Towards $\psi-$extension of Finite Operator Calculus of Rota

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Abstract

ψ- extension of Gian-Carlo Rota's finite operator calculus due to Viskov [1, 2] is further developed. The extension relies on the notion of ∂_ψ-shift invariance and ∂_ψ-delta operators. Main statements of Rota's finite operator calculus are given their ψ-counterparts. This includes Sheffer ψ-polynomials properties and Rodrigues formula - among others. Such ψ-extended calculus delivers an elementary umbral underpinning for q-deformed quantum oscillator model and its possible generalisations.

∂_q-delta operators and their duals and similarly ∂_ψ-delta operators with their duals are pairs of generators of $\psi(q)$- extended quantum oscillator algebras. With the choice $\psi_n(q) = [R(q^n)!]^{-1}$and $R(x) = \frac{1-x}{1-q}$ we arrive at the well known q-deformed oscillator. Because the reduced incidence algebra $R(L(S))$ is isomorphic to the algebra Φ_ψ of ψ -exponential formal power series - the ψ-extensions of finite operator calculus provide a vast family of representations of $R(L(S))$.

KEY WORDS: extended umbral calculus, quantum q-plane
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1 Introduction

The main aim of this paper is a presentation of ψ- extension of Rota‘s finite operator calculus simultaneously with an indication that this is a natural and elementary method for formulation and treatment of q-extended and posibly R-extended or ψ-extended models for quantum-like ψ-deformed oscillators in order to describe [3] eventual processes with parastatistical behavior.

We owe such ψ-extension in operator form to Viskov [1, 2]. This is achieved by considering not only polynomial sequences of binomial type but also of $\{s_n\}_{n\geq 1}$-binomial type where $\{s_n\}_{n\geq 1}$-binomiality is defined with help of the generalized factorial $n_s! = s_1 s_2 s_3 ... s_n$ where $S = \{s_n\}_{n\geq 1}$ is an arbitrary sequence with the condition $s_n \neq 0$, $n \in N$.

A year after Viskov's paper [2] and few years before Roman [4]-[8] Cigler and Kirchenhofer in [9, 10] set up foundations of q-umbral calculus as an extension of the umbral calculus defined in [11] in terms of the algebra of linear functionals P^* - see also [12] two years after [9, 10].

In our presentation here we use the operator formulation as in Rota-Mullin calculus [13]. The ψ- extension Rota‘s finite operator calculus extends the content of Rota‘s devise "much is the iteration of the few" - much of the properties of special polynomials is the application of few basic principles.

Apart from pure mathematical interest there exist also other motivations. We mention here only just one based on the simple observation. Namely q-oscillator algebras generators are the so called ∂_q-delta operators $Q(\partial_q)$ and their duals and these are the basic objects of the q-extended finite operator calculus of Rota to be formulated. (Of course $\partial_q \hat{x} - q\hat{x}\partial_q = id.$)

These q-oscillator algebras generators are encountered explicitly or implicitly in [14, 15] and in many other subsequent references - see [16, 17, 18, 19, 20] and references therein. There q-Laguerre and q-Hermite polynomials appear [18, 19, 20] which are just ∂_ψ-basic polynomial sequences of the ∂_ψ-delta operators $Q(\partial_\psi)$ for $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ and corresponding choice of $Q(\partial_\psi)$ functions of ∂_ψ- see next sections. The case $\psi_n(q) = \frac{1}{R(q^n)!}$: $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ appears implicitly in [21] where advanced theory of general quantum coherent states is beeing developed. In [22] it was noticed that commutation relations for the q-oscillator algebras generators from [14, 15] and others (see also [18, 19, 20, 17]) in appropriate operator variables might be given the form [22]:

$$AA^+ - \mu A^+ A = 1;\ \mu = q^2$$

if appropriate operator variables are chosen [22]. In this connection let us note that various q-deformations of the natural number n for the Fock space representation of normalized eigenstates of $\mid n >$of excitation number operator N are widely used in literature on quantum groups and at least some families of quantum groups may be constructed from q-analogues of Heisenberg-Weyl algebra (see for example: [14, 15, 22, 23, 24, 25]). Note also that q-analogues of Heisenberg-Weyl algebra appear as special cases of general quantum Clifford algebras - understood as Chevalley-Kahler deformations of braided exterior algebras [26, 27, 28].

The known important fact is that the q-commutation relation $AA^{+} - A^{+}A = 1$ leads (see [17]) to the q-deformed spectrum of excitation number operator N and to various parastatistics [3] . More possibilities result from considerations of Wigner [29] extended by the authors of [3]. We therefore hope that the $\psi(q)$-calculus of Rota to be developed here might be useful in a C^{*} algebraic [17] description of "$\psi(q)$-quantum processes" - if any - with various parastatistics [3].

Note also that the "usual" commutation relation $AA^{+} - qA^{+}A = 1$, known since middle of nineteenth century [30] is very useful in such problems as derivation of Bernoulli -Taylor formula with the rest of Cauchy form, inversion of formal power series and Lagrange formula [31, 32]. The ψ-counterparts are expected.

The second purpose of the introduction is to establish notations to be subsequently used . Here and once for all $P \equiv \mathbf{F}[\mathrm{x}]$ denotes the algebra of polynomials over the field F, char $F = 0$. The standard q-deformation of the real number is of the form $x_q \equiv \frac{1-q^x}{1-q}$ and the Jackson's ∂_q- derivative (see: [33]-[38] and [39]) is defined by

$$(\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}$$

We use the following notation for q-factorial

$$n_q! = n_q(n-1)_q!;\ 1_q! = 0_q! = 1;\ n_q! \xrightarrow[q-1]{} n!.$$

As for the further notation we enclose now **the main abbreviations** used in this paper. At first let $\Im$ be the set of function sequences such that

$$\Im = \{\psi; R \supset [a,b]; q \in [a,b]; \psi(q) : Z \to F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N\}.$$

We introduce the notations to be used in the sequel with help of ∂_ψ and n_ψ symbols where $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$; $n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi(n-1)_\psi(n-2)_\psi(n-3)_\psi ... 2_\psi 1_\psi$ and $0_\psi! = 1$. The symbol ∂_ψ is defined as follows. Let $\partial_\psi x^n = n_\psi x^{n-1}; n \geq 0$. The operator ∂_ψ is then linearly extended and we call it the ψ-derivative. There are some special cases on the way. For $\psi_n(q) = \frac{1}{R(q^n)!}$: $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ [21]. If $R(x) = \frac{1-x}{1-q}$ then $n_\psi = n_q$ and $\partial_R = \partial_q$ [9, 10, 11]. Using the above abbreviations we are able now to introduce the next ones.

Abbreviations

1. $(x+_\psi a)^n = \sum_{k\geq 0} \binom{n}{k}_\psi a^k x^{n-k}$ where $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$ and $n_\psi^{\underline{k}} = n_\psi(n-1)_\psi(n-2)_\psi...(n-k+1)_\psi$; $\binom{n}{k}_q$ is the Gauss polynomial in the variable q with integer coefficients and its symbol $\binom{n}{k}_q$ shares [39] main properties typical for $\binom{n}{k}$.

2. $(x+_\psi a)^n \equiv E^a(\partial_\psi)x^n$; $E^a(\partial_\psi) = \sum_{n\geq 0} \frac{a^n}{n_\psi!}\partial_\psi^n$; $E^a(\partial_\psi)f(x) = f(x+_\psi a)$;

   Note however that $(x+_\psi a)^n \neq (x+_\psi a)^{n-1}(x+_\psi a)$.

   For $q=1$ $E^a\left(\frac{d}{dx}\right)$is called the shift operator in [13, 40] while in [11] it is to be the translation operator. In [9, 10] $E^a(\partial_\psi)$ is also named translation operator. The operators $E^y(Q) = \sum_{n\geq 0} p_n(y)\frac{Q^n}{n_\psi!}$ are generalised translation operators [41].

3. $\sum_\psi$ denotes the algebra of ∂_ψ- shift invariant operators where a linear operator $T_{\partial_\psi} : P \rightarrow P$ is ∂_ψ-shift invariant iff $\forall_{\alpha\in F}\left[T_{\partial_\psi}, E^a(\partial_\psi)\right] = 0$.

4. $Q(\partial_\psi)$ denotes the ∂_ψ-delta operator. (It is a specific formal series in ∂_ψ).

5. The symbol $x_{Q(\partial_\psi)}$ denotes the operator dual to the $Q(\partial_\psi)$ operator.
   The map $x_{Q(\partial_\psi)}$ is called an "umbral shift operator" in [11] and in the functional formulation of undeformed umbral calculus [11] it is the operator adjoint to a derivation of P^* the linear space of linear functionals on P - see: Theorem 5 in [11] and see also 1.1.16 in [10]. Of course : for $Q = id$ we have : $x_{Q(\partial_R)} \equiv x_{\partial_R} \equiv \hat{x}$. (see Definition 5.1)

6. The linear operator $\hat{x}_\psi$ is defined accordingly: $\hat{x}_\psi : P \rightarrow P$; $\hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)}x^{n+1} = \frac{n+1}{(n+1)_\psi}x^{n+1}$; $n \geq 0$. In special cases we write: $\hat{x}_R : P \rightarrow P$, $\hat{x}_R x^n = \frac{(n+1)}{R(q^{n+1})}x^{n+1}$, $n\geq 0$; $\hat{x}_q : P \rightarrow P$, $\hat{x}_q x^n = \frac{n+1}{(n+1)_q}x^{n+1}$, $n \geq 0$.

7. The Pincherle ψ-derivative is the linear map ': $\sum_\psi \rightarrow \sum_\psi$ defined by the commutator
   $$T_{\partial_\psi}{}' = T_{\partial_\psi}\hat{x}_\psi - \hat{x}_\psi T_{\partial_\psi} \equiv [T_{\partial_\psi}, \hat{x}_\psi].$$

8. Let now $\{p_n\}_{n\geq 0}$ be the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi) = Q$; then the $\hat{q}_{\psi,Q}$- operator is a liner map $\hat{q}_{\psi,Q} : P \rightarrow P$ defined in the basis $\{p_n\}_{n\geq 0}$ by $\hat{q}_{\psi,Q}p_n = \frac{(n+1)_\psi - 1}{n_\psi}p_n$; $n \geq 0$.
   We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$-*mutator operator*. For $\psi_n(q) = \frac{1}{R(q^n)!}$, $\hat{q}_{R,id}x^n = \frac{R(q^{n+1})-1}{R(q^n)}x^n$, $n\geq 0$. If in addition $R(x) = \frac{1-x}{1-q}$ then $\hat{q}_{R,id}x^n = qx^n$.

9. The $\hat{q}_{\psi,Q}$-mutator of A and B operators reads: $AB - \hat{q}_{\psi,Q}BA \equiv [A,B]_{\hat{q}_{\psi,Q}}$.

The organization of the paper is the following.

In section 2 we observe that the reduced incidence algebra $R(L(S))$ is isomorphic to the algebra Φ_ψ of formal ψ-exponential power series. This makes the link of what follows after - to combinatorics.

Then in section 3 we note - as expected - that the algebra Φ_ψ of ψ-exponential formal power series is isomorphic to the algebra $\sum_\psi$ of ∂_ψ-shift invariant operators which is the basement of $\psi(q)$-extended finite operator calculus. We use this opportunity to introduce there primary objects of the calculus.

In section 4 we develop further this $\psi(q)$- calculus providing few examples of application of principal theorems.

With section 5 we finish our considerations by observing [9, 10] that one can formulate q-extended finite operator calculus with help of "quantum q-plane" q-commuting variables $A, B : AB - qBA \equiv [A,B]_q = 0$.

The question whether one may formulate the ψ-extended finite operator calculus with help of a "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : AB - \hat{q}_{\psi,Q}BA \equiv [A,B]_{\hat{q}_{\psi,Q}} = 0$ is discussed there also.

2 Incidence algebras - primary information and possibility of ψ-extensions

Pierre Simon Laplace (1749-1827) introduced the correspondence between *operations* on sets and on *operations* on formal power series. This has been afterwards developed into the vast activity domain of nowadays' "generatinfunctionology" [42]. One of few turning points in this area was the use of incidence algebras by Rota and his collaborators [13]. These incidence algebras were invented as a still more systematic technique for setting up wider class of generating functions algebras encompassing more than classical algebras of ordinary, exponential, Dirichlet, Eulerian etc. generating functions. Incidence algebras were independently introduced also by Scheid [43] and Smith [44, 45, 46] - see also [47].

In this section we follow the Rota's way of presentation of finite operator calculus as its extensions admit the similar treatment. In order to establish the notation in a selfcontained way and to express our main starting observation let us recall the required notions.

Definition 2.1. *Let $I(P,\mathbf{F}) = \{f; f : P \times P \to \mathbf{F}; f(x,y) = 0;$ unless $x\pi y;$ $x, y \in P\}$ where F is a field; char $F = 0$ and (P,π) is locally finite partially ordered set. Then $(I(P,\mathbf{F}), \mathbf{F}; +; \ ^*; \ ^\circ)$ is called the incidence algebra of P, where "+" and*

"$\circ$" denote the sum of functions and usual multiplication by scalars, while for any $f, g \in I(P, \mathbf{F})$ the following product is defined

$$(f^{*}g)(x, y) = \sum_{z \in P} f(x, z) g(z, y).$$

Let as recall that a partially ordered set is locally finite iff every its segment $[x, y]$ is finite, where $[x, y] = \{z \in P; x \leq z \leq y\}$ hence the summation above is finite as it is over segment $[x, y]$. Here now are some examples [13] for illustration purpose.

