $f(a^{-1}D)$. Finally, if $q_k(x)$ is a basic set, then the basic set of the delta operator QE^a is easily seen from formula (4) of Theorem 4 to be $r_n(x) = xq_n(x - na)/(x - na)$. This generalizes the idea behind the Abel polynomials. Summarizing, we have the following.

PROPOSITION 4. *If $s_n(x)$ is a Sheffer set, so is $a^n s_n(bx)$ for any a and b; if it is a basic set, so are $a^n s_n(bx)$ and $xs_n(x - na)/(x - na)$.*

The preceding result "explains" the so-called "duplication formulas" found in the literature, namely, formulas expressing $p_n(ax)$ as a linear combination of $p_k(x)$. We shall see some instances of this device later.

8. Cross-Sequences

A *cross-sequence* of polynomials, written $p_n^{[\lambda]}(x)$, where λ ranges over the field and n over the nonnegative integers, is defined by the following properties:

(a) for fixed λ, $p_n^{[\lambda]}(x)$ is a polynomial sequence;

(b) for any λ and μ in the field and any x and y, the identity,

$$p_n^{[\lambda+\mu]}(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y), \tag{*}$$

holds for all n.

The theory of cross-sequences (of which several examples are unconsciously present in the literature) parallels in many ways the theory of sequences of binomial type, and we shall shorten the by now familiar devices in the proofs. It will always be assumed that the upper variable ranges over the field and the lower one over the nonnegative integers.

Theorem 8. *A sequence $p_n^{[\lambda]}(x)$ is a cross-sequence if and only if there exists a one-parameter group $P^{-\lambda}$ of shift-invariant operators and a sequence $p_n(x)$ of binomial type such that*

$$p_n^{[\lambda]}(x) = P^{-\lambda} p_n(x). \tag{**}$$

(Thus, for fixed λ a cross-sequence becomes a Sheffer sequence relative to the operator P^{λ}.)

Proof. We first show that every sequence defined by the right side of (**) is a cross-sequence. Recall that the group property states that

$$P^{-(\lambda+\mu)} = P^{-\lambda} P^{-\mu}.$$

Thus, apply $P^{-\lambda}$ to the binomial identity satisfied by the $p_n(x)$, thereby obtaining

$$P^{-\lambda} p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k^{[\lambda]}(x)\, p_{n-k}(y).$$

Now permute x and y, and then apply $P^{-\mu}$ to both sides, to obtain (*). Now to the converse. First, note that the sequence $p_n(x) = p_n^{[0]}(x)$ is of binomial type; setting $\mu = 0$ in (*) and applying Proposition 3 of Section 5, we infer that $p_n^{[\lambda]}(x)$ is a Sheffer set relative to a shift-invariant operator which we shall call P^λ, as in (**). From (*) we have

$$P^{-\lambda} p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}^{[\lambda]}(y),$$

and applying $P^{-\mu}$ to both sides, we infer that

$$P^{-\mu}(P^{-\lambda} p_n(x+y)) = \sum_{k=0}^{n} \binom{n}{k} p_k^{[\mu]}(x)\, p_{n-k}^{[\lambda]}(y).$$

But the right side equals $P^{-\lambda-\mu} p_n(x+y)$, again by (*). This gives $P^{-\mu}P^{-\lambda} = P^{-\mu-\lambda}$ and completes this proof.

Corollary. *If a sequence $p_n^{[\lambda]}(x)$ is a cross-sequence, then there exists delta operators Q and $R^{[\alpha]}$ such that $p_0^{[0]} = c \neq 0$,*

$$\begin{aligned} p_n^{[0]}(0) &= 0, & n &> 0 \\ Q p_n^{[\lambda]}(x) &= n p_{n-1}^{[\lambda]}(x), & n &\geqslant 1, \\ R^{[\alpha]} p_n^{[\lambda]}(x) &= n p_{n-1}^{[\lambda-\alpha]}(x). \end{aligned} \tag{***}$$

Proof. Let Q be the delta operator of $p_n(x)$, and let $R^{[\alpha]} = P^\alpha Q$; then (***) follows from (**).

Proposition 1. *The coefficients $c(n, k, \lambda)$ of a cross-sequence,*

$$P^{-\lambda} p_n(x) = p_n^{[\lambda]}(x) = \sum_{k \geqslant 0} c(n, k, \lambda)\, x^k,$$

are polynomials of degree at most n in the variable λ.

Proof. By Corollary 7 of Theorem 7 we have

$$p_n^{[\lambda]}(x) = \sum_{k \geqslant 0} \frac{x^k}{k!} [P^{-\lambda} Q^k x^n]_{x=0},$$

where Q is the delta operator of the *inverse* of this sequence $p_n(x)$.

Writing $P^{-\lambda} = p(D)^\lambda$ and $q(D) = Q$, we have

$$k!c(n, k, \lambda) = [P^{-\lambda}Q^k x^n]_{x=0} = [D^n p(x)^\lambda\, q(x)^k]_{x=0},$$

whence the conclusion.

The proof does not provide an explicit method for the computation of the coefficients $c(n, k, \lambda)$, but see Proposition 4.

A *Steffensen sequence* $s_n^{[\lambda]}(x)$ relative to a cross-sequence $p_n^{[\lambda]}(x)$, is a sequence satisfying the identities

$$s_n^{[\lambda+\mu]}(x+y) = \sum_{k\geq 0} \binom{n}{k} s_k^{[\lambda]}(x)\, p_{n-k}^{[\mu]}(y),$$

for all n, λ, μ, x, y: Steffensen sequences are characterized by

PROPOSITION 2. *The following conditions are equivalent:*

(a) $s_n^{[\lambda]}(x)$ *is a Steffensen sequence;*

(b) *there exists a delta operator Q and a one-parameter group of shift-invariant operators $P^{-\lambda}$ such that*

$$Qs_n^{[\lambda]}(x) = ns_{n-1}^{[\lambda]}(x),$$

$$P^{-\mu}s_n^{[\lambda]}(x) = s_n^{[\lambda+\mu]}(x);$$

(c) *There exists a cross-sequence $p_n^{[\lambda]}(x)$ and an invertible shift-invariant operator T such that*

$$s_n^{[\lambda]}(x) = T^{-1}p_n^{[\lambda]}(x).$$

The proof follows well trodden paths and is omitted.

PROPOSITION 3. *Let $s_n^{[\lambda]}(x)$ be a Steffensen sequence relative to a shift-invariant operator $T = (Q')^{-1}$, as in the preceding proposition, with $s_0^{[\lambda]}(0) = 1$ for all λ. Then the sequence*

$$xs_n^{[n+1]}(x),$$

is a sequence of binomial type.

Proof. Use Theorem 4. $p_n(x) = xs_n^{[0]}(x)$ is the basic sequence for the operator Q, by the Rodrigues formula.

Writing

$$x s_n^{[n+1]}(x) = xP^{-n-1}x^{-1}p_{n+1}(x)$$

and comparing with (6) of the corollary to Theorem 4, we find that the right side is basic with delta operator $R = PQ$.

PROPOSITION 4. *Suppose that $I - P = Q$, where Q is the delta operator of $p_n(x)$. Then for fixed a and for a Steffensen sequence $p_n^{[\lambda]}(x)$ relative to Q we have that*

$$p_n^{[x-n]}(a) \tag{*}$$

is, for fixed a, a Sheffer sequence relative to the difference operator $\Delta = E - I$.

Proof. We have

$$\begin{aligned} p_n^{[\lambda+1-n]}(x) &- p_n^{[\lambda-n]}(x) \\ &= P^{-\lambda+n-1}(I - P)\, p_n(x) = nP^{-\lambda+n-1}p_{n-1}(x) \\ &= np_{n-1}^{[\lambda-n+1]}(x), \end{aligned}$$

which proves the assertion.

It follows from Corollary 1 to Proposition 2 of Section 5 that any linear combination of polynomials of the form (*) is again a Sheffer set relative to Δ. In particular, the coefficients $c(n, k, \lambda)$ (polynomials, by Proposition 1) of

$$p_n^{[\lambda]}(x) = \sum_{k \geqslant 0} \frac{c(n, k, \lambda)}{k!}\, x^k,$$

have the remarkable property that $c(n, k, x - n)$ is a Sheffer set for Δ. An explicit expression could be constructed. We shall not develop in detail here the theory of umbral composition of Steffensen sets, only a few remarks.

PROPOSITION 5. *For Appell cross sequences, namely of the form $p_n^{[\lambda]}(x) = P^{-\lambda}x^n$, we have the umbral composition*

$$p_n^{[\lambda]}(\mathbf{p}^{[\mu]}(x)) = p_n^{[\lambda+\mu]}(x).$$

Proof. Apply Corollary 2 to Theorem 7.

Every invertible shift-invariant operator P can be written in the form $P = e^F$ for some shift-invariant operator (which is never invertible). Indeed, say that $P = I + S$, where $S1 = 0$. Then $F = \log(I + S)$ is well defined, and $P = e^F$. Thus,

$$P^{-\lambda} = \exp(-\lambda F).$$

Note that F is not necessarily a delta operator, though $F1 = 0$. We call F the *generator* of the cross-sequence $p_n^{[\lambda]}(x)$. Thus, an operator F is the generator of a necessarily unique cross-sequence of polynomials, if and only if $F(1) = 0$.

PROPOSITION 6. (a) *If F and G are the generators of cross-sequences $p_n^{[\lambda]}(x)$ and $q_n^{[\lambda]}(x)$ having the same basic sequence, then $F + G$ is the generator of the cross-sequence*

$$e^{-\lambda G} p_n^{[\lambda]}(x) = e^{-\lambda F} q_n^{[\lambda]}(x).$$

(b) *If P is any invertible operator, then*

$$P^{-\lambda} p_n^{[\lambda]}(x)$$

is a cross-sequence when $p_n^{[\lambda]}(x)$ is one.

9. EIGENFUNCTION EXPANSIONS

It is reasonable to surmise that a Sheffer set of polynomials over the real or complex fields should be obtainable by eigenfunction expansion of differential, difference or other Q-operators in a suitable Hilbert space. We establish the truth of this expectation in the real case. The key step consists in singling out a "natural" inner product associated with a given Sheffer set. To this end, let $s_n(x)$ be a Sheffer set relative to the invertible operator S and the delta operator Q. Let $W : s_n(x) \to x^n$ be the umbral operator sending $s_n(x)$ to x^n. For arbitrary polynomials $f(x)$ and $g(x)$ set

$$(f(x), g(x)) = [(Wf)(Q)\, Sg(x)]_{x=0}\,, \tag{*}$$

we have then the following.

PROPOSITION 1. *The bilinear form $(f(x), g(x))$ defined by * on the vector space of all polynomials is a positive-definite inner product.*

Proof. It suffices to show that $(s_k(x), s_n(x)) = (s_n(x), s_k(x)) = 0$ for $k \neq n$, and $(s_n(x), s_n(x)) > 0$ for all n and k. Now,

$$(s_k(x), s_n(x)) = [Q^k S s_n(x)]_{x=0} = [Q^k p_n(x)]_{x=0} = (n)_k\, p_{n-k}(0) = (n)_k\, \delta_{nk}\,,$$

where $p_n(x)$ are the basic polynomials of Q. This completes the proof.

We shall call (*) the *natural inner product associated with the Sheffer set*

$s_n(x)$. We shall now require some notions of Hilbert space theory, such as one finds in any book on functional analysis.

