Generalized Inverses and Operator Equations

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ABSTRACT

In this paper we obtain conditions under which the operator equations of the types $AX = C$ and $AXA^* = C$ have hermitian and nonnegative definite solutions; here A is assumed to be relatively invertible. In addition we obtain some properties of generalized inverses of operators. Lastly we pose some conjectures; one of them is that the set of all nonzero relatively invertible operators is not connected.

1. INTRODUCTION

Let H be an infinite dimensional complex Hilbert space. Let $\sigma(A)$, $\sigma_r(A)$, $\sigma_l(A)$, $\sigma_p(A)$, $W(A)$, and $\overline{W(A)}$ denote the spectrum, right spectrum, left spectrum, point spectrum, numerical range, and closure of the numerical range of an operator A on H.

Khatri and Mitra [6] consider matrix equations of various types and obtain conditions for hermitian and nonnegative definite solutions. This motivates us to consider operators on an infinite dimensional Hilbert space. We deal with operator equations in Sec. 2. In Sec. 3 we are concerned with the generalized inverses of elements of a Banach algebra. In particular we show that right (or left) invertible elements of a Banach algebra $\mathfrak{B}$ form an open subset of $\mathfrak{B}$. Lastly we talk about generalized inverses of operators on H.

An element a of $\mathscr{B}$ is left (right, relatively) invertible or regular if and only if there is an element b of $\mathscr{B}$ such that $ba = e$ ($ab = e$, $aba = a$, respectively) where e is the identity of $\mathscr{B}$. An element is said to be invertible or regular if it is right as well as left invertible. All these definitions are valid for the operator algebra $\mathscr{B}(H)$ as well.

Halmos [4] has shown that the set of all invertible operators is connected. In this context, what can one say about the set of operators having generalized inverses? Here we pose some conjectures concerning connectedness of the set of nonzero relatively invertible operators on H and the set of operators which are power bounded, along with their generalized inverses.

In what follows $\mathscr{R}(A)$ denotes the range and $\bar{A}$ denotes the relative inverse of an operator A on H. For more details on relative invertibility see Halmos [4] and Koliha [7]. The definitions of hermitian and nonnegative definite operators are the usual ones.

2. HERMITIAN AND NONNEGATIVE DEFINITE SOLUTIONS OF OPERATOR EQUATIONS

Khatri and Mitra [6] have tackled matrix equations of various types and obtained the conditions under which these equations have hermitian and nonnegative definite solutions. In this section we consider operator equations on infinite dimensional Hilbert space and obtain conditions for hermitian and nonnegative definite solutions of these equations. In what follows, by a general hermitian (nonnegative definite) solution X' we mean $X' = X_0 + X_1$, where X_0 is any hermitian (nonnegative definite) solution of the given equation and X_1 is a hermitian (nonnegative definite) solution of the corresponding homogeneous equation.

THEOREM 2.1. *Let A and C be operators on H. If A is relatively invertible, then the equation*

$$AX = C \tag{1}$$

has a hermitian solution if and only if CA^* is hermitian.

Proof. If X is a hermitian solution of (1), then obviously CA^* is hermitian. Conversely, if CA^* is hermitian, $CA^* = AC^*$. Then $X_0 = \bar{A}C + C^*(\bar{A})^* - \bar{A}AC^*(\bar{A})^*$ is a hermitian solution of (1), for

$$AX_0 = A\bar{A}C + AC^*(\bar{A})^* - A\bar{A}AC^*(\bar{A})^* = A\bar{A}C = C.$$

The last equality follows from the fact that $A\overline{A}$ is a projection of H onto $\mathcal{R}(A)$, $\mathcal{R}(A\overline{A}) = \mathcal{R}(A)$ (See Koliha [7]) and $\mathcal{R}(C) \subset \mathcal{R}(A)$. That X_0 is hermitian follows easily by using our hypothesis $CA^* = AC^*$. ∎

COROLLARY 2.1. *The general hermitian solution of* (1) *is given by* $\overline{A}C + C^*(\overline{A})^* - \overline{A}AC^*(\overline{A})^* + (I - \overline{A}A)S(I - \overline{A}A)^*$, *S being an arbitrary hermitian operator on H.*

Proof. Consider the homogeneous equation

$$AX = 0. \tag{2}$$

For any hermitian operator S, $X_1 = (I - \overline{A}A)S(I - \overline{A}A)^*$ satisfies (2). Moreover X_1 is hermitian. Therefore, the hermitian solution of (1) is given by $X = X_0 + X_1$. ∎

THEOREM 2.2. *If Eq.* (1) *has a nonnegative definite solution, then* CA^* *is nonnegative definite. Conversely, if* CA^* *is nonnegative definite and relatively invertible, and* $\mathcal{R}(CA^*) = \mathcal{R}(C)$, *then Eq.* (1) *has a nonnegative definite solution.*

In order to prove Theorem 2.2, we need the following Lemma, the proof of which follows from the properties of relative inverses.