Example 2.1. Let P be the set of nonnegative integers
$P = \{0, 1, 2, 3, 4, 5, 6, 7, ...\}$ and let $\pi \equiv \leq$ then

$$I(P, \mathbf{F}) = \{(a_{ij}); a_{ij} = 0 \ i < j\} \subset M_{\infty}(\mathbf{F})$$

i.e. $(I(P, \mathbf{F}), \mathbf{F}; +; \ ^{*}; \ ^{\circ})$ is represented by the algebra of upper triangular infinite matrices over field $\mathbf{F}$.

Example 2.2. The algebra of formal power series is isomorphic to incidence algebra $R(P)$ where $(P; \pi) \equiv (P; \leq)$ and $P \equiv \mathbf{N} \cup \{0\}$. (As a matter of fact $R(P)$ is the so called standard <u>reduced</u> incidence algebra see - below [13]).
The isomorphism mentioned above is given by the bijective correspondence ϕ

$$\sum_{n \geq 0} a_n z^n \xrightarrow{\varphi} f \equiv \left\{ f_{ij}; f_{ij} = \left\{ \begin{array}{cc} a_{i-j} & i \leq j : \quad i, j \in P \\ 0 & otherwise \end{array} \right. \right\}$$

where $h \equiv f^{*}g$ with $f, g, h \in R(P)$ corresponds to convolution of $\phi^{-1}(f)$ and $\phi^{-1}(g)$ i.e.

$$h(i, j) = \sum_{i \leq k \leq j} f(i, k) g(k, j) = \sum_{i \leq k \leq j} a_{k-i} b_{j-k} \equiv \sum_{r=0}^{n} a_r b_{n-r}$$

after setting $r = k - i$ and $n = j - i$.

Example 2.3. The algebra of formal exponential power series is isomorphic to incidence algebra $R(L(S))$; where $L(S) = \{A; A \subseteq S; |A| < \infty\}$ and S is countable while $(L(S); \subseteq)$ is partially ordered set - ordered by inclusion. As a matter of fact $R(L(S))$ is the so called reduced incidence algebra of the partially ordered set (poset) $L(S)$.
The isomorphism mentioned above is given by the bijective correspondence ϕ:

$$F(z) \equiv \sum_{n \geq 0} \frac{a_n}{n!} z^n \xrightarrow{\varphi} f = \left\{ f(A, B) = \left\{ \begin{array}{ll} a_{|B-A|} & A \subseteq B \\ 0 & otherwise \end{array} \right. ; \quad A, B \in L(S) \right\}$$

where the product $h = f^*g$ with $f, g, h \in R(L(S))$ corresponds to binomial convolution of $F \equiv \phi^{-1}(f)$ and $\phi^{-1}(g) \equiv G$ i.e. for $H(z) \equiv \sum_{n\geq 0} \frac{c_n}{n!} z^n$ $(H \equiv \phi^{-1}(h))$ and $G(z) \equiv \sum_{n\geq 0} \frac{b_n}{n!} z^n$ we get $c_n = \sum_{k\geq 0}^{n} \binom{n}{k} a_k b_{n-k}$.
With "at the point" convergence $I(P; \mathbf{F})$ becomes a topological algebra .

The q-extension or ψ-extension of the above examples and definitions is automatic. It amounts to changes in notation $n \to n_q \to n_\psi$ where - recall it again - $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$;
$n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi(n-1)_\psi(n-2)_\psi(n-3)_\psi ... 2_\psi 1_\psi$ and $0_\psi! = 1$ while $\psi \in \Im$ where $\Im = \{\psi; R \supset [a, b]; q \in [a, b]; \psi(q) : Z \to F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N\}$.
Incidence algebras do characterise their correspondent partially ordered sets as follows.

Theorem 2.1. *Let P, Q be locally finite partially ordered sets. Let $I(P; \mathbf{F})$ and $I(Q; \mathbf{F})$ algebras be isomorphic. Then P and Q are isomorphic.*

Reduced incidence algebras and incidence coefficients are of the more frequent use where the reduced incidence algebras are obtained as quotients of incident algebras segments‘ families and an order compatible equivalence relation. They corresponds to formal series of various kind. The incident coefficients in their turn are generalisation of the binomial coefficients [13].

Definition 2.2. *Let $\sim$ denote an equivalence relation defined on the family $S(P)$ of segments of P - a locally finite partially ordered set. Let $f, g \in I(P; \mathbf{F})$ be such that for $[x, y], [u, v] \in S(P)$ and $[x, y] \sim [u, v]$ equalities $f(x, y) = f(u, v)$ and $g(x, y) = g(u, v)$ take place. If $(f^*g)(x, y) = (f^*g)(u, v)$ $\forall$ $[x, y], [u, v]$ such that $[x, y] \sim [u, v]$ then the relation "$\sim$" is said to be order compatible.*

For detailed information on properties and basic facts about incidence algebras and compatible equivalence relation on locally finite partially ordered sets see [47, 13, 44, 45, 46].

Definition 2.3. *Let P be a locally finite partially ordered set equipped with a compatible equivalence relation $\sim$ on $S(P)$. The set of all functions defined on $S(P)/\sim$ with the product defined below is called the reduced incidence algebra $R(P; \sim)$.*

In order to define the product of $f : S(P)/\sim \to \mathbf{F}$ and $g : S(P)/\sim \to \mathbf{F}$ referred to in the definition above let us denote by $\alpha, \beta, ...$ the nonempty equivalence classes of segments of P i.e. $\alpha, \beta, ... \in S(P)/\sim$ and let us call them [13] *types*.

Definition 2.4. $(Map(S(P)/\sim;\mathbf{F}),\mathbf{F};+;\ ^*;^\circ) \equiv \mathbf{R}(\mathbf{P};\sim)$ *is an algebra under the multiplication "$*$" defined as follows:*

$$Map(S(P)/\sim;\mathbf{F}) \ni f,g \to h := f^*g;$$

$$S(P)/\sim \ni \alpha \to h(\alpha) := \sum_{(\Lambda)} \begin{bmatrix} \alpha \\ \beta\gamma \end{bmatrix} f(\beta)g(\gamma),$$

where the sum $\sum\limits_{(\Lambda)}$ *ranges over all ordered pairs* (β,γ) *of all types and the brackets* $\left[\begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix}\right]$ *are defined below.*

Definition 2.5. $\left[\begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix}\right]$:= *the number of such distinct elements* z *in a segment* $[x,y]$ *of type* α *and such elements* z *that* $[x,z]$ *is of type* β *while* $[z,y]$ *is of type* γ.

One may prove [13] that the reduced incidence algebra $R(P;\sim)$ {i.e. the incidence algebra modulo $\sim$} is isomorphic to a subalgebra of the incidence algebra of P.
Now we may formulate the main observation of this section which links reduced incidence algebras with their representations by ψ-*extended* Rotas' calculus (see: next sections).
For that to see it now and then to explore let us announce ahead the existence (theorem 3.3) of the algebra isomorphism $\sum_\psi \approx \Phi_\psi$ where $\sum_\psi$ denotes the algebra of ∂_ψ -shift invariant operators and Φ_ψ is the algebra of ψ-exponential formal power series. This is to be combined with the following observation.
Starting Observation

The algebra Φ_ψ of ψ-exponential formal power series is isomorphic to the reduced incidence algebra $R(L(S))$. The isomorphism ϕ is given by the bijective correspondence

$$F_\psi(z) = \sum_{n\geq 0} \frac{a_n}{n_\psi!} z^n \xrightarrow{\phi} f = \left\{ f(A,B) = \begin{cases} a_{|A-B|} & A \leq B \\ 0 & \text{otherwise} \end{cases} ; A,B \in L(S) \right\}$$

Here $h := f^*g$ - where $f,g,h \in (R(L(S))$ - corresponds to ψ-binomial convolution i.e. for

$$H_\psi(z) = \sum_{n\geq 0} \frac{c_n}{n_\psi!},\ G_\psi(z) = \sum_{n\geq 0} \frac{b_n}{n_\psi!} z^n \ \textit{and}\ [n]h(z) \equiv [n](f^*g)(z) \equiv c_n;$$

$$c_n = \sum_{k\geq 0}^{n} \binom{n}{k}_\psi a_k b_{n-k}.$$

Remark 2.1. Before coming over to presentation of ψ-extension of finite operator calculus let us remark that variety of enumerative problems are instances [47] of the general problem of inverting indefinite sums [48] ranging over a partially ordered set [47].

The inversion can be carried out with the use of an analogue of difference operator (relative to this partial order).

Next - the indefinite sums ranging in the locally finite partially ordered set P are analogues of indefinite integral - while various difference operators (for example ∂_ψ-delta operators) are analogues of differentiation operator D.
Namely let $f : P \to F$ be a function defined on a locally finite partially ordered set P. Let the indefinite sum g be calculated via $g(x) = \sum\limits_{y \leq x} f(y)$ then the differentiation is being realized by $f(x) = \sum\limits_{y \leq x} g(y)\mu(y,x)$ i.e. by the Möbius inversion formula [47], where $\mu \in I(P,F)$ is the Möbius function on the locally finite partially ordered set P inverse to zeta function $\zeta \in I(P,F)$ determined by $\zeta(x,y) = 1\ for\ x \leq y$ and $\zeta(x,y) = 0\ otherwise$ where of course $x, y \in P$.

In the case of Φ_ψ representation of the reduced incidence algebra $R(L(S))$ the incidence coefficients correspond to ψ-binomial coefficients and the types correspond to ψ-extended integers.

It is known that the binomiality of basic polynomial sequences resulting through isomorphisms $\sum_\psi \approx \Phi_\psi \approx R(L(S))$ has transparent combinatorial interpretation (see: [49, 50] and [42] - page 91. It remains to be an open question for the present author what is the role and scope of possible applications to combinatorics of ψ-binomiality.

Anyhow as we shall see it in the sequel - at least special ψ-extensions supply a mathematical underpinning for $\psi(q)$-deformed quantum oscillator algebras .

3 Finite Operator ψ-Calculus - an Introduction

In sections 3 and 4 we give an exposition of the beginnings of ψ-extension of umbral calculus. This includes among others proofs of The First and The Second Expansion Theorems as well as Rodrigues formula and Spectral Theorem .

Let us however start from the beginning i.e. from the idea.
The objective of [13] was a unified theory of special polynomials. We extend this objective to encompass also correspondent ψ-extended families of polynomials i.e. as a matter of fact - to encompass all those polynomial sequences $\{s_n(x)\}_{n=0}^\infty$,

deg $s_n = n$. (E. Loeb [51]) which may be considered as a Sheffer polynomial sequence with respect the corresponding ∂_ψ-delta operator $Q(\partial_\psi)$. A still further generalisation is to be found in Markowsky paper [40] where generalised Sheffer polynomial sequences are just polynomial sequences $\{s_n(x)\}_{n=0}^{\infty}$; deg $s_n = n$ and the role of ∂_ψ-delta operators $Q(\partial_\psi)$ is taken over by generalised differential operators [40]. Many of the results of ψ-calculus to be developed here may be extended to Markowsky Q-umbral calculus where Q stands for a generalised difference operator i.e. the one lowering the degree of any polynomial by one.
The way to achieve our ψ-goal as in [13] is to exploit the duality between the $\hat{x}$ and $\frac{d}{dx}$ the predecessors of the delta operator notion and its dual.
The technique used and co-invented mostly by [13] is of the past century origin and it is the so called symbolic calculus.

In this section we shall refer all the time to Rota [13] and we shall follow his way of presentation. The language as well as notation have been designed to reflect this resolution. Let us then start with recalling the basic definitions.

Definition 3.1. *Let $E^y(\partial_\psi) \equiv exp_\psi\{y\partial_\psi\} = \sum_{k=0}^{\infty} \frac{y^k \partial_\psi^k}{n_\psi!}$ be the linear operator $E^y(\partial_\psi)$: $P \rightarrow P$. Let $T_{\partial_\psi} : P \rightarrow P$ be a linear operator; then T_{∂_ψ} is ∂_ψ-shift invariant iff*

$$\forall \alpha \in F; \ \left[T_{\partial_\psi}, E^\alpha(\partial_\psi)\right] = 0.$$

Notation: $\sum_\psi$denotes the algebra of F-linear ∂_ψ-shift invariant operators T_{∂_ψ}.

Definition 3.2. *Let $Q(\partial_\psi) : P \rightarrow P$ be a linear operator $Q(\partial_\psi)$ is a ∂_ψ-delta operator iff*

(a) $Q(\partial_\psi)$ *is* ∂_ψ*-shift invariant;*

(b) $Q(\partial_\psi)(id) = const \neq 0$.