THEOREM 9. *For any Sheffer sequence* $s_n(x)$ *with delta operator* Q *and operator* S *there exists a unique operator of the form*

$$A = \sum_{k\geqslant 1} \frac{u_k + xv_k}{(k-1)!} Q^k$$

with the following properties:

(a) *A is essentially self adjoint (and densely defined) in the Hilbert space H obtained by completing the space* $\mathbf{P}$ *of polynomials in the associated inner product* (*);

(b) *The spectrum of A consists of simple eigenvalues at* $0, 1, 2, \ldots$; *the eigenfunction associated with the eigenvalue n is the polynomial* $s_n(x)$;

(c) *the constants u_k and v_k in the previous expression for A are given by*

$$u_k = -\ [(\log S)'\, x^{-1} p_k(x)]_{x=0}; \qquad v_k = p_k'(0),$$

where $p_k(x)$ are the basic polynomials for the delta operator Q.

Proof. We begin by taking the Pincherle derivative of both sides in the expression

$$S^{-1}E^a = \sum_{n\geqslant 0} \frac{s_n(a)}{n!} Q^n,$$

obtained from the second expansion theorem:

$$(S^{-1}E^a)' = \sum_{n\geqslant 1} \frac{s_n(a)}{n!} nQ^{n-1}Q';$$

multiplying by $(Q')^{-1}Q$ and simplifying,

$$(-\ S^{-1}S' + a)\, S^{-1}E^a(Q')^{-1}Q = \sum_{n\geqslant 1} \frac{s_n(a)}{n!} nQ^n = TS^{-1}E^a, \qquad (**)$$

where we have set

$$T = (-\ S^{-1}S' + a)\,(Q')^{-1}Q = (a - (\log S)')\, Q(Q')^{-1}.$$

Next, expand the operator T in powers of Q, that is, compute the coefficients b_k in

$$T = \sum_{k\geqslant 0} \frac{b_k}{k!} Q^k; \qquad b_k = [Tp_k(x)]_{x=0}\,, \qquad (***)$$

as in the first expansion theorem. Set

$$q_{n-1}(x) = x^{-1}p_n(x) \qquad \text{for } n > 0.$$

Rodrigues' formula now reads

$$(Q')^{-1}\, p_n(x) = q_n(x),$$

whence

$$(Q')^{-1}\, Qp_n(x) = nq_{n-1}(x).$$

Thus, for $k = 0$ we have $b_k = 0$, and for $k > 0$

$$\begin{aligned}[Tp_k(x)]_{x=0} &= k[(a - (\log S)')\, q_{k-1}(x)]_{x=0}\\ &= kaq_{k-1}(0) - k[(\log S)'\, q_{k-1}(x)]_{x=0} = kav_k + ku_k\,,\end{aligned}$$

where

$$\begin{aligned}u_k &= -\,[S^{-1}S'q_{k-1}(x)]_{x=0} = -\,[(\log S)'\, q_{k-1}(x)]_{x=0}\,,\\ v_k &= q_{k-1}(0), \qquad k > 0.\end{aligned}$$

Now from (***) we have for any polynomial $f(x)$,

$$TS^{-1}f(x + a) = \sum_{k\geqslant 0} \frac{b_k}{k!}\, Q^k[S^{-1}f(x + a)].$$

But, as remarked previously,

$$S^{-1}f(x + a) = \sum_{n\geqslant 0} \frac{s_n(x)}{n!}\, Q^n f(a),$$

so that placing the right side into the brackets we obtain

$$\begin{aligned}TS^{-1}f(x + a) &= \sum_{k\geqslant 0} \frac{b_k}{k!}\, Q^k \left[\sum_{n\geqslant 0} \frac{s_n(x)}{n!}\, Q^n f(a)\right]\\ &= \sum_{n\geqslant 0} \left[\sum_{k\geqslant 0} \frac{b_k}{k!}\, Q^k s_n(x)\right] \frac{Q^n}{n!}\, f(a),\end{aligned}$$

where we have interchanged the order of summation. Permuting x and a once more, we obtain

$$TS^{-1}E^a = \sum_{n\geqslant 0} \left[\sum_{k\geqslant 0} \frac{b_k}{k!}\, Q^k s_n(a)\right] \frac{Q^n}{n!}\,,$$

and comparing this with the right side of (**), we see that the coefficients of the two expansions must agree. Upon changing a to x, we obtain

$$\sum_{k\geqslant 0} \frac{b_k}{k!} Q^k s_n(x) = n s_n(x), \qquad n \geqslant 0,$$

with

$$b_k = k(u_k + x v_k).$$

The operator

$$A = \sum_{k\geqslant 1} \frac{u_k + x v_k}{(k-1)!} Q^k$$

is clearly well defined on the set of all polynomials. We have shown that $A s_n(x) = n s_n(x)$ for all $n \geqslant 0$, so that the Sheffer set $s_n(x)$ is a set of eigenfunctions of A; since it spans that Hilbert space H we infer that A is an unbounded essentially self-adjoint operator in H having the nonnegative integers as its simple spectrum, with eigenfunctions $s_n(x)$, as we wanted to show.

COROLLARY 1. *Let R be a delta operator with basic polynomials $r_k(x)$. Then the operator A defined previously can be expressed in the form*

$$A = \sum_{k\geqslant 1} \frac{a_k + x b_k}{k!} R^k,$$

with

$$a_k = -\,[(\log S)' Q(Q')^{-1} r_k(x)]_{x=0}\,,$$
$$b_k = [Q(Q')^{-1} r_k(x)]_{x=0}\,.$$

Proof. From the preceding proof we have

$$T = \sum_{k\geqslant 0} \frac{a_k + a b_k}{k!} R^k,$$

whence the conclusion upon interchanging the roles of the variables x and a, as in the proof of Theorem 8.

The computation of the coefficients a_k and b_k is greatly simplified by use of the corollary to Theorem 4 and by various umbral devices.

The generating functions associated with the a_k and b_k are now easily found; they are immediate consequences of the isomorphism theorem:

COROLLARY 2. *Let $Q = \phi(R)$ and $S = \psi(R)$, where ϕ and ψ are formal power series. Then*

$$\sum_{k\geqslant 0} \frac{b_k}{k!} t^k = \frac{\phi(t)}{\phi'(t)} \qquad \textit{and} \qquad \sum_{k\geqslant 0} \frac{a_k}{k!} t^k = -\frac{\psi'(t)}{\psi(t)} \frac{\phi(t)}{\phi'(t)}.$$

By changes of variables, these identities can be recast in a form suitable for computation in any specific case. One question of interest is the following. When is the operator A a *polynomial* in the operator R? The answer is easily found.

COROLLARY 3. *A is a polynomial in R if and only if*

$$\phi(t) = \exp\left(\int p(t)^{-1}\, dt\right),$$

$$\psi(t) = \exp\left(\int q(t)/p(t)\, dt\right),$$

where p and q are polynomials, and $p(0) = 0$ and $p'(0) \neq 0$, as well as $q(0) = 0$.

Proof. From the preceding corollary we find the differential equations

$$\frac{\phi'(t)}{\phi(t)} = \frac{1}{p(t)}, \qquad \frac{\psi'(t)}{\psi(t)} = \frac{q(t)}{p(t)},$$

whence, integrating

$$\phi(t) = \exp\left(\int p(t)^{-1}\, dt\right), \qquad \psi(t) = \exp\left(\int q(t)/p(t)\, dt\right).$$

Now, $\phi(0) = 0$ and $\phi'(0) \neq 0$, because Q and R are delta operators; it follows that the partial fraction expansion of $1/p(t)$ must contain the summand $1/t$, and this happens only if $p(0) = 0$ and $p'(0) \neq 0$. Similarly, $\psi(0) \neq 0$ because the operator S is invertible. This requires that the partial fraction expansion of $q(t)/p(t)$ shall not contain the summand $1/t$, and, in view of $p(0) = 0$, this requires that $q(0) = 0$. Q.E.D.

Another relevant question in the present context is the representability of the inner product (*) by integral operators, evaluations of a function and its derivatives at specific points, etc. It would take us too far afield to treat this question here; suffice it to say that it can be completely answered.

The simplest case of (*) occurs when $S = I$ and $Q = D$, the ordinary derivative. We have then

$$[p(D)\, q(x)]_{x=0} = \frac{1}{\pi} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \overline{p(x+iy)}\, q(x+iy)\, e^{-(x^2+y^2)}\, dx\, dy,$$

an inner product of frequent occurrence in quantum field theory. From the recurrence relations for orthogonal polynomials it is easy to determine (following Sheffer) all Sheffer sets which are orthogonal polynomials over an interval of the real line. Except for linear changes of variable, they are the following:

(a) for $Q = D$, we must have $S = E^a \exp(D^2)$, and we find a generalization of the Hermite polynomials, orthogonal over $(-\infty, \infty)$;

(b) for $Q = D/(D - I)$ we must have $S = (1 - D)^{\alpha+1}$ with $\alpha > -1$, and we find the Laguerre polynomials of order α, treated later;

(c) for $Q = \log(1 + D)$ we must have $S = E^{\alpha}(I + D)^{\rho}$, $\alpha\rho \neq 0$;

(d) for $Q = \log[b(D - c)/(c(D - b))]$, then $S = (1 - D/c)^{\alpha} (1 - D/b)^{\beta}$; $b \neq c$ and $bc \neq 0$.

These are essentially the Pollaczek polynomials. A similar study can be made in the case of discrete orthogonal polynomials. The polynomials under (c) are Sheffer polynomials relative to the exponential polynomials; they seem not to have been studied. It is interesting to speculate on the possible generalizations of the notion of classical orthogonal polynomial that are suggested by the "natural" inner product (*).

10. Hermite Polynomials

We show that classical formulas pertaining to the Hermite polynomials, as found for example in Jackson or Rainville, can be obtained by specializing the preceding results. Define the Hermite polynomials *of variance* v to be the Appell set (as we shall see, the Appell cross-sequence) whose operator is the Weierstrass operator (so dubbed by Hirschman–Widder)

$$W_v p(x) = \frac{1}{(2\pi v)^{1/2}} \int_{-\infty}^{\infty} e^{-t^2/2v} p(x + t)\, dt. \tag{*}$$

The ordinary Hermite polynomials correspond to variance one. Thus,

$$H_n^{(v)}(x) = W_v^{-1} x^n, \qquad DH_n^{(v)}(x) = nH_{n-1}^{(v)}(x),$$

$$H_n^{(v)}(x + y) = \sum_{k \geq 0} \binom{n}{k} y^{n-k} H_k^{(v)}(x), \quad \text{etc.},$$

trivially from Section 5. The indicator of the operator W_v is computed by the first expansion theorem:

$$W_v = \sum_{n \geq 0} \frac{a_n^{(v)}}{n!} D^n,$$

with

$$a_n^{(v)} = \frac{1}{(2\pi v)^{1/2}} \int_{-\infty}^{\infty} e^{-t^2/2v} t^n \, dt = \frac{v^{n/2} n!}{2^{n/2}(n/2)!} \cdot \frac{(1 + (-1)^n)}{2}$$
$$= \begin{cases} v^{n/2} \cdot 1 \cdot 3 \cdot 5 \cdots (n-1) & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \qquad (**)$$

We set $a_n^{(v)} = v^{n/2} b_n$. Thus,

$$W_v = \sum_{n \geqslant 0} \frac{v^n D^{2n}}{2^n \cdot n!} = e^{vD^2/2}. \qquad (***)$$

We infer that $H_n^{(v)}(x) = H_n^{[v]}(x)$ is a *cross-sequence.* Note that the definition of the Weierstrass operator by (*) is valid only for $v > 0$, but (***) always holds. Next,

$$H_n^{[v+w]}(x + y) = \sum_{k \geqslant 0} \binom{n}{k} H_k^{[v]}(x) \, H_{n-k}^{[w]}(y)$$

$$(x + y)^n = \sum_{k \geqslant 0} \binom{n}{k} H_k^{[v]}(x) \, H_{n-k}^{[-v]}(y)$$

setting $y = 0$,

$$x^{2n} = \sum_{j \geqslant 0} \binom{2n}{2j} H_{2j}^{[v]}(x) \, \frac{(v)^{n-j} \, (2n - 2j)!}{2^{n-j}(n - j)!},$$

$$x^{2n+1} = \sum_{j \geqslant 0} \binom{2n + 1}{2j + 1} H_{2j+1}^{[v]}(x) \, \frac{(v)^{n-j} \, (2n - 2j)!}{2^{n-j}(n - j)!},$$

and finally

$$H_n^{[v]}(x) = \sum_{k \geqslant 0} \binom{n}{k} x^{n-k} (-v)^{k/2} \, b_k,$$

where b_n are given previously; whence we glean the simpler expressions in terms of the classical Hermite polynomials

$$H_n^{[v]}(x) = v^{n/2} H_n\left(\frac{x}{(v)^{1/2}}\right),$$

as we could also have done by umbral methods.