LEMMA. *If a nonnegative definite operator S is relatively invertible, then its relative inverse S is nonnegative definite on* $\mathcal{R}(S)$.

Proof of Theorem 2.2. If X is a nonnegative definite solution of (1), then obviously CA^* is nonnegative definite. Conversely, let CA^* be nonnegative definite and relatively invertible. Then $X_0 = C^* \overline{(CA^*)} C$ will be nonnegative definite, since $\mathcal{R}(C) = \mathcal{R}(CA^*)$ and the Lemma is applicable. Also this X_0 satisfies (1), and we are through. ∎

COROLLARY 2.2. *The general nonnegative definite solution of* (1) *is given by*

$$C^* \overline{(CA^*)} C + (I - \overline{A}A)S(I - \overline{A}A)^*,$$

where S is an arbitrary nonnegative definite operator on H.

Proof. For any nonnegative definite operator S, $X_1 = (I - \bar{A}A)S(I - \bar{A}A)^*$ is also nonnegative definite. This X_1 satisfies the equation $AX = 0$. Therefore the general nonnegative definite solution of (1) is given by $X = X_0 + X_1$. ∎

The following theorem involves operators on the direct sum $H \oplus H$.

THEOREM 2.3. *Let* A, B, C, D *be any operators on* H *such that* A *and* B *are relatively invertible. The equations*

$$AX = C \quad and \quad XB = D \tag{3}$$

(I) *have a common hermitian solution if and only if the operator*

$$M = \begin{pmatrix} CA^* & 0 \\ 0 & D^*B \end{pmatrix}$$

is hermitian, in which case the general hermitian solution is

$$\begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \begin{pmatrix} C & 0 \\ 0 & D^* \end{pmatrix} + \begin{pmatrix} C^* & 0 \\ 0 & D \end{pmatrix} \left\{ \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \right\}^*$$

$$- \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- M \left\{ \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \right\}^*$$

$$+ \left\{ I - \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix} \right\} S \left\{ I - \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix} \right\}^*,$$

where S *is an arbitrary hermitian operator on* $H \oplus H$;

(II) *have a common nonnegative definite solution if and only if* M *is nonnegative definite and relatively invertible, and*

$$\mathcal{R}(M) = \mathcal{R} \left\{ \begin{pmatrix} C^* & 0 \\ 0 & D \end{pmatrix} \right\},$$

in which case the general nonnegative definite solution is

$$\begin{pmatrix} C^* & 0 \\ 0 & D \end{pmatrix} \bar{M} \begin{pmatrix} C & 0 \\ 0 & D^* \end{pmatrix}$$

$$+ \left\{ I - \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix} \right\} S \left\{ I - \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}^- \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix} \right\}^*,$$

where S *is an arbitrary nonnegative definite operator on* $H \oplus H$.

Proof. It is to be noted that the necessary and sufficient condition under which the equations (3) will have a common hermitian (nonnegative definite) solution is that the operator equation

$$\begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & X^* \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & D^* \end{pmatrix}$$

on $H \oplus H$ has a hermitian (nonnegative definite) solution. Theorem 2.3 now follows from Theorems 2.1 and 2.2.   ∎

Khatri and Mitra [6] have dealt with matrix equations of the type $AXB = C$, where A and B are nonnegative definite matrices. Also, Douglas [3] has considered operator equations of the type $S^*XT = X$. Here we consider the operator equation $AXA^* = C$.

THEOREM 2.4. *Let A be a relatively invertible operator on H, and C be a nonnegative operator on H. Then the equation*

$$AXA^* = C \tag{4}$$

has a general nonnegative definite solution given by $X = \overline{A}C(\overline{A})^* + (I - \overline{A}A)S(I - \overline{A}A)^*$, where S is an arbitrary nonnegative definite operator on H.