Observation 3.1. *Let $Q(\partial_\psi) \in \sum_\psi$ be the ∂_ψ-delta operator. Then for every constant polynomial $a \in P$ we have $Q(\partial_\psi)a = 0$.*

Proof. Recall that $\forall\ \alpha \in F; \quad [Q(\partial_\psi), E^\alpha(\partial_\psi)] = 0$ then by linearity of $Q(\partial_\psi)$ we have
$(Q(\partial_\psi)E^a(\partial_\psi))$(x) = $Q(\partial_\psi)$ (x+a) = $Q(\partial_\psi)$ (x) + $Q(\partial_\psi)$ (a) = c + $Q(\partial_\psi)$ (a) and at the same time $(E^a(\partial_\psi)Q(\partial_\psi))$(x) = $E^a(\partial_\psi)$ (c) = c. ∎

Observation 3.2. *If $p \in P$, deg $p = n$ then deg $(Q(\partial_\psi)p_n) = n - 1$.*

Proof. The proof goes like in [13]. We "just replace" shift invariance by ∂_ψ-shift invariance .

$Q(\partial_\psi)(E^a(\partial_\psi)(\mathrm{x}^n)) = Q(\partial_\psi)(\mathrm{x} +_\psi \mathrm{a})^n \equiv \sum_{k\geq 0}\binom{n}{k}_\psi a^k Q(\partial_\psi)x^{n-k} = E^a(\partial_\psi)$ $(Q(\partial_\psi)(\mathrm{x}^n)) \equiv r(x +_\psi a)$, where $\binom{n}{k}_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$.

Hence $r(a) = \sum_{k\geq 0}\binom{n}{k}_\psi a^k Q(\partial_\psi)x^{n-k}|_{x=0}$. The coefficient of a^n is therefore equal to zero according to the observation 3.1. At the same time the coefficient of a^{n-1} is equal to $\binom{n}{n-1}_\psi \equiv c \neq 0$ □

Definition 3.3. *A polynomial sequence $\{p_n\}_{n\geq 0}$, deg $p_n = n$ such that*

(a) $p_o(x) = 1$;

(b) $p_n(0) = 0$; $n > 0$;

(c) $Q(\partial_\psi)p_n = n_\psi p_{n-1}$

is called the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$.

The condition b) $p_n(0) = 0$, $n > 0$ used in [13] is supeflouous as from a) and c) ψ-binomiality is easily proven. From the ψ-binomiality property one derives b).

Proposition 3.1. *Every ∂_ψ-delta operator $Q(\partial_\psi)$ has the unique sequence of ∂_ψ-basic polynomials i.e.*

$$Q(\partial_\psi) \quad \begin{array}{c} \xrightarrow{1:1} \\ \xleftarrow[1:1]{} \end{array} \quad \{p_k\}_0^\infty$$

Proof. For $n = 0$ put $p_o(z) = 1$, for $n = 1$ put $p_1(x) = \frac{x}{Q(\partial_\psi)(id)}$. Then inducing on n assume that $\{p_k(z)\}$ have been defined for $k < n$. From this inductive assumption we infer that p_n is defined uniquely. For that to see it is enough to notice that for any $p\in P$, deg $p = n$ i.e.

$$p(z) = az^n + \sum_{k=0}^{n-1} c_k p_k(z) \text{and} a \neq 0$$

we have $Q(\partial_\psi)p(z) = aQ(\partial_\psi)x^n + \sum_{k=1}^{n-1} c_k k_q p_{k-1}(z)$ and $degQ(\partial_\psi)(z^n) = n - 1$. Hence there exist a unique choice of constants c_1, ... ,c_{n-1} for which

$$Q(\partial_\psi)p = n_q p_{n-1;}$$

This determines $p \equiv p_n$ uniquely except for the constant term c_o which is however determined uniquely by the condition $p_n(0) = 0$, $n > 0$. $\square$

Inspired by the predecessors $\hat{x}$ and $\frac{d}{dx}$ of the notions developed in [13] we introduce the next basic notion.

Definition 3.4. *A polynomial sequences* $\{p_n\}_o^\infty$ *is of* ψ*-binomial type if it satisfies the recurrence* $E^y(\partial_\psi)\, p_n(x) \equiv p_n(x +_\psi y) = \sum\limits_{k\geq 0} \left(\begin{array}{c} n \\ k \end{array}\right)_\psi p_k(x)\, p_{n-k}(y)$ *where* $\left(\begin{array}{c} n \\ k \end{array}\right)_\psi \equiv \frac{n_\psi^{\underline{k}}}{k_\psi!}$.

Theorem 3.1. $\{p_n\}_o^\infty$ *is a* ∂_ψ*-basic sequence of some* ∂_ψ*-delta operator* $Q(\partial_\psi)$ *iff it is a sequence of* ψ*-binomial type.*

Proof.

(A) $Q(\partial_\psi)^k p_n(x) = n_\psi^{\underline{k}} p_{n-k}(x)$ therefore for $k = n$ $[Q(\partial_\psi)^n p_n(x)]|_{x=0} = n_\psi!$ while for $0 < k < n$ $\left[Q(\partial_\psi)^k p_n(x)\right]|_{x=0} = 0$ hence $p_n(x) = \sum\limits_{k\geq 0} \frac{p_k(x)}{k_\psi!} \left[Q(\partial_\psi)^k p_n(x)\right]|_{x=0}$ and by linearity argument $\forall p \in P$

$$p(x) = \sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} \left[Q(\partial_\psi)^k p(x)\right]|_{x=0}$$

With the choice $p(x) = p_n(x +_\psi y)$ where y is a parameter - one gets $p_n(x +_\psi y) = \sum\limits_{k\geq 0} \frac{p_k(x)}{k_\psi!} \left[Q(\partial_\psi)^k p_n(x +_\psi y)\right]|_{x=0}$ from which one infers that

$$\begin{aligned} \left[Q(\partial_\psi)^k p_n(x +_\psi y)\right]|_{x=0} &= \left[Q(\partial_\psi)^k E^y(\partial_\psi)\, p_n(x)\right]|_{x=0} = \\ = \left[E^y(\partial_\psi)\, Q(\partial_\psi)^k p_n(x)\right]|_{x=0} &= \left[E^y(\partial_\psi)\, n_\psi^{\underline{k}} p_{n-k}(x)\right]|_{x=0} = \\ &= n_\psi^{\underline{k}} p_{n-k}(x +_\psi y)|_{x=0} = n_\psi^{\underline{k}} p_{n-k}(y). \end{aligned}$$

Altogether

$$p(x +_\psi y) = \sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} n_q^{\underline{k}} p_{n-k}(y) = \sum_{k\geq 0} \left(\begin{array}{c} n \\ k \end{array}\right)_\psi p_k(x)\, p_{n-k}(y) \qquad (*)$$

(B) suppose now that $\{p_n\}_o^\infty$ is a sequence of ψ-binomial type. Setting in (*) $y = 0$ we get

$$p_n(x) = \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x) p_{n-k}(0) \quad =$$

$$p_n(x) p_n(0) + \quad n_\psi p_{n-1}(x) p_1(0) \quad + \quad \binom{n}{2}_\psi p_{n-2}(x) p_2(0) \quad + \quad ...$$

from which we infer that $p_0(x) = 1$ and $p_n(0) = 0$ for $n > 0$. It is sufficient now to define the ∂_ψ-delta operator $Q(\partial_\psi)$ corresponding to $\{p_n\}_o^\infty$. We define it uniquely according to:

(a) $Q(\partial_\psi) p_0(x) = 0$,

(b) $Q(\partial) p_n = n_\psi p_{n-1}$ for $n > 0$,

(c) $Q(\partial_\psi)$is linear.

We now prove that $Q(\partial_\psi)$ is ∂_ψ-shift invariant. For that to do use

$$p_n(x +_\psi y) = \sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} Q(\partial_\psi)^k p_n(y)$$

which by linearity argument extends to any $p \in P$

$$p(x +_\psi y) = \sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} Q(\partial_\psi)^k p(y).$$

Now [13] replace p by $Q(\partial_\psi) p$ and interchange x and y thus getting

$$(Q(\partial_\psi) p)(x +_\psi y) = \sum_{k\geq 0} \frac{p_k(y)}{k_\psi!} Q(\partial_\psi)^{k+1} p(x).$$

However note that
$(Q(\partial_\psi) p)(x +_\psi y)$= $E^y(\partial_\psi)(Q(\partial_\psi) p)(x)$=$E^y(\partial_\psi) Q(\partial_\psi) p(x)$ and

$$\sum_{k\geq 0} \frac{p_k(y)}{k_\psi!} Q(\partial_\psi)^{k+1} p(x) = Q(\partial_\psi) \sum_{k\geq 0} \frac{p_k(y)}{k_q!} Q(\partial_\psi)^k p(x) =$$
$$= Q(\partial_\psi)(p(x +_\psi y)) = Q(\partial_\psi) E^y(\partial_\psi) p(x).$$

$\square$

Theorem 3.2. (First ψ-Expansion Theorem)
Let $T_{\partial_\psi} : P \to P$ be a ∂_ψ-shift invariant operator. Let $Q(\partial_\psi)$ be a ∂_ψ-delta operator with ∂_ψ-basic polynomial sequence $\{p_n\}_o^\infty$. Then

$$T_{\partial_\psi} = \sum_{n\geq 0} \frac{a_n}{n_\psi!} Q(\partial_\psi)^n \quad , \text{ where } a_k = \left[T_{\partial_\psi} p_k(z)\right]|_{z=0} .$$

Proof. Due to Theorem 3.1 the proof is the same as in [13]. Namely the ψ-binomial formula

$$p_n(x +_\psi y) = \sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} Q(\partial_\psi)^k p_n(y)$$

acted upon by ∂_ψ-shift invariant operator T_{∂_ψ} extends by linearity argument to

$$T_{\partial_\psi} p(x +_\psi y) = \sum_{k\geq 0} \frac{T_{\partial_\psi} p_k(x)}{k_\psi!} Q(\partial_\psi)^k p(y) \text{, where} y \text{is treated as a parameter.}$$

Setting above $x = 0$ and exchanging mutually x with y symbols we arrive at

$$T_{\partial_\psi} p(x) = \sum_{k\geq 0} \frac{\left[T_{\partial_\psi} p_k(y)\right]_{y=0}}{k_\psi!} Q(\partial_\psi)^k p(x) .$$

□

Remark 3.1. The first expansion theorem might serve us to express popular q-deformations of delta operators in terms of the others.
(Recall: $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$. For $\psi_n(q) = \frac{1}{R(q^n)!}$: $n_\psi = n_R$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ and if in addition $R(x) = \frac{1-x}{1-q}$ then $n_\psi = n_q$ and $\partial_R = \partial_q$). For example the easy expansion of difference operator $\Delta = \sum\limits_{n\geq 1} \frac{\delta_n}{n!}\frac{d^n}{dx^n}$ where $\delta_n = [\Delta x^n]_{x=0}$ =1 suggests a natural q-deformation of difference operator Δ in the form $\Delta_q = \sum\limits_{n\geq 1} \frac{1_q}{n_q!}\partial_q^n$. Of course every ∂_q-shift invariant operator T_{∂_q} is of the form $T_{\partial_q} = \sum\limits_{n\geq 0} \frac{a_n}{n_q!}\partial_q^n$ and in particular every ∂_q-*delta* operator $Q(\partial_q)$ has an expansion with help of any other one. For example for delta operators $\frac{d}{dx}$ and Δ one has $\frac{d}{dx} = \sum\limits_{k\geq 1} \frac{d_k}{k!}\Delta^k$ where $d_k = \left[\frac{d}{dx}x^{\underline{k}}\right]_{x=0} = (-1)^{k-1}(k-1)!$ and correspondingly : $\partial_q = \sum\limits_{k\geq 1} \frac{d_k^{(q)}}{k_q!}\Delta_q^k$ where $d_k^{(q)}$ might be calculated using the explicit form of ∂_q-basic polynomial sequence of the Δ_q. This form is obtained with help of statement (3) of the theorem 4.1 - see examples at the end of section 4.