Proposition 5 of Section 8 gives the umbral composition formula,

$$H_n^{[v]}(\mathbf{H}^{[w]}(x)) = H_n^{[v+w]}(x), \qquad (*)$$

and in particular the classical

$$H_n(\mathbf{H}(x)) = 2^{n/2} H_n\left(\frac{x}{(2)^{1/2}}\right).$$

The generating function

$$e^{-t^2/2} e^{xt} = \sum_{n \geqslant 0} \frac{H_n(x)}{n!} t^n,$$

is also immediate from Section 5, Proposition 5.

The (classical) Rodrigues formula follows using the Pincherle derivative. Starting with

$$e^{x^2/2v}(vD)\, e^{-x^2/2v} f(x) = (vD - x) f(x)$$

and

$$\begin{aligned} e^{-vD^2/2} x f(x) &= [(e^{-vD^2/2})' + x e^{-vD^2/2}] f(x) \qquad (*) \\ &= (-1)(vD - x)\, e^{-vD^2/2} f(x), \end{aligned}$$

setting $f(x) = x^{n-1}$ and iterating,

$$H_n^{(v)}(x) = (-1)^n\, e^{x^2/2v} (vD)^n\, e^{-x^2/2v},$$

as desired.

Note that this also proves the recurrence formulas, stated for $v = 1$ for convenience,

$$H_n(x) = x H_{n-1}(x) - H_{n-1}'(x) = x H_{n-1}(x) - (n-1) H_{n-2}(x),$$

from which the differential equation can be obtained by application of $H_n'(x) = n H_{n-1}(x)$ and iteration. We prefer, however, to derive the spectral theory directly from the general results of Section 9. Operational identity (*) can also be used to give a quick proof of the formulas of *Burchnall–Feldheim–Watson.* Indeed, from

$$(D - x)^n f(x) = e^{x^2/2} D^n e^{-x^2/2} f(x)$$

we find, upon applying Leibniz's formula, that the right side equals (following Burchnall)

$$e^{x^2/2} \sum_{k=0}^{n} \binom{n}{k} (D^k e^{-x^2/2})\, D^{n-k} f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} H_k(x)\, D^{n-k} f(x),$$

and setting $f(x) = H_j(x)$ we find

$$H_{n+j}(x) = (-1)^n (D - x)^n H_j(x)$$
$$= \sum_{k \geq 0} \binom{n}{k} (-1)^{n-k} (j)_{n-k} H_{j-n+k}(x) H_k(x),$$

as desired. Similarly we can derive a formula for expressing $H_j(x) H_n(x)$ as linear combinations of $H_k(x)$ by Theorem 6.

We find that

$$p(x) = \sum_{n \geq 0} \frac{H_n(x)}{n!} [D^n W_1 p(y)]_{y=0}$$

for any polynomial $p(t)$. Now

$$(D^n W_1) H_j(x) H_k(x) = W_1(D^n(H_j(x) H_k(x)))$$
$$= W_1 \left(\sum_{i=0}^{n} \binom{n}{i} (j)_i H_{j-i}(x) (k)_{n-i} H_{k-n+i}(x) \right)$$
$$= \sum_{i=0}^{n} \binom{n}{i} (j)_i (k)_{n-i} W_1(H_{j-i}(x) H_{k-n+i}(x)).$$

Now (v. below)

$$[W_1 H_r(x) H_s(x)]_{x=0} = [H_r(x), H_s(x)]_1$$
$$= r!\, \delta_{rs}\,.$$

Therefore, if $j \leqslant k$ say, then

$$[(D^n W_1) H_j(x) H_k(x)]_{x=0}$$
$$= \begin{cases} \binom{n}{i} j!\, (k)_{(k-j+n)/2} & \text{if} \quad n \equiv k + j\,(2), \quad i = (j + n - k)/2, \\ & \qquad 0 \leqslant i \leqslant j \\ 0 & \text{otherwise,} \end{cases}$$

and we conclude that

$$H_j(x) H_k(x)$$
$$= \sum_{\substack{n \geq 0 \\ n \equiv (k+j) (\mathrm{mod} 2) \\ n \leqslant k+j \\ n \geqslant k-j}} H_n(x)\, j! k! \frac{1}{\left(\frac{n+j-k}{2}\right)! \left(\frac{n+k-j}{2}\right)! \left(\frac{k+j-n}{2}\right)!}\,.$$

Proposition 1 of Section 9 shows that the Hermite polynomials are orthogonal relative to the inner product

$$(f(x), g(x))_v = [(W_v f)(D)\, W_v g(x)]_{x=0}\,.$$

We next find out when this inner product coincides with the classical inner product

$$[f(x), g(x)]_v = \frac{1}{(2\pi v)^{1/2}} \int_{-\infty}^{\infty} e^{-x^2/2v} f(x)\, g(x)\, dx, \qquad v > 0. \tag{†}$$

By Rodrigues' formula, followed by an integration by parts, we find

$$\begin{aligned}
[H_n^{(v)}(x), g(x)]_v &= \frac{v^n}{(2\pi v)^{1/2}} \int_{\infty}^{\infty} e^{-x^2/2v} D^n g(x)\, dx \\
&= \frac{v^n}{(2\pi v)^{1/2}} \left[\int_{-\infty}^{\infty} e^{-t^2/2v} D^n g(x+t)\, dt\right]_{x=0} \\
&= v^n [D^n W_v g(x)]_{x=0} \\
&= [W_v((W_v T_v H_n)(D)\, g(x))]_{x=0}\,,
\end{aligned}$$

where $T_v: f(x) \to f(xv)$ is an umbral operator. By linearity it follows that

$$[f(x), g(x)]_v = [W_v((W_v T_v f)(D)\, g(x))]_{x=0}\,, \tag{37}$$

for all polynomials f and g. On the other hand, we verify upon replacing f and g by Hermite polynomials that

$$(f(x), g(x))_v = [W_v((W_v f)(D)\, g(x))]_{x=0}\,, \tag{38}$$

so that the two inner products coincide only for $v = 1$. Both inner products, however, are symmetric and nondegenerate for all values of v; for (38) this is true by definition, and for (37) it is verified as follows. Setting

$$f(x) = H_n^{(v)}(x)$$

we find

$$[H_n^{(v)}(x), g(x)]_v = [W_v((vD)^n\, g(x))]_{x=0}\,,$$

and for $g(x) = H_k(x)$ this becomes

$$\begin{aligned}
[H_n^{(v)}(x), H_k^{(v)}(x)]_v &= [W_v((vD)^n\, H_k^{(v)}(x))]_{x=0} \\
&= v^n (k)_n\, [W_v H_{k-n}^{(v)}(x)]_{x=0} \\
&= v^n (k)_n\, [x^{k-n}]_{x=0} \\
&= v^n n!\, \delta_{kn}\,,
\end{aligned}$$

as desired. For $v > 0$ this inner product is positive-definite. However, definition (37) is valid for *arbitrary* v and combined with the results of Section 9 gives a formally valid eigenfunction expansion, whose inner product is nondegenerate but not positive definite in general. On the other hand, the positive-definite inner product (38), as defined in Section 9, gives a Hilbert-space eigenfunction expansion for arbitrary v. The interaction of the two bilinear forms for nonpositive v leads to interesting analytic developments which we are forced to leave to a later publication. There are also interesting applications to Feynman's integral. There remains to be found the operator of which the Hermite polynomials are the eigenfunctions, and this is given at once by Theorem 9. We have $(\log S)' = D$, since $S = W_1$, so that the formulas given there yield $u_2 = -1$, $v_1 = 1$ and all other coefficients 0. We conclude that the Hermite polynomials are a complete sequence of eigenfunctions, with eigenvalues n, of the operator

$$A = D^2 - xD$$

in the Hilbert space which is the closure of the polynomials in (†). That such a closure is the set of all square-integrable functions follows from a (well known) limiting argument. The present treatment shows that, aside from this one fact from analysis, the entire theory of Hermite expansions can be made purely algebraic.

11. Laguerre Polynomials

One of the simplest cross-sequences is

$$M_n^{[\lambda]}(x) = (I - D)^{-\lambda} x^n,$$

or, more explicitly,

$$M_n^{[\lambda]}(x) = \sum_{k\geqslant 0} \binom{n}{k} (\lambda + k - 1)_k \, x^{n-k}.$$

These polynomials seem to have a scarce literature. For $\lambda = 1$ they were considered by Sheffer, with D replaced by $D/2$ they were studied by Peters under the name "Boole polynomials of the second kind." Note that for $\lambda = 1$ they give, after dividing by $n!$, the partial sums of the exponential function.

From the properties of cross-sequences we immediately infer that

$$M_n^{[\lambda+\mu]}(x) = \sum_{k\geqslant 0} \binom{n}{k} (\lambda + k - 1)_k \, M_{n-k}^{[\mu]}(x),$$

as well as

$$M_n^{[\lambda+\mu]}(x+y) = \sum_{k \geqslant 0} \binom{n}{k} M_k^{[\lambda]}(x)\, M_{n-k}^{[\mu]}(y),$$

which explains several classical binomial identities. Moreover, since the $M_n^{[\alpha]}(x)$ are an Appell set, Corollary 2 to Theorem 7 implies the composition law

$$M_n^{[\alpha]}(\mathbf{M}^{[\beta]}(x)) = M_n^{[\beta]}(\mathbf{M}^{[\alpha]}(x)) = M_n^{[\alpha+\beta]}(x).$$

The cross-sequence $M_n^{[\alpha]}(x)$ is related to polynomials of Laguerre type, which are the Sheffer sets relative to the delta operator

$$Kf(x) = -\int_0^\infty e^{-t} f'(x+t)\, dt,$$

called the *Laguerre operators*. From the first expansion theorem we have

$$K = \sum_{n \geqslant 1} \frac{a_n}{n!} D^n; \qquad a_n = -n \int_0^\infty e^{-t} t^{n-1}\, dt = -n!,$$

so that

$$K = -D - D^2 - \cdots = D/(D-I).$$

The basic polynomials of the Laguerre operator are easily computed from Theorem 4, formula (3):

$$L_n(x) = x(D-I)^n x^{n-1}, \tag{*}$$

called the *basic* Laguerre polynomials. From

$$e^x D e^{-x} = D - I \qquad \text{and} \qquad e^x D^n e^{-x} = (D-I)^n,$$

we obtain the classical *Rodrigues formula*,

$$L_n(x) = x e^x D^n e^{-x} x^{n-1}.$$

From formula (*) we find by binomial expansion that

$$L_n(x) = \sum_{k=1}^n \frac{n!}{k!} \binom{n-1}{k-1} (-x)^k,$$

where the coefficients

$$\frac{n!}{k!} \binom{n-1}{k-1}$$

are known as the (signless) *Lah numbers*.