Proof. Let us consider $X_0 = \overline{A}C(\overline{A})^*$. Then it follows that $AX_0A^* = C$. Again $X_1 = (I - \overline{A}A)S(I - \overline{A}A)^*$ is a solution of the homogeneous equation $(A + B)X(A + B) = 0$, for

$$(A + B)X_1(A + B) = (A + B)(I - \overline{A}A)S(I - A^*\overline{A}^*)(A + B)$$

$$= B(I - \overline{A}A)S(A - A^*\overline{A}^*A + B - A^*\overline{A}^*B)$$

$$= B(I - \overline{A}A)S(A + B - A - B) = 0.$$

Therefore the general solution of (4) is given by $X = X_0 + X_1$.   ∎

3. GENERALIZED INVERSES OF ELEMENTS

In this section we describe the classes of left, right, and relatively regular elements of a Banach algebra $\mathfrak{B}$ with identity e.

It is easy to see that the product of two right (left) regular elements of $\mathfrak{B}$ is again right (left) regular. In the case of relatively regular elements of $\mathfrak{B}$, a similar result holds under a certain condition. In fact we have:

RESULT 3.1. *Let x_1 and x_2 in $\mathfrak{B}$ be relatively regular with relative inverses y_1 and y_2 respectively. Then $x_1 x_2$ is relatively regular if and only if $y_1 x_1 x_2 y_2$ is relatively regular.*

Proof. We have $x_1 y_1 x_1 = x_1$ and $x_2 y_2 x_2 = x_2$. Let $x_1 x_2$ be relatively regular with u as relative inverse. Then

$$(x_1 x_2) u (x_1 x_2) = x_1 x_2. \tag{5}$$

Consider

$$(y_1 x_1 x_2 y_2)(x_2 u x_1)(y_1 x_1 x_2 y_2)$$

$$= y_1 x_1 (x_2 y_2 x_2) u (x_1 y_1 x_1) x_2 y_2$$

$$= y_1 x_1 (x_2 u x_1) x_2 y_2 = y_1 x_1 x_2 y_2,$$

from which it is clear that $y_1 x_1 x_2 y_2$ is relatively regular. Conversely, let $y_1 x_1 x_2 y_2$ be relatively regular with v as the relative inverse. Then we have $(y_1 x_1 x_2 y_2) v (y_1 x_1 x_2 y_2) = y_1 x_1 x_2 y_2$, which implies that $(x_1 y_1 x_1)(x_2 y_2 v y_1 x_1)(x_2 y_2 x_2) = (x_1 y_1 x_1)(x_2 y_2 x_2)$. That is, $x_1 x_2 (y_2 v y_1)(x_1 x_2) = x_1 x_2$.

This shows that $x_1 x_2$ is relatively regular. ∎

REMARK. We have used simple arguments to prove the above result which was proved by Koliha [7] for operators.

THEOREM 3.2. *The set $\mathfrak{B}_r$ (or $\mathfrak{B}_l$) of right (or left) regular elements of $\mathfrak{B}$ is an open subset of $\mathfrak{B}$. More specifically, if $x \in \mathfrak{B}_r$, then $\{ y : \| y - x \| \leqslant \| x_1 \|^{-1} \} \subset \mathfrak{B}_r$, x_1 being the right inverse of x.*

Proof. If $\| y - x \| \leqslant \| x_1 \|^{-1}$, then we have

$$\| e - y x_1 \| = \| x x_1 - y x_1 \| \leqslant \| x - y \| \, \| x_1 \| < 1,$$

which shows regularity of yx_1; from this it is clear that y is right regular. ∎

It is to be noted that for regular elements of $\mathscr{B}$, Berberian [1, p. 214] has given a similar version.

4. GENERALIZED INVERSES FOR OPERATORS

Halmos (see [4, problem 143]) deals with two operators whose difference is a compact operator, one of them being invertible. Here we weaken the condition of invertibility to right invertibility and obtain some interesting corollaries. The technique used is essentially that of Halmos.

THEOREM 4.1.   *If A and B are operators such that $A - B$ is compact and if $\lambda \in \sigma_r(A) - \sigma_p(A)$, then $\lambda \in \sigma_r(B)$.*

Proof.   Translate by λ and reduce the assertion to this: "If A is not right invertible, $\ker A = \{0\}$, then B is not right invertible." Contrapositively, if B is right invertible, then either $\ker A \neq \{0\}$ or A is right invertible. Let B_1 be the right inverse of B, i.e., $BB_1 = 1$. Now A can be expressed as $A = B\{I + B_1(A - B)\}$. Since $A - B$ is compact, obviously $B_1(A - B)$ is also compact. Write $S = B_1(A - B)$. For this compact S, either -1 is an eigenvalue of S or $I + S$ is invertible. If -1 is an eigenvalue of S, then there exists a nonzero x in H such that $Sx = -x$, yielding $Ax = 0$. This will imply that $\ker A \neq \{0\}$. On the other hand, if $I + S$ is invertible, then $(I + S)^{-1}B_1$ acts as the right inverse of $B(I + S) = A$. Thus we arrive at our conclusion. ∎

Following is a consequence of the very well-known Wold decomposition of isometries and also of Theorem 4.1.