Theorem 3.3. $\Sigma_\psi \approx \Phi_\psi$

Let $Q(\partial_\psi)$ be a ∂_ψ-delta operator and let Φ_ψ be the algebra of formal exp_ψ series of $t \in F$ over the same field F for which $Q(\partial_\psi)$ is defined. Then there exists an isomorphism φ:
$\varphi : \Phi_\psi \to \Sigma_\psi$ of the algebra Φ_ψ onto the algebra Σ_ψ of ∂_ψ-shift invariant operators T_{∂_ψ} which carries

$$f_\psi(t) = \sum_{k\geq 0} \frac{a_k t^k}{k_\psi!} \xrightarrow{into} T_{\partial_\psi} = \sum_{k\geq 0} \frac{a_k}{k_\psi!}\partial_\psi^k.$$

Proof. The proof goes like in [13] without significant changes. Indeed, with φ being obviously linear and onto by the first expansion theorem, it is enough to show that φ preserves products.
In the algebra Φ_ψ the product is given by the ψ-binomial convolution i.e. we have

$$\left(\sum_{k\geq 0} \frac{a_k}{k_\psi!}x^k\right)\left(\sum_{k\geq 0} \frac{b_k}{k_\psi!}x^k\right) = \left(\sum_{k\geq 0} \frac{c_k}{k_\psi!}x^k\right)$$

where

$$c_n = \sum_{k\geq 0} \binom{n}{k}_\psi a_k b_{n-k}.$$

Therefore with

$$\Phi_\psi \ni s(x) = \sum_{k\geq 0} \frac{b_k}{k_\psi!}x^k \quad \xrightarrow{on} \quad \sum_{k\geq 0} \frac{b_k}{k_\psi!}Q(\partial_\psi)^k \quad = S_{\partial_\psi} \in \Sigma_\psi$$

it is enough to show that

$$\left[T_{\partial_\psi} S_{\partial_\psi} p_n(x)\right]_{x=0} \; = \; c_n = \sum_{k\geq 0} \binom{n}{k}_\psi a_k b_{n-k}$$

and this is the case because - due to $p_n(0) = 0$ for $n > 0$ & $p_0(x) = 1$ we have

$$\left[T_{\partial_\psi} S_{\partial_\psi} p_n(x)\right]_{x=0} \; = \; \left[\left(\sum_{k\geq 0} \frac{a_k}{k_\psi!}Q(\partial_\psi)^k \sum_{r\geq 0} \frac{b_r}{r_\psi!}Q(\partial_\psi)^r\right) p_n(x)\right]_{x=0} =$$

$$= \left[\sum_{k\geq 0}\sum_{r\geq 0} \frac{a_k}{k_\psi!}\frac{b_r}{r_\psi!}Q(\partial_\psi)^{k+r} p_n(x)\right]_{x=0} =$$

$$= \left[\sum_{k\geq 0} \frac{a_k b_{n-r}}{k_\psi! \left(n-r\right)_\psi!} Q\left(\partial_\psi\right)^n p_n\left(x\right) \right]_{x=0} = \left[\sum_{k\geq 0} \frac{a_k b_{n-r}}{k_\psi! \left(n-r\right)_\psi!} n_\psi! p_0\left(x\right) \right]_{x=0} =$$

$$= \sum_{k\geq 0} \binom{n}{k}_\psi a_k b_{n-k}$$

□

Remark 3.2. In vain of the proof above one easily notices that the algebra Φ_ψ of ψ-exponential formal power series is isomorphic to the reduced incidence algebra R(L(S)) where the isomorphism φ is given by the bijective correspondence:

$$F_\psi\left(z\right) = \sum_{n\geq 0} \frac{a_n}{n_\psi!} z^n \xrightarrow{\varphi} f = \left\{ f\left(A,B\right) = \left\{ \begin{array}{l} a_{|B-A|};\ A \leq B \\ 0;\quad otherwise \end{array} \right. ;\quad A, B \in L\left(S\right) \right\}$$

where $f, g, h \in R(L(S))$ and $h := f * g$ corresponds to ψ-binomial convolution i.e. for

$$H_\psi\left(z\right) = \sum_{n\geq 0} \frac{c_n}{n_\psi!} \text{ and } G_\psi\left(z\right) = \sum_{n\geq 0} \frac{b_n}{n_\psi!} z^n\ ,\ [n]W(z) \equiv [n](f^* g)(z) \equiv c_n \text{ we get}$$

$$c_n = \sum_{k\geq 0}^{n} \binom{n}{k}_\psi a_k b_{n-k}.$$

This remark and the isomorphism $\Sigma_\psi \approx \Phi_\psi$ constitute expected link of finite operator ψ-calculus with incident algebras . Namely- the ψ- extension of Rota's operator calculus is a general representation of the algebra structure of $R(L(S))$. In any kind of algebra the principal question is to find out which elements have their inverse. In the case of the algebra Σ_ψ we know the answer due to the isomorphism Theorem 3.3. Indeed, in the algebra Φ_ψ of ψ-exponential formal power series its' element $\sum\limits_{k\geq 0} \frac{a_k}{k_\psi!} x^k$ is invertible iff $a_0 \neq 0$ because then and only then the infinite system of linear equations $a_0 b_0 = 1\,;\ a_0 b_1 + a_1 b_0 = 0\,;\ a_0 b_2 + a_1 b_1 + a_2 b_0 = 0...$ has solution and then this solution is unique.

Corollary 3.1. *Operator $T_{\partial_\psi} \in \Sigma_\psi$ has its' inverse $T^{-1}_{\partial_\psi} \in \Sigma_\psi$ iff $T_{\partial_\psi} 1 \neq 0$.*

Proof. Take $n = 0$ in $\left[T_{\partial_\psi} S_{\partial_\psi} p_n\left(x\right)\right]_{x=0} = \sum\limits_{k\geq 0} \binom{n}{k}_\psi a_k b_{n-k}$ and use the Theorem 3.3 where $T_{\partial_\psi} = \sum\limits_{n\geq 0} \frac{a_n}{n_\psi!} \partial_\psi^n$ and $S_{\partial_\psi} = \sum\limits_{n\geq 0} \frac{b_n}{n_\psi!} \partial_\psi^n$. □

Remark 3.3. The operator $E^a(\partial_\psi) = exp_\psi\{a\partial_\psi\}$ is invertible in Σ_ψ but it is not ∂_ψ-delta operator. From Corollary 3.1 we infer that no one of ∂_ψ-delta operators $Q(\partial_\psi)$ is invertible.
¿From the stated above and the first expansion theorem we have the next corollary.

Corollary 3.2. *Operator $R_{\partial_\psi} \in \Sigma_\psi$ is a ∂_ψ-delta operator iff $a_0 = 0$ and $a_1 \neq 0$, where $R_{\partial_\psi} = \sum_{n\geq 0} \frac{a_n}{n_\psi!} Q\left(\partial_\psi\right)^n$ or equivalently : $r(0) = 0$ & $r'(0) \neq 0$ where $r(x) = \sum_{k\geq 0} \frac{a_k}{k_\psi!} x^k$ is the correspondent of R_{∂_q} under the isomorphism Theorem 3.3.*

Now note that every ∂_ψ-delta operator $Q(\partial_\psi)$ is a function Q of ∂_ψ with the expansion

$$Q(\partial_\psi) = \sum_{n\geq 1} \frac{q_n}{n_\psi!} \partial_\psi^n$$

while ∂_ψ-delta operator ∂_ψ is a function of ∂_ψ-delta operator $Q(\partial_\psi)$ with the expansion

$$\partial_\psi = Q^{-1}(Q(\partial_\psi)) = \sum_{n\geq 0} \frac{q_n}{n_\psi!} Q^{-1}(Q(\partial_\psi))^n \quad ;$$

$$Q \circ Q^{-1} = id.$$

We shall now derive an expression for the ψ-exponential generating function for ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi)$ using notation established above.

Remark 3.4. $exp_\psi\{zx\}$ is the ψ-exponential generating function for ∂_ψ-basic polynomial sequence $\{x^n\}_{n=0}^\infty$ of the ∂_ψ operator.

Corollary 3.3. *The ψ-exponential generating function for ∂_ψ-basic polynomial sequence $\{p_n(x)\}_{n=0}^\infty$ of the ∂_ψ-delta operator $Q(\partial_\psi)$ is given by the following formula*

$$\sum_{k\geq 0} \frac{p_k(x)}{k_\psi!} z^k = exp_\psi\{xQ^{-1}(z)\}.$$

Proof. ¿From $E^a(\partial_\psi) = \sum_{k\geq 0} \frac{a_k}{k_\psi!} Q(\partial_\psi)^k$ where $a_k = [E^a(\partial_\psi) p_k(x)]_{x=0} = p_k(a)$ we get

$$E^a(\partial_\psi) = \sum_{k\geq 0} \frac{p_k(a)}{k_\psi!} Q(\partial_\psi)^k.$$

Applying now the isomorphism Theorem 3.3 we have

$$exp_\psi(xa) = \sum_{k\geq 0} \frac{p_k(a)}{k_\psi!} Q(x)^k.$$

After the substitution $Q(x) = z$ and $a = x$ we get the thesis. $\square$

4 The Pincherle ψ-Derivative and Sheffer ψ-polynomials

We again deliberately follow the way of Rota‘s systematic presentation as the finite operator calculus is already well suited for its generalizations encompassing ψ-extension. We mostly use Rota‘s notation with indispensable changes.

4.1 Pincherle ψ-derivative

In order to define the Pincherle ψ-derivative "’" as a linear map
’ : $\Sigma_\psi \quad \rightarrow \quad \Sigma_\psi$ we define $\hat{x}_\psi$.

Definition 4.1. (compare with (2) in [2] and see also [40] and [10])
$\hat{x}_\psi : P \rightarrow P; \quad \hat{x}_\psi x^n = \frac{\psi_{n+1}(q)(n+1)}{\psi_n(q)} x^{n+1} = \frac{(n+1)}{(n+1)_\psi} x^{n+1}, \quad n \geq 0.$

Definition 4.2. (compare with (17) in [2])
The Pincherle ψ-derivative is the linear map ’ : $\Sigma_\psi \quad \rightarrow \quad \Sigma_\psi$ defined according to

T_{∂_ψ} ’ $= T_{\partial_\psi} \hat{x}_\psi - \hat{x}_\psi T_{\partial_\psi} \equiv [T_{\partial_\psi}, \hat{x}_\psi]$.

It is easy to see that T_{∂_ψ} ’ $\in \Sigma_\psi$ for $T_{\partial_\psi} \in \Sigma_\psi$ i.e.
$\forall$ a $\in F$; $[E^a(\partial_q), [T_{\partial_\psi}, \hat{x}_\psi]] = 0$ The later follows inmediately from $[\partial_\psi, [T_{\partial_\psi}, \hat{x}_\psi]] = 0$ and the expansion $E^a(\partial_\psi) = \sum_{k\geq 0} \frac{a^k}{k_\psi!}\partial_\psi^k$ due to ∂_ψ ’$= 1$ (as direct verification in the $\{x^n\}_{n=0}^\infty$ basis shows). Therefore the linear map ’: $\Sigma_\psi \quad \rightarrow \quad \Sigma_\psi$’ is well defined. Because ∂_ψ’$=1$ one may introduce in a formal sense the appealing notation ’ $\equiv \frac{d}{d\partial_\psi}$. It is easy to see that the usual rules hold for Pincherle ψ-derivative as for example

$$(\partial_\psi^n)\text{'} = \partial_\psi^{n-1} \quad \Leftrightarrow \quad \frac{d}{d\partial_\psi}\partial_\psi^n = n\partial_\psi^{n-1}.$$

Pincherle ψ-derivative is therefore the true derivation with Leibnitz rule (see: Proposition 4.1)

Definition 4.3. *In accordance with the isomorphism theorem every ψ-shift invariant operator $T_{\partial_\psi} \in \Sigma_\psi$ of the form $T_{\partial_\psi} = \sum_{n\geq 0} \frac{a_n}{n_\psi!}\partial_\psi^n$, has as its unique correspondent the ψ-exponential formal power series $t(z) = \sum_{k\geq 0} \frac{a_k}{k_\psi!} z^k$. We shall call $t(z)$ the indicator of T_{∂_ψ} operator.*

Corollary 4.1. *Let* $t(z) = \sum_{k\geq 0} \frac{a_k}{k_\psi!} z^k$ *be the indicator of* $T_{\partial_\psi} \in \Sigma_\psi$*. Then* $t'(z) = \sum_{k\geq 1} \frac{ka_k}{k_\psi!} z^{k-1}$ *is the indicator of* $T_{\partial_\psi}{}' \in \Sigma_\psi$*.*

Due to the isomorphism theorem and the Corollary 4.1 the Leibniz rule holds which is obvious after one notices that "'" is the derivation in the commutative algebra Σ_ψ of ∂_ψ-shift invariant linear operators on the commutative algebra P of polynomials.

Proposition 4.1. $(T_{\partial_\psi} S_{\partial_\psi})' = T_{\partial_\psi}{}' S_{\partial_\psi} + S_{\partial_\psi} T_{\partial_\psi}{}'$; $T_{\partial_\psi}, S_{\partial_\psi} \in \Sigma_\psi$.

As an immediate consequence of the Proposition 4.1 we get

$$(S^n_{\partial_\psi})' = n\, S_{\partial_\psi}{}' S^{n-1}_{\partial_\psi} \quad \forall S_{\partial_\psi} \in \Sigma_\psi.$$

Like in [13] one infers from the isomorphism theorem that the following is true.

Proposition 4.2. $Q(\partial_\psi)$ *is the* ∂_ψ*-delta operator iff there exists invertible* $S_{\partial_\psi} \in \Sigma_\psi$ *such that*

$$Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}.$$

Proof.
$\Rightarrow$Let $S_{\partial_\psi} = \sum_{k\geq 0} \frac{s_k}{k_\psi!} \partial_\psi^k$ and $\exists\ S^{-1}_{\partial_\psi}$. Then $\partial_\psi S_{\partial_\psi}$ is a ∂_ψ-delta operator because $s_0 \neq 0$.