We shall be concerned with Laguerre type sets relative to the operators (Laguerre operators of order α):

$$K_\alpha = I/(I - D)^{\alpha+1}.$$

Let us note here that for $\alpha > -1$,

$$K_\alpha f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty t^\alpha e^{-t} f(x + t)\, dt,$$

as is easily verified by the first expansion theorem. The Sheffer sets relative to these operators are polynomial sets $L_n^{[\alpha]}(x)$, classically known as ***Laguerre polynomials of order*** α. (Note that our definition of Laguerre polynomials differs from that used by many authors by a factor of $n!$. It does, however, agree with Jackson's notation.)

Again, by definition of the Sheffer polynomials we have

$$L_n^{(\alpha)}(x) = (I - D)^{\alpha+1} L_n(x),$$
$$(1 - D)^\beta L_n^{(\alpha)}(x) = L_n^{(\alpha+\beta)}(x).$$

We infer from (*) the identity

$$L_n^{(\alpha)}(x) = (I - D)^{\alpha+1}\, x(D - I)^n\, x^{n-1}.$$

Using the Pincherle derivative identity

$$(D - I)^n\, x - x(D - I)^n = ((D - I)^n)' = n(D - I)^{n-1},$$

we simplify this expression to

$$\begin{aligned} L_n^{(\alpha)}(x) &= (-1)^n (I - D)^{\alpha+n}\, x^n = (-1)^n \sum_{k\geqslant 0} (-1)^k \frac{(\alpha + n)_k}{k!} D^k x^n \\ &= (-1)^n \sum_{k\geqslant 0} (-1)^k \binom{n}{k} x^{-\alpha} D^k x^{n+\alpha} \\ &= x^{-\alpha}(D - I)^n\, x^{n+\alpha} = x^{-\alpha} e^x D^n e^{-x} x^{n+\alpha}, \end{aligned}$$

which is the classical Rodrigues formula.

Expanding the third formula on the right of the string of identities gives the coefficients of the Laguerre polynomials

$$\begin{aligned} L_n^{(\alpha)}(x) &= (-1)^n \sum_{k\geqslant 0} (-1)^k \frac{(\alpha + n)_k}{k!} (n)_k\, x^{n-k} \\ &= \sum_{i\geqslant 0} \frac{n!}{i!} \binom{\alpha + n}{n - i} (-x)^i. \end{aligned}$$

The binomial theorem for Sheffer polynomials (Proposition 2 of Section 5) yields the identity

$$L_n^{(\alpha)}(x+y) = \sum_{k\geqslant 0} \binom{n}{k} L_k(x) L_{n-k}^{(\alpha)}(y);$$

whence, upon applying the operator $(1 - D)^{\beta+1}$ to both sides, we obtain the *first composition law*

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k\geqslant 0} \binom{n}{k} L_k^{(\beta)}(x) L_{n-k}^{(\alpha)}(y).$$

Further properties follow from the fact that

$$L_n^{(\alpha)}(x) = (-1)^n M_n^{[-\alpha-n]}(x), \qquad \text{or} \qquad M_n^{[\alpha]}(x) = L_n^{[-\alpha-n]}(x)(-1)^n.$$

Next we apply Theorem 7 to study the umbral composition of two Laguerre polynomials. A trivial identification of the various operators at hand yields

$$\begin{aligned} L_n^{(\alpha)}(\mathbf{L}^{(\beta)}(x)) &= (I-D)^{\beta-\alpha} x^n = M_n^{[\alpha-\beta]}(x) \\ &= (-1)^n L_n^{(\beta-\alpha-n)}(x). \end{aligned}$$

For $\beta = \alpha$ we obtain the remarkable identity

$$L_n^{(\alpha)}(\mathbf{L}^{(\alpha)}(x)) = x^n,$$

showing that all the Laguerre polynomials are self-inverse sets. This is true even of the basic Laguerre polynomials, which correspond to the case $\alpha = -1$.

So far we have considered only the umbral composition of $L_n^{(\alpha)}(x)$ with $L_n^{(\beta)}(x)$ and of $M_n^{[\alpha]}(x)$ with $M_n^{[\beta]}(x)$. Umbral composition of $M_n^{(\alpha)}(x)$ with $L_n^{(\beta)}(x)$ gives, by an application of Theorem 7, the Sheffer set relative to—oh surprise!—the delta operator $D/(D-I)$ and the operator $I/(I-D)^{(\alpha+\beta+1)}$, that is, the Laguerre polynomials again! In symbols,

$$\begin{aligned} M_n^{(\alpha)}(\mathbf{L}^{(\beta)}(x)) &= M_n^{(\beta)}(\mathbf{L}^{(\alpha)}(x)) \\ &= L_n^{(\alpha+\beta)}(x). \end{aligned}$$

Piecing together these results on umbral composition, we are led to the following remarkable *second composition law* for Laguerre polynomials:

$$\begin{aligned} &L_n^{(\alpha_1)}(\mathbf{L}^{(\alpha_2)}(\mathbf{L}^{(\alpha_3)}(\cdots \mathbf{L}^{(\alpha_k)}(x)) \cdots) \\ &\quad = \begin{cases} L_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), & k \text{ odd}, \\ M_n^{[\alpha_1-\alpha_2+\alpha_3-\cdots-\alpha_k]}(x), & k \text{ even}. \end{cases} \end{aligned}$$

When expanded in powers of x, this equation leads to several binomial identities, of which we only give a sampling:

If k is even,

$$\binom{-\alpha_1+\alpha_2-\alpha_3+\cdots+\alpha_k}{m}$$

$$= \sum_{r_1,\ldots,r_{k-1}\geqslant 0} (-1)^{r_1+r_3+\cdots+r_{k-1}} \binom{\alpha_1}{r_1}\binom{\alpha_2-r_1}{r_2}\binom{\alpha_3-r_1-r_2}{r_3}\cdots$$

$$\times \binom{\alpha_{k-1}-r_1-r_2-\cdots-r_{k-2}}{r_{k-1}}\binom{\alpha_k-r_1-\cdots-r_{k-1}}{m-r_1-\cdots-r_{k-1}},$$

and if k is odd,

$$\binom{\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k}{m}$$

$$= \sum_{r_1,\ldots,r_{k-1}\geqslant 0} (-1)^{r_2+r_4+\cdots+r_{k-1}} \binom{\alpha_1}{r_1}\binom{\alpha_2-r_1}{r_2}\cdots\binom{\alpha_k-r_1-\cdots-r_{k-1}}{m-r_1-\cdots-r_{k-1}}.$$

The so-called "duplication formulas" for Laguerre polynomials (see, e.g. Rainville) are trivial consequences of Theorem 7; we shall only derive one of them to indicate the method. We are to express $L_n(ax)$ as a linear combination of $L_k(x)$. By Section 7, the sequence $L_n(ax)$ is basic to the operator $a^{-1}D/(a^{-1}D-I)$. We are, therefore, to find a formal power series $f(t)$ such that $a^{-1}t/(a^{-1}t-1)=f(t/(t-1))$. An easy computation gives $f(t)=t/[(1-a)\,t+a]$. Now, the basic polynomials for $f(D)$ are computed by Theorem 4; they are

$$\begin{aligned} p_n(x) &= x[(1-a)\,D+aI]^n\,x^{n-1} \\ &= \sum_{k=0}^{n-1}\binom{n}{k}(1-a)^k\,a^{n-k}(n-1)_k\,x^{n-k} \\ &= \sum_{k=1}^{n}\frac{n!}{k!}\binom{n-1}{k-1}(1-a)^{n-k}\,(ax)^k. \end{aligned}$$

If we now apply the umbral operator $V: x^k \to L_k(x)$, then by Proposition 1 of Section 7 the sequence $Vp_n(x)$ will be basic for the delta operator

$$\begin{aligned} Vf(D)\,V^{-1} &= f(VDV^{-1}) = f(K) = f(D/(D-I)) \\ &= \frac{a^{-1}D}{a^{-1}D-I}, \end{aligned}$$

whose basic sequence is, as we have remarked, $L_n(ax)$. Thus, we are led to Erdelyi's formula

$$L_n(ax) = \sum_{k=1}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (1-a)^{n-k} a^k L_k(x).$$

For the Laguerre polynomials of order α, Proposition 5 of Section 5 gives us the generating function

$$\sum_{n\geq 0} \frac{L_n^{(\alpha)}(x)}{n!} t^n = \frac{1}{(1-t)^{\alpha+1}} e^{xt/(t-1)}.$$

Since the generating function of $M_n^{[\alpha]}$ is easily seen to be

$$\sum_{n\geq 0} \frac{M_n^{[\alpha]}(x)}{n!} t^n = \frac{1}{(1-t)^{\alpha}} e^{xt}$$

and

$$M_n^{[\alpha]}(x) = (-1)^n L_n^{(-\alpha-n)}(x),$$

we obtain the following interesting relation:

$$\sum_{n\geq 0} \frac{L_n^{(\alpha-n)}(x)}{n!} t^n = (1+t)^\alpha e^{-xt}.$$

We will now generalize these relations and obtain generating functions for the sequences $L_n^{(\alpha+bn)}(x)$, where b is any fixed complex number. For b an integer these were first obtained by Brown, and Carlitz later generalized them to any b.

A routine calculation shows that $L_n^{(\alpha+bn)}(x)$ is Sheffer relative to the delta operator $Q_b = -D(I-D)^{-b-1}$. Since, by formula (2) of Theorem 4, the basic polynomials for Q_b are

$$(-1)^n (I-D)^{bn+n-1} (I+bD) x^n,$$

we discover that $L_n^{(\alpha+bn)}(x)$ is Sheffer relative to the invertible shift-invariant operator

$$S_{b,\alpha} = \frac{I+bD}{(I-D)^{\alpha+1}}.$$

If we now let $Q_b = q_b(D)$, $S_{b,\alpha} = s(b, \alpha, D)$, and $q_b^{-1}(t) = A(b, t)$, then by Proposition 5 of Section 5 we obtain

$$\sum_{n\geq 0} \frac{L_n^{(\alpha+bn)}(x)}{n!} t^n = (s(b, \alpha, A(b, t)))^{-1} e^{xA(b,t)},$$

which is the desired generating function. Further, since $A(b, t)$ is the (unique) formal power series solution to

$$\frac{-A}{(1-A)^{b+1}} = t,$$

an easy calculation shows that

$$A(-b-1, t) = \frac{-A(b, -t)}{1 - A(b, -t)}.$$

Similarly, we discover that

$$s(-b-1, -\alpha, A(-b-1, t)) = s(b, \alpha, A(b, -t)) \cdot (1 - A(b, -t)).$$

The spectral theory of Laguerre polynomials can only be sketched here. The classical inner product,

$$[f(x), g(x)]_\alpha = \int_0^\infty x^\alpha e^{-x} f(x)\, g(x)\, dx,$$

can be redefined so as to make sense not only for $\alpha > 0$, but for all α (except when α is a negative integer). Indeed, as with the Hermite polynomials we find

$$\begin{aligned}
&\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x)\, g(x)\, dx \\
&\quad = \int_0^\infty D^n(x^{\alpha+n} e^{-x})\, g(x)\, dx = \int_0^\infty (-1)^n\, x^{\alpha+n} e^{-x} D^n g(x)\, dx \\
&= \left[\int_0^\infty (-1)^n t^{\alpha+n} e^{-t} D^n g(x+t)\, dt\right]_{x=0} = \Gamma(\alpha+n+1)[K^n K_\alpha g(x)]_{x=0},
\end{aligned}$$

whereas, the inner product given by Proposition 1 of Section 9 is

$$[KK_\alpha(f(K)\, g(x))]_{x=0} = (f(x), g(x))_\alpha.$$

The two inner products do not coincide. The second inner product is, however, positive definite for all α; whereas, the first is symmetric for all α and gives

$$[L_n^{(\alpha)}(x), L_n^{(\alpha)}(x)]_\alpha = n!\, \Gamma(\alpha+n+1),$$

so that it is well defined, whenever α is not a negative integer. Nevertheless, the eigenfunction expansion still makes sense, and Theorem 9 readily yields the differential equation

$$L_n^{(\alpha)''}(x) + (\alpha + 1 - x)\, L_n^{(\alpha)'}(x) + nL_n(x) = 0.$$

Again we must leave a detailed analysis of these inner products to a later publication.