COROLLARY 1.   *For no nonunitary isometry U can $\mathrm{Im}\, U$ be compact.*

REMARK.   The result stated in Corollary 1 seems to be related to the known fact that the hyponormal operator with compact imaginary part is normal (see [9, p. 583]).

COROLLARY 2.   *A right invertible operator (in particular, the adjoint of unilateral shift) cannot be perturbed by a compact operator to a one–to–one quasinilpotent operator.*

Halmos [5] has proved that if U is a isometry and A is quasinilpotent, then $\|U - A\| \geq 1$. In the following theorem we observe that a similar conclusion holds in the case where U is coisometric.

THEOREM 4.2.   *If U is coisometric and A is quasinilpotent, then $\|U - A\| \geqslant 1$.*

The proof of Theorem 4.2 follows on the lines of Halmos [5], and hence we omit it. Again by using the arguments due to Halmos (see [4, Problem 119]), we have the following:

THEOREM 4.3.   *If U is unilateral shift and if V is any unitary operator, then $N = (V^* U - 1)$ is normaloid. This N is dissipative also.*

Halmos [4] has shown that if a partial isometry is sufficiently near an isometry, then it is an isometry. Analogously we have:

THEOREM 4.4.   *Let U and V be relatively invertible and left invertible operators, respectively, with $\|U - V\| < k$ (where k is a lower bound of V). Then U is left invertible.*

Proof.   Since V is left invertible, it is bounded from below; i.e., there exists a real nonzero number k such that $\|Vx\| \geqslant k\|x\|$ for all $x \in H$. In view of the hypothesis, it follows that $\ker U = \{0\}$. For if not, then $\exists x \in H$ with $Ux = 0$, $x \neq 0$. Now $k\|x\| \leqslant \|Vx\| = \|Ux - Vx\| \leqslant \|U - V\| \, \|x\| < k\|x\|$, a contradiction. Thus U has trivial kernel. Moreover U is relatively invertible. In view of Halmos [4, Problem 70], it follows that U is left invertible.   ∎

Halmos [4, Problem 100] deals with the closedness and connectedness of the set of all nonzero partial isometries. Naturally this motivates us to similar considerations in case of relatively invertible operators. Also, we try to replace isometries and coisometries by left and right invertible operators respectively.

Halmos (see [4, solution to Problem 119]) remarks that the set of all isometries is not only closed but also open in the set of all partial isometries, and hence the set of all nonzero partial isometries is not connected. This suggests:

CONJECTURE 1.   The set of all nonzero relatively invertible operators is not connected.

In order to have positive answer to this conjecture (in view of Theorem 3.2) it will suffice to show that the set of all left invertible operators is closed in the set of all nonzero relatively invertible operators.

It is obvious that each operator similar to a unitary operator is power bounded, i.e., if $A = S^{-1}US$, then $\|A^n\| \leqslant C$ for every positive integer n (where $C = \|S^{-1}\| \, \|S\|$) (see Halmos [5]). It will be interesting to consider an operator A which is similar to isometry, coisometry, or partial isometry. In

these cases A will be left, right, and relatively invertible, respectively, and both A and its respective inverse will be power bounded.

Sz.-Nagy [2] has proved that if A is invertible and both A and A^{-1} are power bounded, then A is similar to a unitary operator. (See Halmos [5] also.) What will be the implication of power boundedness of A and its generalized inverses? Patel [8] has proved that if T is a left invertible operator with a left inverse T_1, and if there exists an operator S such that $T^* = S^{-1}T_1 S$, and $0 \notin \overline{W(S)}$, then T is similar to an isometry. One can show that if T is relatively invertible operator with a relative inverse T_1 and if there exists a positive operator S such that $T^* = S^{-1}T_1 S$, $0 \notin \overline{W(S)}$, then T is L-similar to a partial isometry, i.e., there exist operators B and R_1 such that $T = RBR_1$, where $R_1 = R^* S^{-1}$, $RR^* = S$ (obviously here R is right invertible) and B is partial isometry.

This motivates us to consider the following conjecture.

CONJECTURE 2.   If A and B are power bounded, where B is the left, right, or relative inverse of A, then A is similar to a isometry, coisometry, or partial isometry respectively.

The authors are grateful to the referee for his kind suggestions.

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