$\Leftarrow$ Let $Q(\partial_\psi) = \sum_{k\geq 1} \frac{q_k}{k_\psi!} \partial_\psi^k$, $q_1 \neq 0$ be a ∂_ψ-delta operator. Then { "$S = Q/\partial_\psi$" }

$$S_{\partial_\psi} = \sum_{k\geq 0} \frac{q_{k+1}}{(k+1)_\psi!} \partial_\psi^k \equiv \sum_{k\geq 0} \frac{s_k}{k_\psi!} \partial_\psi^k \quad ; \quad s_0 = q_1 \neq 0.$$

□

Example 4.1. $\Delta_q = \partial_q S \equiv \partial_q \sum_{k\geq 0} \frac{\partial_q^k}{(k+1)_q!} = E^a(\partial_q) - id.$

$\Delta_\psi = \partial_\psi S \equiv \partial_\psi \sum_{k\geq 0} \frac{\partial_\psi^k}{(k+1)_\psi!} = E^a(\partial_\psi) - id.$

The Pincherle ψ-derivative notion appears very effective in formulating expressions for ∂_ψ-basic polynomial sequences of the given ∂_ψ-delta operator $Q(\partial_\psi)$. This is illustrated by (compare with 1.1.37. in [10]) the theorem that follows.

Theorem 4.1. (ψ-Lagrange and ψ-Rodrigues formulas)
Let $\{p_n(x)\}_{n=0}^\infty$ *be* ∂_ψ*-basic polynomial sequence of the* ∂_ψ*-delta operator* $Q(\partial_\psi)$*:* $Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}$*. Then for* $n > 0$*:*

(1) $p_n(x) = Q(\partial_\psi)' \, S_{\partial_\psi}^{-n-1} \, \mathrm{x}^n$;

(2) $p_n(x) = S_{\partial_\psi}^{-n}\mathrm{x}^n - \frac{n_\psi}{n} \, (S_{\partial_\psi}^{-n})'\mathrm{x}^{n-1}$;

(3) $p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi S_{\partial_\psi}^{-n}\mathrm{x}^{n-1}$;

(4) $p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi(Q(\partial_\psi)')^{-1}p_{n-1}(x)$ (Rodrigues ψ-formula).

Proof. Temporarily we use in the proof the following abbreviations: $Q(\partial_\psi) = Q$; $S_{\partial_\psi} = S$

I. We shall prove that right-hand sides of (1) and (2) determine the same polynomial sequence:

$Q'S^{-n-1} = (\partial_\psi S)' \, S^{-n-1} = (S + \partial_\psi S')S^{-n-1} = S^{-n} + S'S^{-n-1}\partial_\psi = S^{-n} - (1/n)(S^{-n})' \, \partial_\psi$

due to this $Q'S^{-n-1} \, \mathrm{x}^n = S^{-n} \, \mathrm{x}^n - \frac{n_\psi}{n}(S^{-n})'\mathrm{x}^{n-1}$

II. We now prove that right-hand sides of (1) and (2) and (3) determine the same polynomial sequence:

$S^{-n}\mathrm{x}^n - \frac{n_\psi}{n}(S^{-n})'\mathrm{x}^{n-1} = S^{-n}\mathrm{x}^n - \frac{n_\psi}{n}(S^{-n}\hat{x}_\psi - \hat{x}_\psi S^{-n})\mathrm{x}^{n-1} = \frac{n_\psi}{n}\hat{x}_\psi S^{-n} \mathrm{x}^{n-1}$

III. We denote this polynomial sequence by $q_n(x) = Q'S^{-n-1} \, \mathrm{x}^n$

We prove that the sequence $q_n(x)$ satisfies the requirements of the ∂_ψ-basic polynomial sequence

(a) $Qq_n(x) = \partial_\psi SQ'S^{-n-1}\mathrm{x}^n = Q'S^{-n}\partial_\psi \, \mathrm{x}^n = n_\psi q_{n-1}(x)$;

(b) $q_n(x) = \frac{n_\psi}{n}\hat{x}_\psi S^{-n}\mathrm{x}^{n-1} \quad \Rightarrow \quad q_n(0) = 0;\, n > 0$;

(c) $q_n(x) = \frac{n_\psi}{n}\hat{x}_\psi S^{-n}\mathrm{x}^{n-1} \quad \Rightarrow \quad q_0(\mathrm{x}) = 1$.

Therefore in our notation : $q_n(x) \equiv p_n(x)$.

IV. We now derive the Rodrigues formula. According to Corollary 4.1 and Corollary 3.1 there exists $(Q')^{-1}$. Hence $Q'S^{-n-1} \, \mathrm{x}^n = p_n(x)$ may be rewritten as $\mathrm{x}^n = S^{n+1} \, (Q')^{-1} \, p_n(x)$. Inserting $\mathrm{x}^{n-1} = S^n(Q')^{-1} \, p_{n-1}(x)$ into (3) $p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi \, S_{\partial_q}^{-n} \, \mathrm{x}^{n-1}$ one gets the Rodrigues formula (4) $p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi \, (Q')^{-1} \, p_{n-1}(x)$.

□

Comment 4.1. Let $\{p_n\}_{n\geq 0}$ be the normal polynomial sequence i.e. $p_0\,(x)= 1$, $\deg p_n = n$ and $p_n\,(0)= 0$; $n \geq 1$ [40]. Then we call $\{p_n\}_{n\geq 0}$ the ψ-basic sequence of the operator Q and Q in its turn is called generalized differential operator [40] if in addition $Q\,p_n = n_\psi p_{n-1}$. Paralelly we define a linear map $\hat{x}_Q$: $P \rightarrow P$ such that $\hat{x}_Q p_n = \frac{(n+1)}{(n+1)_\psi} p_{n+1}$; $n \geq 0$. One notices immediately that the Theorem 4.1 holds also for the above operators Q and $\hat{x}_Q$ (see Theorem 4.3. in [40]).

Example 4.2. Put in Rodrigues formula $Q\,(\partial_\psi) = \partial_\psi$. Then $p_n(x) = x^n$, $n = 0, 1, 2, ...$ because $\partial_\psi' = 1$ and $p_0(x) = 1$.

Corollary 4.2. *Let $Q\,(\partial_\psi) = \partial_\psi S_{\partial_\psi}$ and $R\,(\partial_\psi) = \partial_\psi P_{\partial_\psi}$ be ∂_ψ-delta operators with ∂_ψ-basic polynomial sequences $\{p_n\,(x)\}_{n=0}^{\infty}$ and $\{r_n\,(x)\}_{n=0}^{\infty}$ respectively. Then*

(5) $p_n(x) = R\,(\partial_\psi)'(Q\,(\partial_q)')^{-1} S_{\partial_\psi}^{-n-1}\, P_{\partial_\psi}^{n+1} r_n(x)\ n \geq 0$;

(6) $p_n(x) = \hat{x}_\psi (P_{\partial_\psi}\ S_{\partial_\psi}^{-1})^n \hat{x}_\psi^{-1} r_n(x); n > 0$.

Proof. ad.(5) use (1) from the Theorem 4.1 for obvious substitution; ad.(6) use (3) from the Theorem 4.1 for obvious substitution. □

To this end we shall present another characterization of ∂_ψ-basic polynomial sequences of ∂_ψ-delta operators $Q\,(\partial_\psi)$.

Theorem 4.2. *Let $P_{\partial_\psi} \in \Sigma_\psi$ be an invertible ∂_ψ-shift invariant operator. Let $\{p_n\,(x)\}_{n=0}^{\infty}$ be ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q\,(\partial_\psi)$ satisfying*

$$\left[x^{-1} p_n\,(x)\right]_{x=0} \ = \ n_\psi \left[P_{\partial_q}^{-1} p_{n-1}\,(x)\right]_{x=0} \ n > 0.$$

Then $\{p_n\,(x)\}_{n=0}^{\infty}$ is ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator of the form: $Q\,(\partial_\psi) = \partial_\psi P_{\partial_\psi}$.

Proof. The linear operator $Q\,(\partial_\psi)$ is defined completely by $Q\,(\partial_\psi)\,1 = 0$ and $Q\,(\partial_\psi)\,p_n(x) = n_\psi p_{n-1}(x)$. It is easy to see that $Q\,(\partial_\psi) \in \Sigma_\psi$. The condition $\left[x^{-1} p_n\,(x)\right]_{x=0} = n_\psi \left[P_{\partial_\psi}^{-1} p_{n-1}\,(x)\right]_{x=0}$ $n > 0$ may be rewritten as follows: $\left[x^{-1} p_n\,(x)\right]_{x=0} = \left[P_{\partial_\psi}^{-1} Q\,(\partial_\psi)\,p_n\,(x)\right]_{x=0}$ $n > 0$. By linearity argument the above extends to arbitrary polynomial i.e. we have
$\left[x^{-1} p\,(x)\right]_{x=0} = \left[P_{\partial_\psi}^{-1} Q\,(\partial_\psi)\,p\,(x)\right]_{x=0}$ $n > 0$. At the same time for all polynomial such that $p(0) = 0$ the obvious identity holds:

$$\left[x^{-1} p\,(x)\right]_{x=0} \ = \ \left[\partial_\psi p\,(x)\right]_{x=0}$$

due to which we have the identity

$$\left[P_{\partial_\psi}^{-1} Q\left(\partial_\psi\right) p\left(x\right)\right]_{x=0} \quad = \quad \left[\partial_\psi p\left(x\right)\right]_{x=0}$$

satisfied now for all polynomials (including those with $p(0) \neq 0$)
Put now $p(x) = q(\mathrm{x} +_\psi a)$ and use the fact : P_{∂_ψ} , $Q\left(\partial_\psi\right) \in \Sigma_\psi$ in order to obtain

$$\partial_\psi q(a) = \left[P_{\partial_\psi}^{-1} Q\left(\partial_\psi\right) E^a\left(\partial_\psi\right) q\left(x\right)\right]_{x=0} = \left[E^a\left(\partial_\psi\right) P_{\partial_\psi}^{-1} Q\left(\partial_\psi\right) q\left(x\right)\right]_{x=0} =$$
$$= P_{\partial_\psi}^{-1}\, Q\left(\partial_\psi\right) q(a)$$

$\forall \mathrm{a} \in F$. Hence $Q\left(\partial_\psi\right) = \partial_\psi P_{\partial_\psi}$. ☐

Conclusion 4.1. One already sees that Rota‘s finite operator calculus is ready and well suited for natural almost automatic generalizations encompassing ψ-extensions ("ψ-deformations") and Q-extensions (see Comment 4.1). We have seen that the use Rota‘s calculus notion and and ideas of proofs with indispensable changes leads to this kind of extension.

We confirm it further considering a very important notion of Sheffer ψ-polynomials or correspondingly generalized Sheffer ψ-polynomials - see Comment 4.1 and [40].

This later notion provides ultimate generalization of umbral calculus [7, 11] in that sense that we deal now with all $\{p_n\}_o^\infty$, $\deg p_n = n$ polynomial sequences (see: [40] and [51]). Umbral calculus for other functions (not necessarily polynomials) is well known and has vast literature (see: recent one [51])

4.2 Sheffer ψ-polynomials

Definition 4.4. *A polynomial sequence $\{s_n\left(x\right)\}_{n=0}^\infty$ is called the Sheffer ψ-polynomials sequence of the ∂_ψ-delta operator $Q\left(\partial_\psi\right)$ iff*

(1) $s_0\left(x\right) = c \neq 0,$

(2) $Q\left(\partial_\psi\right) s_n\left(x\right) = n_\psi s_{n-1}\left(x\right).$

The following proposition relates Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q\left(\partial_\psi\right)$ to the unique ∂_ψ-basic polynomial sequence of $Q\left(\partial_\psi\right)$.

Proposition 4.3. *Let $Q\left(\partial_\psi\right)$ be a ∂_ψ-delta operator with ∂_ψ-basic polynomial sequence $\{q_n\left(x\right)\}_{n=0}^\infty$. Then $\{s_n\left(x\right)\}_{n=0}^\infty$ is a sequence of Sheffer ψ-polynomials of $Q\left(\partial_\psi\right)$ iff there exists an invertible ∂_ψ-shift invariant operator S_{∂_ψ} such that $s_n\left(x\right) = S_{\partial_\psi}^{-1}\, q_n\left(x\right)$.*

Proof.
$\Rightarrow$ Let there exist a ∂_ψ-shift invariant operator S_{∂_ψ} such that $s_n(x) = S^{-1}_{\partial_\psi} q_n(x)$. Of course $[S^{-1}_{\partial_\psi}, Q(\partial_\psi)] = 0$. Hence $Q(\partial_\psi) s_n(x) = n_\psi s_{n-1}(x)$. Indeed;
$Q(\partial_\psi) s_n(x) = Q(\partial_\psi) S^{-1}_{\partial_\psi} q_n(x) = S^{-1}_{\partial_\psi} Q(\partial_\psi) q_n(x) = n_\psi s_{n-1}(x)$. Also $s_0(x) = S^{-1}_{\partial_\psi} q_n(x) = S^{-1}_{\partial_\psi} 1 = \mathrm{c} \neq 0$.
$\Leftarrow$ Let $\{s_n(x)\}_{n=0}^{\infty}$ be a Sheffer ψ- sequence of $Q(\partial_\psi)$. The linear operator S_{∂_ψ} is defined via: $S_{\partial_\psi} : s_n(x) \to q_n(x)$. S_{∂_q} is of course invertible because $\deg s_n(x) = \deg q_n(x)$ and $s_0(x) = c \neq 0$. Also $S_{\partial_\psi} \in \Sigma_\psi$. Indeed. $[S_{\partial_\psi}, Q(\partial_\psi)] = 0$, as easily checked in $\{s_n(x)\}_{n=0}^{\infty}$ basis. Then $[E^a(\partial_\psi), S_{\partial_\psi}] = 0$, $\forall a \in F$ because $E^a(\partial_\psi) = \sum_{n\geq 0} \frac{a_n}{n_\psi!} Q(\partial_\psi)^n$. □

Conclusion 4.2. The family of Sheffer ψ-polynomials' sequences $\{s_n(x)\}_{n=0}^{\infty}$ corresponding to a fixed ∂_ψ-delta operator $Q(\partial_\psi)$ is labeled by the abelian group of all invertible operators $S_{\partial_\psi} \in \Sigma_\psi$. This families of Sheffer ψ-polynomials are orbits of such groups contained in the algebra Σ_ψ.
Naming: Sheffer ψ-polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ labeled by S_{∂_ψ} is refered to as the sequence of Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ *relative to* S_{∂_ψ}.