We shall now generalize slightly the Laguerre operator K and consider the delta operators

$$L_{\alpha,\beta} = l_{\alpha,\beta}(D) = \frac{\alpha D}{1 - \beta D}, \qquad \alpha \neq 0.$$

The Laguerre operator corresponds, of course, to $\alpha = -\beta = -1$. From formula (2) of Theorem 4 we find that the basic polynomials $J_n^{(\alpha,\beta)}(x)$ for $L_{\alpha,\beta}$ are given by

$$\begin{aligned} J_n^{(\alpha,\beta)}(x) &= \alpha^{-n}(1 - \beta D)^{n-1} x^n \\ &= \alpha^{-n} \sum_{k=1}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-\beta)^{n-k} x^k. \end{aligned}$$

Since

$$l_{\alpha,\beta}(l_{\alpha',\beta'}(D)) = l_{\alpha\alpha',\beta\alpha'+\beta'}(D),$$

we see that the $L_{\alpha,\beta}$ form a group under convolution and that this group is in fact isomorphic to the multiplicative group of matrices

$$\begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix}, \qquad \alpha \neq 0.$$

This enables us to easily compute the umbral composition of the $J_n^{(\alpha,\beta)}(x)$. Thus, for example, we obtain

$$J_n^{(\alpha,\beta)}(\mathbf{J}^{(\gamma,\delta)}(x)) = J_n^{(\alpha\gamma,\beta\gamma+\delta)}(x),$$

which yields the binomial identity. Deeper properties can be obtained by developing the theory of Sheffer sets relative to these operators.

12. Vandermonde Convolution

The difference analogs of Abel polynomials, with delta operator $E^{-b}\Delta$, may be called the *Gould polynomials* and denoted by $G_n(x, b)$. By the corollary to Theorem 4, we readily find the explicit expressions for the $G_k(x, b)$;

$$\begin{aligned} A_k(x, b) = G_k(x, b)/k! &= \frac{x}{x + bk} (x + bk)_k/k! \\ &= \frac{x}{x + bk} \binom{x + bk}{k}. \end{aligned}$$

We refer to Gould's papers for comparison. The identity expressing that these polynomials are of binomial type is sometimes known as the *Vandermonde convolution*, though the name is also applied to other identities. Gould's (1961, 1.1) is the generating function, a special case of Corollary 3 to Theorem 2. The binomial identity can be strengthened to

$$\sum_{k=0}^{n} \binom{n}{k} (p + qk)\, G_k(x, b)\, G_{n-k}(c, b) = \frac{p(x + c) + qxn}{x + c}\, G_n(x + c, b).$$

Gould's inverse relations are straightforward applications of Theorem 2. Since

$$(E^{-b}\Delta)^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} E^{j-nb},$$

we find that

$$F(n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j - nb)$$

is the inverse of

$$f(x) = \sum_{n \geqslant 0} \frac{x}{x + bn} \binom{x + bn}{n} F(n) = \sum_{n \geqslant 0} A_n(x, b)\, F(n),$$

which can be considered as the basic inversion formulas associated with Vandermonde convolution (a recasting of Gould (1962, 3.1 and 3.2)). Several special cases are discussed by Gould, in particular, his Theorem 2 (1960).

We next obtain the connection constants of $G_n(x, c)$ in terms of $G_n(x, c - b)$. This is done most simply by expanding the first set in terms of the second. Now,

$$\begin{aligned} E^{b-c}\Delta G_n(x, c) &= E^b(E^{-c}\Delta G_n(x, c)) \\ &= E^b n G_{n-1}(x, c) = nG_{n-1}(x + b, c), \end{aligned}$$

and, therefore,

$$(E^{b-c}\Delta)^k\, G_n(x, c) = (n)_k\, G_{n-k}(x + kb, c);$$

whence, by Theorem 2

$$G_n(x + a, c) = \sum_{k\geqslant 0} \frac{G_k(a, c - b)}{k!} (E^{b-c} \Delta)^k G_n(x, c),$$

or

$$G_n(x + a, c) = \sum_{k\geqslant 0} \binom{n}{k} G_k(a, c - b) G_{n-k}(x + bk, c);$$

or, in Gould's notation

$$A_n(x + a, c) = \sum_{k\geqslant 0} A_k(x, c - b) A_{n-k}(a + bk, c).$$

For convenience we also write the inverse formulas, obtained by a change of parameters:

$$A_n(x + a, c - b) = \sum_{k=0}^{n} A_k(x, c) A_{n-k}(a - bk, c - b).$$

In more classical notation, this pair yields the inversion formulas:

$$f_n(x + a) = \sum_{k=0}^{n} F_k(x) A_{n-k}(a - bk, c - b),$$

$$F_n(x + a) = \sum_{k=0}^{n} f_k(x) A_{n-k}(a + bk, c).$$

This implies Gould's main theorem (1962, 5.3 and 5.4) and has the advantage of a simpler formulation. Next, the polynomials $(x + bk)_k$ are Sheffer relative to the delta operator $E^{-b}\Delta$. Hence, the binomial theorem for Sheffer polynomials (Proposition 2 of Section 5) gives

$$\binom{x + y + bn}{n} = \sum_{k\geqslant 0} \binom{x + bk}{k} \frac{y}{y + b(n - k)} \binom{y + b(n - k)}{n - k},$$

which is slightly deeper than the identity, obtained from the fact that the

$$E^a G_k(x, b) = \frac{x + a}{x + a + bk} (x + a + bk)_k$$

are a cross-sequence, namely

$$\frac{x+a+c}{x+a+c+bn}\binom{x+a+c+bn}{n}$$
$$=\sum_{k\geqslant 0}\frac{x+a}{x+a+bk}\binom{x+a+bk}{k}\frac{x+c}{x+c+b(n-k)}\binom{x+c+b(n-k)}{n-k}.$$

A similar identity follows from the fact that $E^a(x+bk)_k$ are a Steffensen sequence. These identities also give the connection constants for expressing $G_n(x, b)$ as a linear combination of $G_k(x, c)$. In short, the previous form reads

$$A_n(x+a+c, b)=\sum_{k\geqslant 0} A_k(x+a, b)\, A_{n-k}(x+c, b),$$

and the Steffensen form is

$$\binom{x+a+c+bn}{n}=\sum_{k\geqslant 0} A_{n-k}(x+a, b)\binom{x+c+bk}{k}.$$

The inverse set of the $G_n(x, b)$, call it $J_n(x, b)$, is easily computed by Theorem 7.

Consider the umbral operator W sending x^n to $(x)_n$, and, thus, $WDW^{-1}=\Delta$. The inverse operator sends $E^{-b}\Delta$ to $D(1+D)^{-b}$, a delta operator whose basic polynomials are

$$p_n(x)=x(1+D)^{nb}\, x^{n-1}=xe^{-x}D^{nb}e^x x^{n-1}$$
$$=\sum_{k\geqslant 0}\binom{bn}{k}(n-1)_k\, x^{n-k},$$

which are polynomials of Laguerre type.

Gould's summation formula 5.5 and Bateman's alternating convolution can also be obtained from the expansion theorem. We have thus "explained" most identities for the polynomials $A_n(x, b)$ given in Gould's two papers.

13. Examples and Applications

Appell Polynomials.

As already remarked, these are Sheffer polynomials relative to D. It is impossible to summarize here the immense literature on these sets; a few pertinent remarks must suffice.

If $p_n(x)=T^{-1}x^n$, then an easy computation gives

$$p_n(x)=((T^{-1})'\, T+x)\, p_{n-1}(x),$$

a useful recurrence formula which yields various classical formulas (for example, the recurrence for Hermite polynomials).

Expansion of the product $p_n(ax)\,g_k(bx)$ of Appell sets in terms of a third set were considered by Carlitz (1963); his results are special cases of those of Section 5.

By far the most widely studied class of Appell polynomials are the Bernoulli polynomials (see Nörlund). They correspond to the operator J^a, where

$$Jp(x) = \int_x^{x+1} p(t)\,dt.$$

(Since $DJ = \Delta$, J^a is also defined by $J^a = (\Delta/D)^a = [(e^D - I)/D]^a$.) For $a = 1$, we have $J^{-1}x^n = B_n(x)$, the familiar Bernoulli polynomials, whose elementary property can be gleaned from Section 5. The second expansion theorem yields the Euler–MacLaurin sum formula; generalizations (Nörlund) are obtained by taking the $B_n^{[a]}(x) = J^{-a}x^n$. From (3) of Theorem 4 we easily infer that the sequence $xB_{n-1}^{[na]}(x)$ is basic for the operator DJ^a. This fact, combined with the general results given previously, yields all of Nörlund's identities. The umbral properties of these polynomials are remarkable, but require an extensive separate treatment.

Appell sets with the Bernoulli–like property,

$$p_n(-x-1) = (-1)^n\,p_n(x),$$

were studied by Nielsen; Ward considered the more general functional equation,

$$p_n(ax + b) = c_n p_n(x), \tag{*}$$

and called such Appell sequences *regular*. If a is not a root of unity, the only regular sequence is

$$K_n(x) = c_n[x + b/(a-1)]^n; \qquad c_n = a^n.$$

When a is a root of unity, however, we find a wealth of possibilities, as follows: let a be a primitive rth root of unity, then every Appell set satisfying (*) can be uniquely represented in the form

$$p_n(x) = s_0K_n(x) + s_rK_{n-r}(x) + \cdots + s_{tr}K_{n-tr}(x),$$

and conversely.

Another extensively studied (by Nörlund) class of Appell polynomials is

$$E_n^{[a]}(x) = [1 + (\Delta/2)]^{-a}\,x^n,$$

and again their "properties" become special cases of the previous result. Again (Steffensen) the sequence $xE_{n-1}^{[na]}(x + na/2)$ is basic for the operator D cosh $(D/2)$. These sequences are variously called "Euler polynomials," an honor which is, however, bestowed upon a great many other polynomial sequences. For $a = -1$ we obtain, apart from a constant factor, the *Genocchi polynomials* $G_n(x)$, and $G_n(0)$ are the *Genocchi numbers*. The second expansion theorem applied to the Euler polynomials yields the *Boole* summation formula.

Inverse Relations.

Given two polynomial sequences $p_n(x)$ and $q_n(x)$, suppose we can determine the connection constants

$$p_n(x) = \sum_{k=0}^{n} c_{nk} q_k(x),$$

$$q_n(x) = \sum_{k=0}^{n} d_{nk} p_k(x),$$

then we can derive a pair of inverse relations. Given any sequence a_n, set $L(q_n(x)) = a_n$; this defines a linear functional L on the space $\mathbf{P}$. If $b_n = L(p_n(x))$, we have

$$b_n = \sum_{k=0}^{n} c_{nk} a_k, \qquad a_n = \sum_{k=0}^{n} d_{nk} b_k. \tag{*}$$

By specializing to suitable sets of Sheffer polynomials, a great many of the inverse relations in the literature can be explained. In this context, Theorem 7 will help find the inverse of certain infinite matrices.

The simple inverse relations in Riordan (pp. 43–49) fall under the present scheme. Glancing at Table 2.1 (Riordan, p. 49), we recognize that 1. and 2. reduce to Theorem 2 for Δ and the backward difference ∇, and the rest result from an umbral interpretation of the foregoing identities for Laguerre polynomials. For example, 6 follows from the fact that the basic Laguerre polynomials are self-inverse.

For the sake of clarity we discuss the simplest of all inverse relations, namely

$$a_n = \sum_{k \geqslant 0} (-1)^k \binom{n}{k} b_k; \qquad b_n = \sum_{k \geqslant 0} (-1)^k \binom{n}{k} a_k. \tag{*}$$

This is immediately understood by defining the linear functional $L(x^k) = b_k$, which by the first identity gives $a_n = L((1 - x)^n)$. Hence,

$$b_n = L((1 - (1 - x))^n),$$

which is the second identity.

Klee's identity (Riordan, p. 13),

$$\sum_{k\geqslant 0} (-1)^k \binom{n}{k} \binom{n+k}{m} = (-1)^n \binom{n}{m-n},$$

is another simple example of the use of such umbral techniques. Variants of the two inversion formulas derived previously are discussed by Riordan (pp. 49–54) and summarized in his Table 2.2 (p. 52). These inverse relations can be treated by the methods developed here.

Generating Functions.

To relate a generating function identity in the literature to the present techniques, we compare with the generating function of basic and Sheffer polynomials, thereby identifying the operators involved. Take, say Example 2 of Riordan (p. 100). Changing variables,

$$\frac{1}{(1-t)^x} = e^{x \log(1-t)^{-1}} = \sum_{n\geqslant 0} \frac{t^n}{n!} p_n(x),$$

where $p_n(x)$ is basic relative to backward difference; the inversion formula

$$a_n = \sum_{k\geqslant 0}^{n} \binom{p+k}{k} b_{n-k}; \qquad b_n = \sum_{k=0}^{n} (-1)^k \binom{p+k}{k} a_{n-k}$$

is, therefore, the umbral version of the expansion formula for ∇. Again following Riordan (p. 101), taking

$$e^{x \log(1-t-t^2)^{-1}} = \sum_{n\geqslant 0} \frac{p_n(x)}{n!} t^n,$$

we find that $p_n(x)$ are basic for $Q = [(5 - 4E^{-1})^{1/2} - 1]/2$ and $p_n(1)/n!$ are the Fibonacci numbers, whence a host of identities, and so forth to include Riordan (pp. 99–106).

The case of exponential generating functions (Riordan, pp. 106–114) is simpler; most of the examples treated there reduce to Appell sets and their inverse. The same can be said of the theory of Lagrange series (Riordan, pp. 146–151).

The solution of transcendental equations is sometimes effectively carried out by operator methods. Suppose we are to find a solution t of $y = q(t)$. Letting $Q = q(D)$ (so that we require $q(0) = 0$ and $q'(0) \neq 0$), we find from

$$E^a = \sum_{n \geqslant 0} \frac{p_n(a)}{n!} Q^n$$

that the solution t (Theorem 3) is

$$e^{at} = \sum_{n \geqslant 0} \frac{p_n(a)}{n!} y^n.$$

The Heaviside Calculus

Although the name should be Boole's, the term is usually applied to the study of shift-invariant operators which are polynomials in D (the analog for Δ, although easily derived, does not seem to appear in any treatise on finite differences). There are two main applications. Any differential equation $p(D) f(x) = g(x)$ with $p(0) \neq 0$ has a *unique* polynomial solution for every polynomial $g(x)$, as follows immediately from Corollary 1 to Theorem 3 (this fact has been the point of departure for generalizations to functions of exponential type), and the inverse operator can be written in closed form using the Laguerre operator K and its iterations, which are easily simplified by the Riemann–Liouville formula.

The second (and less well known) is the theory of expansions of formal power series $f(t)$ in powers of a given polynomial $p(t)$ with $p(0) = 0, p'(0) \neq 0$:

$$f(t) = \sum_{n \geqslant 0} \frac{a_n}{n!} (p(t))^n.$$

How are the coefficients a_n to be determined? There is a unique inverse power series $p^{-1}(t)$ of the polynomial $p(t)$. Suppose a delta operator Q can be found for which both $R = p^{-1}(Q)$ and $f(R)$ have a simple enough form. Then $a_n = [f(R)\, p_n(x)]_{x=0}$, where $p_n(x)$ are the basic polynomials of Q, by Theorem 2. This technique works more often than it appears; we illustrate it with an example from the literature.

It was reputedly proved by Schur that in the expansion

$$\sin \pi x = \sum_{n=1}^{\infty} \frac{a_n}{n!} (x(1-x))^n \qquad (*)$$

the coefficients a_n are positive, but no explicit expression was found. Carlitz (1966) found an explicit formula for the coefficients, but it is not clear from his result that the a_n are positive.

Now, it is obvious from (*) that the delta operator in question is $Q = D(I - D)$, whose basic polynomials are $p_n(x)$, computed by

$$p_n(x) = x(I - D)^{-n} x^{n-1},$$

that is,

$$(I - D)^{-n} = \left(\frac{1}{I - D}\right)^n = (I + D + D^2 + D^3 + \cdots)^n$$

$$= I + nD + \binom{n+1}{2} D^2 + \cdots,$$

$$p_n(x) = \sum_{i=0}^{n-1} \binom{n+i-1}{i} (n-1)_i \, x^{n-i}$$

thus (Theorem 3)

$$e^{ax} = \sum_{n \geqslant 0} \frac{p_n(a)}{n!} (x(1-x))^n.$$

Setting $A_n = [p_n(\pi i) - p_n(-\pi i)]/2in!$, Carlitz's explicit expression is obtained. The polynomials $p_n(x)$ and the coefficients a_n can be expressed in the closed form

$$p_n(x) = \frac{x}{(n-1)!} \int_0^{\infty} e^{-t}[t(x+t)]^{n-1} \, dt,$$

$$a_n = \frac{\pi}{(n-1)!} \int_0^{\pi/2} [y(\pi - y)]^{n-1} \sin y \, dy$$

easily derived from the integral form of $(1 - D)^{-n}$. From this, the positivity of a_n can be inferred.

The well known Bessel polynomials $y_n(x)$ of Krall and Frink are not a Sheffer set, but the related set $f_n(x) = x^n y_{n-1}(x^{-1})$ is one. Its delta operator is $Q = D - D^2/2$. This makes some of the results in Carlitz (1957) special cases of the present theory. For instance, the generating function, (Carlitz's 2.5)

$$\sum_{n \geqslant 0} \frac{f_n(x)}{n!} t^n = e^{x[1-(1-2t)^{1/2}]},$$

the property of being of binomial type (2.7); and Carlitz's (2.8) are obtained by computing the connection constants with x^n. The formulas expressing the derivatives of $f_n(x)$ as linear combinations of the $f_n(x)$ follow from the expansion theorems (2.10, 2.12) as do (3.1, 3.2). Burchnall's $\theta_n(x)$ are the Sheffer set relative to $Q' = 1 - D$; this gives $(-2)^n \theta_n(x/2) = L_n^{(-2n-1)}(x)$ by an easy umbral computation. Carlitz's (4.4) gives the connection constants between $L_k^{(\alpha)}(2x)$ and $f_n(x)$, which follow from Theorem 7, and (4.6) connects $\theta_n(x)$ with $f_n(x)$.

Difference Polynomials

They are the Sheffer sets associated with the difference operator $\Delta = E - I$, having the basic polynomials $(x)_n = x(x-1) \cdots (x-n+1)$. (The closely related backward difference operator, $\nabla = I - E^{-1}$, has the basic polynomials $x^{(n)} = x(x+1) \cdots (x+n-1)$. Curiously, the connection constants of $x^{(n)}$ with $(x)_n$ are, apart from sign, the coefficients of the basic Laguerre polynomials (an easy computation using Theorem 7).)

The generating function of a set of difference polynomials can be written in the suggestive form $s(t)^{-1}(1+t)^x$.

The first expansion theorem applied to Δ gives the *Newton* expansion. The expansion of the Bernoulli operator J in powers of Δ is *Gregory's formula.*

Newton's expansion, combined with the identity,

$$\Delta^n = \sum_{k \geqslant 0} \binom{n}{k} (-1)^{n-k} E^k,$$

gives a pair of inverse relations which could simplify many a calculation in the literature (e.g. Carlitz (1952)). Notable difference sets (cf. Boas and Buck) are:

(a) *Poisson-Charlier* polynomials, with $S = E$ (apart from a parameter);

(b) *Narumi* polynomials, with $S = D^k/(\log(I + D))^k$;

(c) *Boole* polynomials, with $S = I + (I + D)^k$;

(d) *Peters* polynomials, with $S = (I + (I + D)^k)^\lambda$;

(e) *Bernoulli* polynomials of the second kind $b_n(x) = J(x)_n$, extensively studied by Jordan.

(f) The *Stirling polynomials* $N_n(x)$, introduced by Nielsen (p. 72), are the basic set inverse to the upper factorial powers $x^{(n)}$. They are, therefore, easily reduced to the exponential polynomials. Nielsen's notation $\psi_n(x)$ is related to the present notation by $(x+1)\psi_n(x) = N_n(-x-1)/n!$. The central difference operator $S = [E^{1/2} - E^{-1/2})/2$ has an extensive literature (but see Riordan, pp. 212–217); it is a special case of an Abel operator. Its basic polynomials are written $x^{[n]}$; their connection constants (the central factorial coefficient) with x^n were computed by Carlitz and Riordan, and their results are derived from Theorem 7 and its corollaries. Expansions in powers of S, such as the formulas of Lubbock and Woolhouse, are heuristically derived by Steffensen; they can, of course, be verified by Theorem 2, whose application becomes particularly useful when the sign of a square root is to be chosen.

It does not seem to have been realized that Newton's expansion and its variants obtained from Theorem 6 yield a powerful technique for proving *binomial identities*. We give a sampling, taken from Riordan (pp. 1–18).

The original Vandermonde formula (3a),

$$\binom{n+p}{m} = \sum_{k\geqslant 0} \binom{n}{k}\binom{p}{m-k},$$

follows from the expansion of $(n+p)_m$ in terms of the basic polynomials $(n)_k$. Grosswald's identity (Example 7),

$$\sum_{k=0}^{2p} (-2)^{-k} \binom{n}{m+k}\binom{n+m+k}{k} = (-1)^p 2^{-2p} \binom{n}{p}, \qquad n-m=2p,$$

becomes clear when one replaces m by $m-n$:

$$\sum_{k=0}^{2p} (-2)^{-k} \binom{n}{m-n+k}\binom{m+k}{k} = (-1)^p 2^{-2p} \binom{n}{p},$$

with $2n-m=2p$. Again replacing k by $2p-k$ on the left, this reduces to

$$\sum_{k=0}^{2p} (-2)^{k-2p} \binom{n}{k}\binom{m+2p-k}{2p-k} = (-1)^p 2^{-2p} \binom{n}{p},$$

and this is clearly a Newton expansion relative to the basic polynomials $(n)_k$; the computation of the coefficient is routine.