The following theorem is valid for invertible S_{∂_ψ} such that the sequence of Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ is relative to S_{∂_ψ}.

Theorem 4.3. (Second ψ- Expansion Theorem)
Let $Q(\partial_\psi)$ be the ∂_ψ-delta operator $Q(\partial_\psi)$ with the ∂_ψ-basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$. Let S_{∂_ψ} be an invertible ∂_ψ-shift invariant operator and let $\{s_n(x)\}_{n=0}^{\infty}$ be its sequence of Sheffer ψ-polynomials. Let T_{∂_ψ} be any ∂_ψ-shift invariant operator and let p(x) be any polynomial. Then the following identity holds :

$$\forall\, y \in F \wedge \forall\, p \in P \quad T_{\partial_\psi}\, p(x +_\psi y) = \sum_{k\geq 0} \frac{s_k(y)}{k_\psi!} Q(\partial_\psi)^k S_{\partial_\psi} T_{\partial_\psi}\, p(x) \ .$$

Proof. According to the first expansion theorem

$$E^y(\partial_\psi) \quad = \quad \sum_{k\geq 0} \frac{q_k(y)}{k_\psi!} Q(\partial_\psi)^k$$

from which - by acting on $p(x)$ - we have
$p(x +_\psi y) = \sum_{k\geq 0} \frac{q_k(y)}{k_\psi!} Q(\partial_\psi)^k p(x)$ or equivalently
$p(x +_\psi y) = \sum_{k\geq 0} \frac{q_k(x)}{k_\psi!} Q(\partial_\psi)^k p(y)$.

Of course $\quad S_{\partial_\psi}^{-1}\, p\left(x +_\psi y\right) = \sum_{k\geq 0} \frac{s_k(x)}{k_\psi!} Q\left(\partial_\psi\right)^k p\left(y\right)$ $\quad$ or equivalently

$$p\left(x +_\psi y\right) = \sum_{k\geq 0} \frac{s_k\left(y\right)}{k_\psi!} Q\left(\partial_\psi\right)^k S_{\partial_\psi}\, p\left(x\right).$$

Application of $T_{\partial_\psi} \in \Sigma_\psi$ to both sides yields the thesis. $\square$

The theorem (4.3) - called after Rota [13] The Second Expansion Theorem - has as an important outcome the following Corollary.

Corollary 4.3. (ψ- expansion)
Let $\left\{s_n\left(x\right)\right\}_{n=0}^{\infty}$ be the sequence of Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q\left(\partial_\psi\right)$ relative to S_{∂_ψ}. Then

$$S_{\partial_\psi}^{-1} \quad = \quad \sum_{k\geq 0} \frac{s_k\left(0\right)}{k_\psi!} Q\left(\partial_\psi\right)^k.$$

Proof. Set $y = 0$ and take $T_{\partial_\psi} = S_{\partial_\psi}^{-1}$ in the second expansion theorem thesis. $\square$

The property of ψ-binomiality of ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q\left(\partial_\psi\right)$ has its straightforward counterpart for $Q\left(\partial_\psi\right)$'s Sheffer ψ-sequence relative to S_{∂_ψ}.

Theorem 4.4. (The Sheffer ψ-Binomial Theorem)
Let $Q\left(\partial_\psi\right)$ be the ∂_ψ-delta operator with the ∂_ψ-basic polynomial sequence $\left\{q_n\left(x\right)\right\}_{n=0}^{\infty}$. Let $\left\{s_n\left(x\right)\right\}_{n=0}^{\infty}$ be the sequence of Sheffer ψ-polynomials of $Q\left(\partial_\psi\right)$ relative to an invertible ∂_ψ-shift invariant operator S_{∂_ψ}. Then the following identity is true

$$s_n\left(x +_\psi y\right) = \sum_{k\geq 0} \left(\begin{array}{c} n \\ k \end{array}\right)_\psi s_k\left(x\right) q_{n-k}\left(y\right).$$

Proof. In order to get the thesis apply $S_{\partial_\psi}^{-1}$ to both sides of ψ-binomial formula

$$q_n\left(x +_\psi y\right) = \sum_{k\geq 0} \left(\begin{array}{c} n \\ k \end{array}\right)_\psi q_k\left(x\right) q_{n-k}\left(y\right).$$

$\square$

Analogously to the undeformed case [13] Sheffer ψ-polynomials are completely determined by their constant terms as seen from the successive Corollary.

Corollary 4.4. *For* $\{q_n(x)\}_{n=0}^{\infty}$ *and* $\{s_n(x)\}_{n=0}^{\infty}$ *defined as in Theorem (4.4) the following holds* $s_n(x) = \sum_{k\geq 0} \binom{n}{k}_{\psi} s_k(0)\, q_{n-k}(x)$.

Proof. Obvious. □

Analogously to the undeformed case [13] also the converse of the second ψ-expansion theorem is true.

Proposition 4.4. *Let* $Q(\partial_\psi)$ *be a* ∂_ψ*-delta operator. Let* S_{∂_ψ} *be an invertible* ∂_ψ*-shift invariant operator. Let* $\{s_n(x)\}_{n=0}^{\infty}$ *be a polynomial sequence. Let*

$$\forall\, a \in F \wedge \forall\, p \in P \quad E^a(\partial_\psi)\, p(x) = \sum_{k\geq 0} \tfrac{s_k(a)}{k_\psi!} Q(\partial_\psi)^k\, S_{\partial_\psi}\, p(x)\ .$$

Then the polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ *is the sequence of Sheffer* ψ*-polynomials of the* ∂_ψ*-delta operator* $Q(\partial_\psi)$ *relative to* S_{∂_ψ}.

Proof. The above formula may be recasted in the form

$$E^a(\partial_\psi)\, p(x) = \sum_{k\geq 0} \frac{S_{\partial_\psi} s_k(a)}{k_\psi!} Q(\partial_\psi)^k\, p(x)$$

whereupon we take now $p(x) = q_n(x)$ and where $\{q_n(x)\}_{n=0}^{\infty}$ is the the ∂_ψ-basic polynomial sequence of $Q(\partial_\psi)$ thus arriving at

$$q_n(x +_\psi a) = \sum_{k\geq 0} \frac{S_{\partial_\psi} s_k(a)}{k_\psi!} n_\psi^{\underline{k}} q_{n-k}(x)$$

where we set $x = 0$ thus getting

$$q_n(a) = \sum_{k\geq 0} \binom{k}{n}_{\psi} S_{\partial_\psi}\ s_n(a)\, q_{n-k}(0)$$

hence $\forall$ a$\in F \quad q_n(a) = S_{\partial_\psi}\ s_n(a)$. □

Similarily as in nonextended case [13] further correspondent constructs, propositions and theorems hold also for the ψ- extension of the finite operator calculus. For example due to the above Proposition 4.4 we get another proof of the Rodrigues formula

$$p_n(x) = \tfrac{n_\psi}{n} \hat{x}_\psi\ (Q(\partial_\psi)'\ )^{-1}\ p_{n-1}(x).$$

Proposition 4.5. *Let* $\{p_n(x)\}_{n=0}^{\infty}$ *be the* ∂_ψ*-basic polynomial sequence of the* ∂_ψ*-delta operator* $Q(\partial_\psi)$*. Then*

$$p_n(x) = \tfrac{n_\psi}{n}\hat{x}_\psi (Q(\partial_\psi)')^{-1} p_{n-1}(x).$$

Proof. Due to $E^a(\partial_\psi) = \sum_{k\geq 0} \frac{a_k}{k_\psi!} Q(\partial_\psi)^k$ where $a_k = [E^a(\partial_\psi) p_k(x)]_{x=0} = p_k(a)$ we have

$$\sum_{n\geq 0} \frac{a^n}{n_\psi!}\partial_\psi^n = E^a(\partial_\psi) = \sum_{k\geq 0} \frac{p_k(a)}{k_\psi!} Q(\partial_\psi)^k.$$

The Pincherle derivative beeing applied to both sides yields

$$a\sum_{k\geq 0} \frac{a^k \partial_\psi^k}{k_\psi!} \frac{(k+1)}{(k+1)_\psi} = \hat{a}_\psi \sum_{k\geq 0} \frac{a^k \partial_\psi^k}{k_\psi!} =$$

$$= \hat{a}_\psi E^a(\partial_\psi) = \sum_{k\geq 0} \frac{p_{k+1}(a)}{k_\psi!} \frac{(k+1)}{(k+1)_\psi} Q(\partial_\psi)^k Q(\partial_\psi)'.$$

Compare this with $E^a(\partial_\psi) = \sum_{k\geq 0} \frac{s_k(a)}{k_\psi!} Q(\partial_\psi)^k S_{\partial_\psi}$ (Proposition 4.4) and recall that
$\hat{x}_\psi x^n = \frac{(n+1)}{(n+1)_\psi} x^{n+1}$; $\quad n \geq 0$ and $\hat{x}_\psi^{-1} x^n = \frac{n_\psi}{n} x^{n-1}$; $\quad n > 0$ in order to notice that from $E^a(\partial_\psi) = \sum_{k\geq 0} \frac{\hat{a}_\psi^{-1} p_{k+1}(a)}{k_\psi!} \frac{(k+1)}{(k+1)_\psi} Q(\partial_\psi)^k Q(\partial_\psi)'$ and from the Proposition 4.4. it follows that $\{\frac{(k+1)}{(k+1)_\psi} \hat{x}_\psi^{-1} p_{k+1}(x)\}_{k=0}^{\infty}$ is a sequence of Sheffer ψ-polynomials of ∂_ψ-delta operator $Q(\partial_\psi)$ - a sequence relative to the invertible ∂_ψ-shift invariant operator $Q(\partial_\psi)'$. Hence
$\frac{n}{n_\psi} \hat{x}_\psi^{-1}$ *$p_n(x)$* $= (Q(\partial_\psi)')^{-1}$ *$p_{n-1}(x)$* according to the proposition 4.3. $\square$

In what follows let us remark that the notion and properties of umbral composition, cross-sequences etc [13] straihtfowardly spread also to the ψ-extended calculus case.
Here - for the sake of future examples - we shall formulate few more propositions for which - with indispensable modifications - proofs of Rota can be adopted.

Proposition 4.6. *Let* $Q(\partial_\psi)$ *be a* ∂_ψ*-delta operator. Let* S_{∂_ψ} *be an invertible* ∂_ψ*-shift invariant operator. Let q(t) and s(t) be the indicators of* $Q(\partial_\psi)$ *and* S_{∂_ψ} *operators. Let* $q^{-1}(t)$ *be the inverse* ψ*-exponential formal power series inverse to q(t). Then the* ψ*-exponential generating function of Sheffer* ψ*-polynomials sequence* $\{s_n(x)\}_{n=0}^{\infty}$ *of* $Q(\partial_\psi)$ *relative to* S_{∂_ψ} *is given by*

$$\sum_{k\geq 0} \frac{s_k(x)}{k_\psi!} z^k = \frac{1}{s(q^{-1}(z))}\, exp_\psi\{x q^{-1}(z)\}.$$

In addition we have the following characterization of Sheffer ψ-polynomials:

Proposition 4.7. *A sequence $\{s_n(x)\}_{n=0}^{\infty}$ is the sequence of Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ with the ∂_ψ-basic polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ iff*

$$s_n(x +_\psi y) = \sum_{k\geq 0} \binom{n}{k}_\psi s_k(x)\, q_{n-k}(y).$$

Also the recurrence formula from [13] ψ-extends straightforwardly.

Proposition 4.8. *Let $\{p_n(x)\}_{n=0}^{\infty}$ be a polynomial sequence with $p_0(x) = c \neq 0$. If in adition it is Sheffer ψ-sequence then for any ∂_ψ-delta operator A there exists the sequence of constants $\{s_n\}_{n=0}^{\infty}$such that*

$$Ap_n(x) = \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)\, s_{n-k} \quad n \geq 0.$$

If the above recurrence holds for some ∂_ψ-delta operator A and some sequence of constants $\{s_n\}_{n=0}^{\infty}$ then $\{p_n(x)\}_{n=0}^{\infty}$ is a Sheffer ψ-sequence. (Operator A is not necessarily associated with $\{p_n(x)\}_{n=0}^{\infty}$).