The expansion of a product of two binomial coefficients (10),

$$\binom{n}{p}\binom{n}{g} = \sum_{k\geqslant g} \binom{k}{g}\binom{g}{k-p}\binom{n}{k}, \tag{*}$$

follows the same reasoning. Because of its importance, we derive it in full. Jordan's formula,

$$\Delta^k(uv) = \sum_{j=0}^{k} \binom{k}{j} \Delta^j u \Delta^{k-j} E^j v,$$

gives, when $u = (x)_p$ and $v = (x)_g$ and $g \geqslant p$,

$$\begin{aligned}[\Delta^k((x)_p\,(x)_g)]_{x=0} &= \binom{k}{p} p![\Delta^{k-p}E^p(x)_g]_{x=0} \\ &= \binom{k}{p} p!(g)_{k-p}\,[E^p(x)_{g-k+p}]_{x=0} \\ &= \binom{k}{p} p!(g)_{k-p}\,(p)_{g+p-k} = \binom{k}{g}\binom{g}{k-p} p!g!,\end{aligned}$$

as desired.

Shanks' result that

$$\binom{x}{i}^k = \sum_{j=1}^{ik-i+1} A^i_{kj} \binom{x+j-1}{ik}$$

with $A^i_{kj} > 0$, can be established in the same way, but the literature on the A^i_{kj} is scarce.

Abel polynomials

They are the basic polynomials for the delta operator $Q = E^{\alpha}D$, given by (3) of Theorem 4 as

$$A_n^{(\alpha)}(x) = x(x - n\alpha)^{n-1}.$$

Expansions into Abel polynomials have an extensive theory (Hurwitz, Salié, Boas, and Buck). The polynomials have notable statistical and combinatorial significance. Identities for the Abel polynomials, as well as for the related Sheffer polynomials $(x - (n+1)\,a)^n$, follow the same pattern as those for the Gould polynomials. All identities in Riordan (pp. 18–23) can be obtained either by one of the expansion theorems or by umbral composition (sometimes by both methods). Similarly, the Abel inverse relations of Riordan (pp. 92–99) can be obtained by either of the foregoing methods or by recognizing a cross-sequence. As we have already described the techniques in deriving Gould's inversion formulas, we shall not repeat them here. As a simple example of an inverse pair, we quote the following, due to Clarke:

$$b_n = \sum_{k \geqslant 0} \binom{n}{k} kn^{n-k-1} a_k\,,$$

$$a_n = \sum_{k \geqslant 0} (-1)^{n+k} \binom{n}{k} k^{n-k} b_k\,,$$

which the reader will readily identify.

Abel's identity,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} (y + ka)^{n-k} x(x - ka)^{k-1},$$

is nothing but an instance of the first expansion theorem as is the superficially remarkable identity in Bernoulli and Abel polynomials

$$B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(y + ka) x(x - ka)^{k-1},$$

and many similar formulas stated by Nörlund, Steffensen, and others. The inverse set to the Abel polynomials does not seem to have been considered, though they have a combinatorial significance, and we shall briefly derive its properties here. Let

$$\begin{aligned} B_n^{(a)}(x) &= \sum_{k\geqslant 0} \binom{n}{k} x^k (ka)^{n-k} \\ &= \sum_{k\geqslant 0} \frac{x^k}{k!} [E^{ka} D^k x^n]_{x=0}; \end{aligned}$$

from the summation formula we recognize that these are indeed the inverses of the Abel polynomials. Their umbral recursion formula is

$$\mathbf{B}^{(a)}(x) (\mathbf{B}^{(a)}(x) - na)^{n-1} = x^n,$$

and the identity stating that the two sets are inverse is

$$x^n = \sum_{k\geqslant 0} \binom{n}{k} (ka)^{n-k} x(x - ka)^{k-1}.$$

The summation formula (Corollary 7 of Theorem 7) becomes

$$f(\mathbf{B}^{(a)}(x)) = \sum_{k\geqslant 0} \frac{x^k}{k!} f^{(k)}(ka).$$

This identity gives ample evidence of the simplicity of the umbral method.

Various authors have considered basic polynomials relative to the operator $Q = E^a(1 + D)^b D$. The connection constants with the Abel polynomials are easily found by Theorem 4:

$$p_n(x) = \sum_{k=0}^{n-1} (-1)^k \binom{nb + k - 1}{k} (n - 1)_k A_{n-1-k}^{(a)}(x).$$

For $Q = E^{\alpha} e^{D^2/2} D$ we find a generalization of the Hermite polynomials considered by Steffensen. The theory of crosssequences expresses them at once in terms of the Hermite polynomials $H_n(x)$, that is,

$$n^{(n-1)/2} x H_{n-1}\left(\frac{n\alpha - x}{n^{1/2}}\right) = p_n^{(\alpha)}(x).$$

The connection constants with x^n can be computed by the summation formula, in view of the fact that the inverse polynomials can be expressed in terms of the inverses of the Abel polynomials. This gives

$$p_n^{(\alpha)}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} n^{k/2} H_k[\alpha(n^{1/2})]\, x^{n-k}.$$

The inverse connection constants can also be computed by Theorem 7; for $\alpha = 0$ we have

$$x^{2n} = \sum_{k=1}^{n} \frac{(2n)_{2n-2k}}{(n-k)!} k^{n-k} p_{2k}^{(0)}(x),$$

$$x^{2n+1} = \sum_{k=0}^{n} \frac{(2n+1)_{2n-2k}}{(n-k)!} \left(\frac{2k+1}{2}\right)^{n-k} p_{2k+1}^{(0)}(x),$$

two Hermite-reminding identities.

Cotlar Polynomials

An interesting class of Sheffer operators associated with the difference operator Δ has been studied by Cotlar. It is easy to see that a polynomial sequence $p_n(x)$ has the property that $p_n(k) = p_k(n)$ for all nonnegative integers k and n, if and only if it can be written in the form

$$p_n(x) = \sum_{i=0}^{n} \binom{n}{i} \lambda_i (x)_i .$$

for some sequence $\lambda_i \neq 0$. Such sequences of polynomials are said to be ***permutable***. There is one and only one permutable Sheffer set—except for a parameter; it must be a Sheffer set for the delta operator Δ and the invertible operator $(I - a\Delta)^{-1}$; it has the explicit expression

$$a^n (x)_n + \binom{n}{1} a^{n-1} (x)_{n-1} + \cdots + 1 = p_n(x).$$

Again, all Sheffer sets $p_n(x)$ such that the sequence $g_n(x) = p_n(x)/n!$ is permutable can be classified (Cotlar). The delta operator is $\log(1 + aD/(D - I))$

and the invertible operator is $(1 - D)^{-1}$. In particular for $a = -2$ one obtains a sequence of Sheffer polynomials $M_n(x)$ enjoying the remarkable properties

$$\begin{aligned} M_n(-x-1) &= (-1)^n M_n(x), & \\ M_n(k) &= M_k(n), & k, n \geqslant 0, \\ (-1)^k M_{k-1}(-n) &= (-1)^n M_{n-1}(-k); & k, n \geqslant 1. \end{aligned}$$

It can be shown that the three foregoing properties uniquely determine the sequence $M_n(x)$, which is in fact explicitly given by

$$M_n(x) = \frac{2^n}{n!}(x)_n + \frac{2^{n-1}}{(n-1)!}(x)_{n-1} + \cdots + 1.$$

The inverse set of the $M_n(x)$ can be expressed in terms of Bernoulli polynomials.

Exponential polynomials

Also of statistical origin are the *exponential polynomials* $\phi_n(x)$, introduced by Steffensen and studied further by Touchard and others. Some of their properties were developed in III. We recall that they are the basic polynomials for the delta operator $\log(I + D)$, and that they are inverse to $(x)_n$, so that

$$\phi(\phi - 1) \cdots (\phi - n + 1) = x^n,$$

and

$$\begin{aligned} \phi_n(x) &= \sum_{k \geqslant 0} \frac{x^k}{k!} [\Delta^k x^n]_{x=0} \\ &= \sum_{k \geqslant 0} S(n, k)\, x^k, \end{aligned}$$

where, following Riordan's notation, the $S(n, k)$ denote the Stirling numbers of the second kind (and $s(n, k)$ those of the first). Also, the Rodrigues formula ((4) of Theorem 4) says that

$$\phi_n(x) = x(\phi_{n-1}(x) + \phi'_{n-1}(x)).$$

The generalized Dobinsky formula follows most easily by umbral methods. Let $p_n(x) = (x)_n$. Then

$$p_n(\phi(x)) = x^n = e^{-x} \sum_{k \geqslant 0} \frac{p_n(k)}{k!} x^k,$$

and, hence, by linearity

$$p(\phi(x)) = e^{-x} \sum_{k \geqslant 0} \frac{p(k)}{k!} x^k,$$

for every polynomial $p(x)$. Setting $p(x) = x^n$ we obtain finally

$$\phi_n(x) = e^{-x} \sum_{k \geqslant 0} \frac{k^n x^k}{k!}.$$

Similarly one establishes the recursion

$$\phi_{n+1}(x) = x(\phi(x) + 1)^n.$$

We shall add to the properties developed in III the generating function,

$$\sum_{n \geqslant 0} \frac{\phi_n(x)}{n!} t^n = e^{x(e^t - 1)},$$

and Rodrigues' formula, implicitly established in III, that

$$\phi_n(x) = e^{-x}(\mathbf{x}D)^n e^x,$$

which shows the roots of these polynomials to be real. Also, recall that the connection constants with x^n are the Stirling numbers of the second kind. The connection constants between x^n and $\phi_n(x)$ are the Stirling numbers of the first kind, since the $\phi_n(x)$ are the inverse set of the $(x)_n$.

As an example of computation of a "new" set of connection constants, we shall connect the Laguerre polynomials with the polynomials $\phi_n(-x)$. It is easy to see that the $\phi_n(-x)$ are basic for the delta operator $\log(I - D)$. Thus, we must find a formal power series $f(t)$ such that $f(\log(1 - t)) = t/(t - 1)$. Clearly $f(t) = 1 - e^{-t}$ is the desired series. The connection constants are therefore given by the coefficients of the basic sequence for the backward difference operators $\nabla = I - E^{-1}$, namely the polynomials $x(x + 1) \cdots (x + n - 1)$. In symbols,

$$\begin{aligned} L_n(t) &= \phi(x)\,(\phi(x) + 1) \cdots (\phi(x) + n - 1) \\ &= \sum_{k=0}^{n} |\, s(n, k)|\, \phi_k(x). \end{aligned}$$

Riordan's treatment of operators (pp. 200–205) furnishes a further batch of examples of the present theory. We shall now briefly develop some of the properties of the polynomials

$$\psi_n(x) = \sum_{k \geqslant 0} s(n, k)\,(x)_k,$$

which are the difference analogs of the exponential polynomials. The umbral theory of these two sets of polynomials can be used to systematically develop identities for the Stirling numbers.

If V is the umbral operator defined by $V(x)_n = \psi_n(x)$, then by Proposition 1 of Section 7 $\psi_n(x)$ is basic for $V\Delta V^{-1}$. But $Vx^k = (x)_k$, since

$$(x)_n = \sum_{k\geqslant 0} s(n, k)\, x^k,$$

and so $VDV^{-1} = \Delta$. Therefore, $\psi_n(x)$ is basic for

$$Q = V\Delta V^{-1} = V(e^D - I)\, V^{-1} = e^\Delta - I.$$

But then, by Theorem 7,

$$\psi_n(\phi(x)) = \phi_n(\Psi(x)) = (x)_n\,,$$

which give orthogonality relations for the Stirling numbers. The reader should convince himself that Stirling number identities can be inferred from identities relating the $\phi_n(x)$ and the $\psi_n(x)$. We give a sampling, leaving the umbral proofs as exercises.

(1) $\psi_n(x) = e^\Delta \left(\dfrac{\Delta}{e^\Delta - I}\right)^{n+1} (x)_n\,.$

(2) $\phi_{n+1}(x) = x(\phi(x) + 1)^n$ gives

$$S(n + 1, k) = \sum_{i\geqslant 0} \binom{n}{i} S(i, k - 1).$$

(3) $\phi_n(\Psi(x)) = (x)_n$ gives

$$\sum_{k\geqslant 0} S(n, k)\, s(k, i) = \delta_{ni}\,.$$

$\psi_n(\phi(x)) = (x)_n$ gives

$$s(n, k) = \sum_{k,i\geqslant 0} s(n, k)\, s(k, i)\, S(i, j).$$

(4) $\phi_n(x) = e^{-x} \sum\limits_{k\geqslant 0} \dfrac{x^k k^n}{k!}\,.$

Taylor's expansion gives

$$e^x\phi_n(x) = \sum_{k\geqslant 0} \frac{x^k}{k!} [D^k e^t \phi_n(t)]_{t=0},$$

which implies

$$k^n = \sum_{i\geqslant 0} \binom{k}{i} i!\, S(n, i).$$

Also by Taylor,

$$e^{-x} \sum_{k\geqslant 0} \frac{x^k k^n}{k!} = \sum_{k\geqslant 0} \frac{x^k}{k!} \left(D^k \sum_{i\geqslant 0} \frac{e^{-t} t^i i^n}{i!}\right)_{n=0}$$

$$= \sum_{k\geqslant 0} \frac{x_k}{k!} \sum_{i\geqslant 0} \binom{k}{i} (-1)^{k-i}\, i^n,$$

which implies

$$S(n, k) = \frac{1}{k!} \sum_{i\geqslant 0} \binom{k}{i} (-1)^{k-i}\, i^n.$$

(5) $\phi_n(x)$ of binomial type gives

$$\binom{i+j}{i} S(n, i+j) = \sum_{k\geqslant 0} \binom{n}{k} S(k, i)\, S(n-k, j).$$

(6) $\psi_n(x)$ of binomial type gives

$$\binom{i+j}{i} s(n, i+j) = \sum_{k\geqslant 0} \binom{n}{k} s(k, i)\, s(n-k, j).$$

14. Problems and History

We have assembled in random order some open questions suggested by the preceding theory. Other problems are mentioned in the text.

(1) The present work unifies and extends the identities given by Riordan (pp. 1–23, 43–54, 92–116, 128–131, 141–152, 200–205, 212–217), that is, 82 out of 146 pages of text or 56%. We have excluded the exercises for reasons of time. Notable exceptions are Riordan's theory of Chebychev and Legendre inversions, the Bell polynomials, and differential operators of the type xD. Each of these topics calls for a development along a similar line but with a different invariance property than shift-invariance.

(2) Expansions of products of polynomials of one set in terms of those of another can be carried out by the foregoing methods but with difficulty. Indications from special identities (e.g. Hermite, Laguerre) are that there should be a general technique, which could apply more successfully to summing multiple binomial coefficients.

(3) Let $Qx = 1$ for the delta operator Q. Then Q can be embedded in a one-parameter group of operators $Q^{(t)}$ whose indicators satisfy the functional equation

$$q^{(t)}(q^{(s)}(x)) = q^{(s+t)}(x).$$

The corresponding basic sets satisfy

$$q_n^{(t)}(\mathbf{q}^{(s)}(x)) = q_n^{(s+t)}(x).$$

Develop the theory of such sets. How can the "infinitesimal generator" be computed? The simplest example of this is the basic Laguerre set.

(4) It has been suggested by Gould that some of the identities in Vandermode convolution are analogous to Kapteyn series. Several other analogies with classical eigenfunction expansions can be noted, which suggest an extension of the theory to classes of special functions. Truesdell's theory is helpful in this connection. Another possible extension is to exponential polynomials.

(5) Statistical, probabilistic and combinatorial interpretations of the identities are worthwhile. Several special sets, e.g. Abel, are connected with particular distributions of statistics (see e.g. Dwass, Pyke). There are at least three possibilities; interpretation as compound Poisson processes; interpretation through stationary stochastic processes, as in the relation of Hermite polynomials to Brownian motion of the Poisson–Charlier polynomials by the Poisson process, and, finally, the combinatorial interpretation through counting binomial type structures such as reluctant functions (see III). Very little is known about combinatorial interpretation of Sheffer polynomials; occasionally (Laguerre) they arise in counting permutations with restricted position. A major step forward would be a combinatorial or probabilistic interpretations of Bernoulli numbers; we surmise that the fact that these are, apart from a factor, the cumulants of the uniform distribution is relevant.

(6) One of the most difficult open problems is that of estimating the remainder after n terms in the expansion formulas. Little is known except in the Appell case. For p-adic convergence, the results are comparatively simple (see LeVeque, p. 55ff.), but undeveloped.

(7) Which Sheffer sets are orthogonal relative to some weight function in some region of the complex plane? Such a region is probably related to the convergence region of Boas and Buck.

(8) Another approach to the present theory is through the techniques of Hopf algebras. The algebra of polynomials in the variable x is a Hopf algebra, with diagonal map

$$\Delta\colon x^n \to \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}.$$

The dual Hopf algebra is the algebra of differential operators with constant coefficients, the pairing between the two being given by

$$\langle p(D), q(x)\rangle = [p(D)\, q(x)]_{x=0}\,.$$

An umbral operator can be defined as one that commutes with the diagonal map, for example. The greater elegance of this approach is evident, as are some of its advantages: one can consider differential operators acting or polynomials or polynomials $p(x)$ as operators on operators. In addition, this point of view should point the way to a generalization to several variables, to the exterior algebra (in infinite dimensions) and to more general Hopf algebras. The theory of spherical harmonics should fit in one such generalization.

(9) There is a curious relationship between the coefficients of the expansion of a probability distribution into Hermite polynomials, and the cumulants. If the mean is zero and the variance one, the two coincide up to $n = 5$; this led Jordan (1972) to mistakenly conclude (p. 150) that they all coincide, but see Kendall and Stuart (p. 158). At any rate, the relationship between the two sets of coefficients seems fairly simple and should be worked out, especially in view of the mystery underlying the cumulants. Note that one can define cumulants relative to any sequence of binomial type, e.g. the factorial cumulants (Kendall and Stuart). Do these lend themselves to easier interpretations?

(10) There is no special reason for choosing polynomials instead of trigonometric polynomials; various identities relating Fourier and Dirichlet expansions might become clearer, for example the relationship between Bernoulli numbers and the values of the zeta function.

(11) Work out formulas for $p_n(Q)$, when $p_n(x)$ is a Sheffer set relative to the delta operator Q.

(12) There are several relationships between the factorization of differential operators with polynomial coefficients (of which no general theory exists) and Sheffer sets, see e.g. the last chapter of Riordan and various papers of Klamkin and Newman. One should begin by developing the theory of xD; for example, $L_n(xD)$ has a simple expression (why?). See also Rainville (1941), Carlitz (1930), and Carlitz (1932).

(13) The Laguerre polynomials are formally related to the gamma distributions as the Hermite to the normal, the Poisson–Charlier to the Poisson; nevertheless, a specific construction of the corresponding stochastic process or a group of transformations relative to which they are the "spherical harmonics" seems to still be missing.

(14) Various representations of the inner product making the Sheffer polynomials orthogonal are possible, and they should be investigated. The classical theory of orthogonal polynomials may have extensions to inner products "involving derivatives." In what sense is the inner product of Section 9 "natural"? The inner product for the Hermite polynomials with negative or imaginary variance is particularly interesting, in view of possible connection with the Feynman integral.

(15) The explicit representation of umbral operators leads to operator-differential equations in the Pincherle derivative, and is an untouched subject of great interest.

(16) The theory of factorial series (see e.g. Nörlund or Nielsen) indicates that expansions in series of the form $\sum_{n\geqslant 0} a_n/p_n(x)$ are at least possible in some cases. Is it possible to extend the present theory in this direction?

(17) In the same vein, the divided difference operation,

$$\Delta : f(x) \to \frac{f(x)-f(y)}{x-y},$$

is easily checked to be coassociative. This suggests that the theory be best developed in the context of coalgebras (Sweedler) and that a suitable notion of shift-invariance may be at hand. The same may be said of Thiele's inverse differences (Nörlund).

(18) An operational calculus, as understood in the last fifty years, is an isomorphism of a function algebra into an algebra of operators. In this respect, the isomorphism in the present calculus possesses one extra feature: it preserves functional composition, in fact, it gives meaning to it in terms of an operation on operators. Can this feature be carried over to other operational calculi?

(19) Work out representations of shift-invariant operators analogous to Post's inversion formula for the Laplace transform.

(20) Under what conditions are the zeros of a Sheffer set real?

(21) Evidently the kind of umbral composition we have considered is not as general as it should be, as it does not explain why $H_{2n}(x)$ is a constant multiple of $L_n^{(-1/2)}(x^2)$.

(22) The analogy between the functions e^{ax} and $(a-x)^{-1}$ suggests that there should be a theory of operators where shift-invariance is replaced by the functional equation

$$(a-x)^{-1}-(a-y)^{-1}=(x-y)\,(a-x)^{-1}\,(a-y)^{-1}.$$

This suggests parametrized families T_x of operators such that

$$T_xT_y=(T_x-T_y)/(x-y).$$

Some work of Redheffer supports this feeling.

(23) It is easy to see that a polynomial $p(x)$ is positive for all integer values of x if and only if its expansion in a Newton series has nonnegative coefficients. We conjecture that analogous results exist for Laguerre and Hermite polynomials and relate to the position of the zeros of these polynomials.

History

It is impossible to account for the detailed development of the Heaviside calculus from its beginnings; we shall only mention the works that relate to the present approach. Perhaps the most striking feature of this subject is that each author in the past would develop one approach to the exclusion of others. Thus, Carlitz, Riordan, and Steffensen, while feeling at home with generating functions, are somewhat ill-at-ease when handling operators, called by Steffensen "symbols." Pincherle, on the other hand, is fully aware of the abstract possibilities of the concept of operator, but ignorant of the nitty-gritty of numerical analysis, where he would have found a fertile ground for his ideas. Sheffer also uses power series in preference to operators, with a resulting lack of completeness.

The characterizations of basic polynomials, Sheffer polynomials and cross-sequences in terms of a binomial property (Theorems 1 and 8, and Proposition 6 of Section 5) are new. Other authors have used characterizations in terms of operators, thereby missing one of the main techniques. The two expansion theorems may also be said to be new, although various partial versions may be found in the literature from Pincherle on. The notions of a delta operator and basic sets are due to Steffensen (who, however, did not give them a name and did not realize that they were one and the same as sequences of binomial type) as is that of a cross-sequence (again unnamed and uncharacterized). The isomorphism theorem was at least intuited by Pincherle, and has been tacitly—and often unrigorously—used by several authors.

The idea of applying the Pincherle derivative (the name is ours) in the present context is new; it greatly simplifies the proof of Theorem 4 (first guessed by Steffensen) as well as the theory of Laguerre and Hermite polynomial, to name only a few instances. Theorem 5 is new (first stated in III). The recurrence formulas are due to Sheffer, as are the eigenfunction expansion formulas, with the exception of the explicit inner products; his proofs, however, use power series. Section 7 is new, as are most of the results in Section 8. In the examples, detailed references are given.

An extended bibliography has been appended as a hunting ground for further applications and extensions of the present methods. Items cited in the bibliography of Mullin–Rota will not be repeated here.

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