Proof. Adopt Rota‘s proof [13] - with obvious replacements - and use the identity

$$(n-k)_\psi \binom{n}{k}_\psi = n_\psi \binom{n-1}{k}_\psi.$$

□

In effect we have another characterization of Sheffer ψ-polynomials.

Corollary 4.5. *A sequence $\{p_n(x)\}_{n=0}^{\infty}$ is the sequence of Sheffer q-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ with the ∂_ψ-basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$ iff there exists such a ∂_ψ-delta operator A (not necessarily associated with $\{p_n(x)\}_{n=0}^{\infty}$) and the sequence $\{s_n\}_{n=0}^{\infty}$ of constants such that*

$$Ap_n(x) = \sum_{k\geq 0} \binom{n}{k}_\psi p_k(x)\, s_{n-k}, \quad n \geq 0.$$

As in the undeformed case the natural inner product may be assosiated with the sequence $\{s_n(x)\}_{n=0}^{\infty}$ of Sheffer ψ-polynomials of the ∂_ψ-delta operator $Q(\partial_\psi)$ relative to an invertible ∂_ψ-shift invariant operator S_{∂_ψ} . For that purpose define the linear umbral operator W : $s_n(x) \rightarrow x^n$.

Definition 4.5. *Let S_{∂_ψ} be a ∂_ψ-shift invariant operator. Let W be the linear operator such that*

$W : s_n(x) \rightarrow \ x^n$ and then extended by linearity. We define the following bilinear form

$$(f(x),g(x))\psi := [(Wf)(Q(\partial_\psi))S_{\partial_\psi} \ g(x)]_{x=0} \ f,g \in P.$$

It is easy to state the important property of this bilinear form now on the reals.

Proposition 4.9. *The bilinear form over reals*

$$(f(x),g(x))_\psi := [(Wf) \ (Q(\partial_\psi))S_{\partial_\psi} \ g(x)]_{x=0} \ f,g \in P$$

is a positive definite inner product.

Proof.

$$(s_k(x) \ , \ s_n(x))_\psi = [Q(\partial_\psi)^k S_{\partial_\psi} \ s_n(x)]_{x=0} = [Q(\partial_\psi)^k p_n(x)]_{x=0} =$$
$$= n_\psi^{\underline{k}} p_{n-k}(0) = n_\psi! \delta_{nk}$$

where $\{p_n(x)\}_{n=0}^{\infty}$ is ∂_ψ-basic polynomial sequence of the operator $Q(\partial_\psi)$. ☐

We shall call the scalar product $(\ ,\ )_\psi : \aleph \times \aleph \rightarrow \boldsymbol{R}$ - the natural inner product associated with the sequence $\{s_n(x)\}_{n=0}^{\infty}$ of Sheffer ψ-polynomials. Naturally unitary space $(P \ ; (\ ,\ )_\psi\ )$ may be completed to the unique Hilbert space $\aleph = \overline{P}$. If so then the following Theorem is valid also in ψ-extended case of finite operator calculus.

Theorem 4.5. (Spectral Theorem)
Let $\{s_n(x)\}_{n=0}^{\infty}$ be the sequence of Sheffer ψ-polynomials relative to an invertible ∂_ψ-shift invariant operator S_{∂_ψ} for the ∂_ψ-delta operator $Q(\partial_\psi)$ with the ∂_ψ-basic polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$. Then there exists a unique essentially self adjoint operator $A_\psi : \aleph \rightarrow \aleph$ given by

$$A_\psi = \sum_{k \geq 1} \frac{u_k + \hat{v}_k(x)}{(k-1)_\psi!} Q(\partial_\psi)^k$$

such that the spectrum of A_ψ consists of $n = 0, 1, 2, 3, ...$ where $A_\psi s_n(x) = n s_n(x)$. The quantities u_k and $\hat{v}_k$ are calculated according to :

$$u_k \ = \ -\left[\left(logS_{\partial_\psi}\right)^{\text{`}} \hat{x}_\psi^{-1} p_k(x)\right]_{x=0} \ \textit{and} \ \ \hat{v}(x)_k = \hat{x}_\psi \left[\tfrac{d}{dx} p_k\right](0).$$

Proof. ¿From the second expansion theorem one has

$$S_{\partial_\psi}^{-1}\ E^a(\partial_\psi) \quad = \sum_{n\geq 0}\frac{s_n(a)}{n_\psi!}Q\left(\partial_\psi\right)^n .$$

Taking the Pincherle derivative of both sides in the above we get

$$(S_{\partial_\psi}^{-1}\ E^a(\partial_\psi))\text{'} = \sum_{n\geq 1}\frac{s_n(a)}{n_\psi!}nQ\left(\partial_\psi\right)^{n-1}Q(\partial_\psi)\text{'}.$$

After multiplication of both sides by $(Q(\partial_\psi)\text{'}\ )^{-1}Q(\partial_\psi)$ we arrive at

$$\left(-S_{\partial_\psi}^{-1}\ S_{\partial_q}\ +\hat{a}_\psi\right)S_{\partial_\psi}^{-1}\ E^a(\partial_\psi)\left(Q(\partial_\psi)\text{'}\right)^{-1}Q(\partial_\psi)\equiv$$

$$\equiv \quad T_{\partial_\psi}\ S_{\partial_\psi}^{-1}\ E^a(\partial_\psi) = \sum_{n\geq 1}\frac{s_n(a)}{n_\psi!}nQ(\partial_\psi)^n \quad (1)$$

where

$$T_{\partial_q}\ \equiv(-S_{\partial_\psi}^{-1}\ S_{\partial_\psi}\ \text{'} + \hat{a}_\psi)(Q(\partial_\psi)\text{'}\ )^{-1}Q(\partial_\psi) = (\hat{a}_\psi - (\log S_{\partial_\psi}\ \text{'})\,(Q(\partial_\psi)\text{'}\ )^{-1}Q(\partial_\psi)$$

Using the Rodrigues formula $p_{n+1}(x) = \frac{(n+1)_\psi}{n+1}\hat{x}_\psi\left(Q(\partial_\psi)\text{'}\right)^{-1}p_n(x)$ we define $q_n(x)$:

$$q_n(x): \quad = \quad \frac{(n+1)}{(n+1)_\psi}\hat{x}_\psi^{-1}p_{n+1}(x) = \left(Q(\partial_\psi)\text{'}\right)^{-1}p_n(x)$$

for $n\geq 0$ whence

$$Q(\partial_\psi)\,q_n(x) \quad = \left(Q(\partial_\psi)\text{'}\right)^{-1} \quad (\partial_\psi)\,p_n(x) = n_\psi q_{n-1}(x)$$

for $n>0$. Consider now $\quad T_{\partial_\psi} = \sum_{n\geq 0}\frac{\hat{b}_n}{n_\psi!}Q(\partial_\psi)^n \quad$ where $\quad \hat{b}_k = \left[T_{\partial_\psi}p_k(x)\right]\big|_{x=0}$. Easy calculation yields for $k\in N$: $\hat{b}_k = \left[T_{\partial_\psi}p_k(x)\right]\big|_{x=0} = k\left(\hat{v}_k + u_k\right)$ where

$$u_k = -\left[\left(\log S_{\partial_\psi}\right)^{\text{`}}\hat{x}_\psi^{-1}p_k(x)\right]_{x=0} \text{and } \hat{v}_k(a) = \left[\hat{a}_\psi\hat{x}_\psi^{-1}p_k(x)\right]_{x=0} = \hat{a}_\psi\left[\frac{d}{dx}p_k\right](0).$$

In order to find out the operator A_ψ ; $A_\psi s_n(x) = ns_n(x)$ it is enough to consider now an expansion of the operator $T_{\partial_\psi}\ S_{\partial_\psi}^{-1}\ E^a(\partial_\psi)$. For that purpose note that

$$T_{\partial_\psi}\ S_{\partial_\psi}^{-1}\ f(x+_\psi a) = \sum_{k\geq 1}\frac{\hat{b}_k}{k_\psi!}Q\left(\partial_\psi\right)^k\left[S_{\partial_q}^{-1}\ f(x+_\psi a)\right]\forall f\in P$$

and recall the formula we have started the proof with doe to which we have $S_{\partial_\psi}^{-1}\, E^a\left(\partial_\psi\right)$

$$S_{\partial_\psi}^{-1}\, f\left(x +_\psi a\right) = \sum_{n\geq 0}\frac{s_n\left(x\right)}{n_\psi!}Q\left(\partial_\psi\right)^n f\left(a\right).$$

After the obvious inserting we obtain

$$T_{\partial_\psi}\, S_{\partial_\psi}^{-1}\, f\left(x +_\psi a\right) = \sum_{k\geq 1}\tfrac{\hat{b}_k}{k_\psi!}Q\left(\partial_\psi\right)^k\left[\sum_{n\geq 0}\tfrac{s_n(a)}{n_\psi!}Q\left(\partial_\psi\right)^n f\left(a\right)\right]\quad \forall\, f\in P \text{ or}$$

$$T_{\partial_\psi}\, S_{\partial_\psi}^{-1}\, f\left(x +_\psi a\right) = \sum_{n\geq 0}\left[\sum_{k\geq 1}\tfrac{\hat{b}_k}{k_\psi!}s_n\left(x\right)Q\left(\partial_\psi\right)^k\right]\tfrac{Q\left(\partial_\psi\right)^n}{n_\psi!}f\left(a\right)\quad \forall\, f\in P.$$

Permuting up there a and x one gets

$$T_{\partial_\psi}\, S_{\partial_\psi}^{-1}\, E^a\left(\partial_\psi\right) = \quad \sum_{n\geq 0}\left[\sum_{k\geq 1}\frac{\hat{b}_k}{k_\psi!}s_n\left(a\right)Q\left(\partial_\psi\right)^k\right]\frac{Q\left(\partial_\psi\right)^n}{n_\psi!}.$$

Comparing now this with [see: (1)]

$$T_{\partial_\psi}\, S_{\partial_\psi}^{-1}\, E^a\left(\partial_\psi\right) = \sum_{n\geq 1}\frac{s_n\left(a\right)}{n_\psi!}nQ\left(\partial_\psi\right)^n$$

we conclude that upon chanching again a to x we end up with

$$\sum_{k\geq 1}\frac{\hat{b}_k}{k_\psi!}Q\left(\partial_\psi\right)^k s_n\left(x\right)\ =\ ns_n\left(x\right)\quad \text{for } n\geq 0\quad (b_0 = 0)$$

where $\hat{b}_k = \left[T_{\partial_\psi}p_k\left(x\right)\right]\big|_{x=0} = k\left(\hat{v}_k + u_k\right)$, $k\in N$. The operator $A_\psi : \aleph \rightarrow \quad \aleph$; $A_\psi s_n\left(x\right) = ns_n\left(x\right)$ has then the form

$$A_\psi = \sum_{k\geq 1}\frac{u_k + \hat{v}_k\left(x\right)}{\left(k-1\right)_\psi!}Q\left(\partial_\psi\right)^k\ .$$

The sequence $\left\{s_n\left(x\right)\right\}_{n=0}^{\infty}$of Sheffer ψ-polynomials spans the Hilbert space $\aleph$ We therefore state that the operator $A_\psi : \aleph \rightarrow \aleph$ is the unique essentiall self adjoint unbounded operator having nonnegative integers as its spectrum with corresponding eigenfuctions $\left\{s_n\left(x\right)\right\}_{n=0}^{\infty}$. $\square$

To this end let us draw one general conclusion and let us make a link to Hopf algebra in a form of a remark with indication of references for further readings.

Conclusion 4.3. One has experienced the way in which finite operator calculus gains its natural ψ-extended representation. We have seen that the use of Rota's calculus notions and ideas of proofs with indispensable changes leads to this kind of extension.

In this context it should be noted that the notion of generalised Sheffer ψ-polynomials (see Comment 4.1 and [40]) constitutes the ultimate generalisation of umbral calculus in that sense that this calculus deals now with *all polynomial* sequences (see: [40] and [51]).

Remark 4.1. (general Hopf algebras' remark)
Exactly as in the undeformed case of umbral calculus [51] the generalized translation operator $E^y(\partial_\psi) = exp_\psi\{y\partial_\psi\}$ is *an example* of comultiplication over the space of polynomials F[x] (F[x] = P) i.e. a map from F[x] to F[x,y] satisfying the conditions of coassociativity and counicity ; indeed - with notation $E^y(\partial_\psi)\,p(x) =: p(x +_\psi y)$ we have $p((x +_\psi y) +_\psi z) = p(x +_\psi (y +_\psi z))$ and generalized translation operator fixes all constants.

Hence with natural antipode the algebra of polynomials may be seen as a graded Hopf algebra [39]. Coalgebra maps [51] S are in their turn exactly umbral substitution maps as defined in [49] - (see there Theorem 5 then and section 7). The ψ-extension of this Theorem 5 from [49] with corresponding "ψ-proof" is automatic. In this connection note: there are other than $E^y(\partial_\psi)$ comultiplications however all these are of the form

$E^y(\Omega) = \sum\limits_{n\geq 0} \omega_n(y)\frac{\Omega^n}{n_\psi!}$ – where $\{\omega_n(x)\}_{n\geq 0}$ is a basic sequence of Ω.

A polynomial sequence $\{s_n(x)\}_{n\geq 0}$ is then said to be a Sheffer *with respect to* $E^y(\Omega)$ comultiplication *and a pair* of $E^y(\Omega)$ invariant operators Q and S (see Proposition 4.3).

At the end this section we give few examples - postponing the systematic application of the ψ-calculus of Rota to the subsequent publication . Meanwhile let us indicate reference [7]- as the elegant presentation of umbral calculus using objects of triple meaning.

There [7] a lot of examples and also some q-examples are elaborated. The latter encompass Gegenbauer and Jacobi polynomials hence also Chebyshev polynomials. See also [11]. For q-Abel polynomials see [33], [34, 35]. For q-Hermite polynomials see [9, 10] while for q-Laguerre polynomials see [52, 53]. Also quantum q-oscillator algebra provides a natural setting for q-Laguerre polynomials and q-Hermite polynomials [54, 55, 56, 57].

It seems to be interesting and perhaps important question to ask whether these quantum like models might be ψ-extended.

Examples

1. Let $Q(\partial_\psi) = \partial_\psi$ then $\partial_\psi\text{'} = id$. Apply now the Rodrigues ψ-formula in a recurrent way

$$p_n(x) = \frac{n_\psi}{n}\hat{x}_\psi\left(\partial_\psi\text{'}\right)^{-1}p_{n-1}(x) = \frac{n_\psi}{n}\hat{x}_\psi\left(\frac{(n-1)_\psi}{(n-1)}\hat{x}_\psi p_{n-2}(x)\right) = \ldots =$$
$$= \frac{n_\psi!}{n!}\left(\hat{x}_\psi\right)^n[1] = x^n.$$

2. Let $Q(\partial_q) = \Delta_q := E(\partial_q) - id$. Then $\Delta_q\text{'} = \hat{1}_q E^1(\partial_q)$ where $\hat{1}_q x^n = \frac{(n+1)}{(n+1)_q}x^n$ in accordance with $\hat{x}_q x^n = \frac{(n+1)}{(n+1)_q}x^{n+1}$. Let us observe here that ψ-extended case is covered in this example just by replacement $q \to \psi$.

Recall now the Rodrigues q- formula $p_n(x) = \frac{n_q}{n}\hat{x}_q\left(\Delta_q\text{'}\right)^{-1} p_{n-1}(x) = \frac{n_q}{n}\hat{x}_q E^{-1}(\partial_q)\hat{1}_q^{-1}p_{n-1}(x)$ where $\hat{1}_q^{-1}x^n = \frac{n_q}{n}x^n$ for $n > 0$ and $\hat{1}_q^{-1}[1] = 1$ in order to apply this Rodrigues q- formula in a recurrent way as follows:

$$p_n(x) = \frac{n_q}{n}\hat{x}_q\, E^{-1}(\partial_q)\,\hat{1}_q^{-1}p_{n-1}(x) =$$
$$= \frac{n_q}{n}\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\left(\frac{(n-1)_q}{(n-1)}\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}p_{n-2}(x)\right) =$$
$$= \frac{n_q}{n}\frac{(n-1)_q}{(n-1)}\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\left(E^{-1}(\partial_q)\,\hat{1}_q^{-1}p_{n-2}(x)\right) = \ldots =$$
$$= \frac{n_q!}{n!}\left\{\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)\circ\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)\circ\ldots\circ\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)\right\}[1] \equiv$$
$$\equiv \quad \frac{n_q!}{n!}\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)^n[1] = p_n(x).$$

Due to $E^{-1}(\partial_q) = \sum\limits_{n\geq 0}\frac{(-1)^n}{n_q!}\partial_q^n$ we have

$$E^{-1}(\partial_q)\,x^n = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_q (-1)^k x^{n-k} \equiv (x -_q 1)^n.$$

Hence

$$\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)[1] = x \text{ and}$$
$$\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)[x] = \; = \frac{2}{2_q}x^2 - x \underset{q\to 1}{\longrightarrow} x(x-1) = x^{\underline{2}}$$

and by induction we get the expression:

$$p_n(x) = \frac{n_q!}{n!}\left(\hat{x}_q E^{-1}(\partial_q)\,\hat{1}_q^{-1}\right)^n[1]$$

which may be given a simpler form

$$p_n(x) = \frac{n_q!}{n!}\hat{x}_q\left(E^{-1}(\partial_q)\,\hat{x}\right)^n[1]\,\text{due to }\hat{1}_q^{-1}\hat{x}_q = \hat{x}.$$

Of course $p_n(x) = \frac{n_q!}{n!}\hat{x}_q\left(E^{-1}(\partial_q)\,\hat{x}\right)^n[1]\underset{q\to 1}{\longrightarrow} x^{\underline{n}}$ where $\{x^{\underline{n}}\}_{n\geq 0}$ is the basic polynomial sequence of the delta operator Δ ; $\Delta_q \underset{q\to 1}{\longrightarrow} \Delta$.
Let us observe here that ψ-extended case is covered in this example just by replacement $q \to \psi$.
One may give also the explicit expression of ∂_q-basic polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ of the ∂_q-delta operator Δ_q using the statement (3) $p_n(x) = \frac{n_q}{n}\hat{x}_q\ S_{\partial_q}\ x^{n-1}$ of the Theorem 4.1. Namely - we see that

$$\Delta_q = \partial_q S_{\partial_q}\ \equiv \partial_q \sum_{k\geq 0}\frac{\partial_q^k}{(k+1)_q!} = E^a(\partial_q) - id\ \text{i.e.}S_{\partial_q}\ = \sum_{k\geq 0}\frac{\partial_q^k}{(k+1)_q!}$$

and for $n > 0$

$$p_n(x) = \frac{n_q}{n}\hat{x}_q\sum_{k\geq 0}\frac{\partial_q^k}{(k+1)_q!}x^{n-1} = \frac{n_q}{n}\hat{x}_q\sum_{k\geq 0}^{n-1}\frac{(n-1)_q^{\underline{k}}}{(k+1)_q!}x^{n-k-1}$$

and finally

$$p_n(x) = \frac{n_q}{n}\sum_{k\geq 0}^{n-1}\frac{(n-1)_q^{\underline{k}}}{(k+1)_q!}\frac{(n-k)}{(n-k)_q}x^{n-k}\ .$$

Let us observe again that ψ-extended case is covered in this example just by replacement $q \to \psi$.

3. Let $Q(\partial_q) = \nabla_q := id - E^{-1}(\partial_q)$. Then $\nabla_q' = \hat{1}_q E^{-1}(\partial_q)$. Similarly as in example (2) $p_n(x) = \frac{n_q!}{n!}\left(\hat{x}_q E(\partial_q)\,\hat{1}_q^{-1}\right)^n[1]$ or due to $\hat{1}_q^{-1}\hat{x}_q = \hat{x}$: $p_n(x) = \frac{n_q!}{n!}\hat{x}_q\left(E(\partial_q)\,\hat{x}\right)^n[1]$.
Of course $p_n(x) = \frac{n_q!}{n!}\hat{x}_q\left(E(\partial_q)\,\hat{x}\right)^n[1] \underset{q\to 1}{\longrightarrow} x^{\overline{n}}$ where $\left\{x^{\overline{n}}\right\}_{n\geq 0}$ is the basic polynomial sequence of the delta operator ∇; $\nabla_q \underset{q\to 1}{\longrightarrow} \nabla$.

Let us observe then that ψ-extended case is covered in this example just by replacement $q \to \psi$.

One may give also the explicit expression of ∂_q-basic polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ of the ∂_q-delta operator ∇_q using the statement

$$p_n(x) = \frac{n_q}{n}\hat{x}_q S_{\partial_q}\, x^{n-1}.$$

Namely

$\nabla_q = \partial_q S_{\partial_q} \equiv \partial_q \sum_{k\geq 0} \frac{(-1)^k \partial_q^k}{(k+1)_q!} = id - E^{-1}(\partial_q)$ i.e. $S_{\partial_q} = \sum_{k\geq 0} \frac{(-1)^k \partial_q^k}{(k+1)_q!}$

and for $n > 0$

$p_n(x) = \frac{n_q}{n}\hat{x}_q \sum_{k\geq 0} \frac{(-1)^k \partial_q^k}{(k+1)_q!} x^{n-1} = \frac{n_q}{n}\hat{x}_q \sum_{k\geq 0}^{n-1} (-1)^k \frac{(n-1)_q^{\underline{k}}}{(k+1)_q!} x^{n-k-1}$. Finally

$$p_n(x) = \frac{n_q}{n} \sum_{k\geq 0}^{n-1} (-1)^k \frac{(n-1)_q^{\underline{k}}}{(k+1)_q!} \frac{(n-k)}{(n-k)_q} x^{n-k}.$$

Naturally ψ-extended case is covered in this example just by replacement $q \to \psi$.

4. Let $A(\partial_q) = \partial_q E^a(\partial_q)$ be the "q-Abel operator". Recall now from the Theorem 4.1. the statement (3) $p_n(x) = \frac{n_q}{n}\hat{x}_q S^{-n} \mathrm{x}^{n-1}$ for $n > 0$ where $\{p_n(x)\}_{n=0}^{\infty}$ is the ∂_q-basic polynomial sequence of the ∂_q-delta operator $Q(\partial_q)$ of the form : $Q(\partial_q) = \partial_q S_{\partial_q}$. In the case of q-Abel operator we then have for $n > 0$

$$A_{n,q}(x) = \frac{n_q}{n}\hat{x}_q E^{-na}(\partial_q)\, x^{n-1}$$

where we shall call the ∂_q-basic polynomial sequence $\{A_{n,q}(x)\}_{n\geq 0}$ of the q-Abel operator - Abel q-polynomials sequence. With our convention we may write the formula for Abel q-polynomials in the form which mimics the undeformed case form

$$A_{n,q}(x) = \frac{n_q}{n}\hat{x}_q (x -_q na)^{n-1} \underset{q\to 1}{\longrightarrow} A_n(x) = (x - na)^{n-1}$$

where $(x -_q na)^{n-1} \equiv \sum_{k=0}^{n-1} \binom{n-1}{k}_q (-na)^k x^{n-k-1}$ or explicitly written

$$A_{n,q}(x) = \frac{n_q}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}_q (-na)^k \frac{(n-k)}{(n-k)_q} x^{n-k} .$$

Again note: ψ-extended case is covered in this example just by replacement $q \to \psi$.

5. Let $L(\partial_q) = -\sum_{k=0}^{\infty} \partial_q^{k+1} \equiv \frac{\partial_q}{\partial_q - 1} \equiv -\left[\partial_q + \partial_q^2 + \partial_q^3 + \partial_q^4 + \partial_q^5 + ...\right]$ be the "q-Laguerre operator". In this case for $n > 0$

$$L_{n,q}(x) = \frac{n_q}{n}\hat{x}_q \left[\frac{1}{\partial_q - 1}\right]^{-n} x^{n-1} = \frac{n_q}{n}\hat{x}_q \left(\partial_q - 1\right)^n x^{n-1} =$$

$$= \frac{n_q}{n}\hat{x}_q \sum_{k=1}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right)_q \partial_q^{n-k} x^{n-1} =$$

$$= \frac{n_q}{n}\sum_{k=1}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array}\right)_q (n-1)_q^{\underline{n-k}} \frac{k}{k_q} x^k = \frac{n_q}{n}\sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \frac{(n-1)_q^{\underline{n-k}}}{(n-k)_q!} \frac{k}{k_q} x^k.$$

So finally

$$L_{n,q}(x) = \frac{n_q}{n} \sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right)_q \frac{k}{k_q} x^k.$$

We shall call the ∂_q-basic polynomial sequence$\{L_{n,q}(x)\}_{n\geq 0}$of the q-Laguerre operator $L(\partial_q)$ - Laguerre q-polynomials sequence- and note again that ψ-extended case is covered in this example just by replacement $q \to \psi$.

Final Remark The way is now opened to find out ψ-analogues of typical identities derived from explicit form of the above ∂_ψ-basic polynomial sequences due to the Corollary 4.2. as in [58] (p.147 - for example Lah numbers) . Also q-analogues or ψ-analogues of Hermite and Pollaczek polynomials apart from Laguerre polynomials are not difficult to be find out so as to give rise to orthogonal over an interval of real line sequences $\{s_n(x)\}_{n=0}^{\infty}$ of Sheffer ψ-polynomials with appropriately chosen ∂_ψ-shift invariant invertible S_{∂_ψ} operators.

As stated earlier - in this section we only give few examples - postponing the systematic application of the ψ-calculus of Rota to the subsequent publication.

Let us then recall again that quantum q-oscillator algebra provides a natural setting for Laguerre q-polynomials and Hermite q-polynomials [54, 55, 56, 57].

5 No ψ-analogue extension of quantum q-plane formulation?

The idea to use "q-commuting variables" goes back at least to Cigler (1979) [9, 52, 53] (see formula (7),(11) in [9]) and also to Kirchenhofer - see [10] for further systematic development. In [10] Kirchenhofer defined the polynomial sequence $\{p_n\}_o^\infty$ of q-binomial type by

$$p_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q p_k(A)\, p_{n-k}(B) \text{ where } [B,A]_q \equiv BA - qAB = 0.$$

A and B might be interpreted here as co-ordinates on quantum q-plane (see [39] Chapter 4). For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. If so then the following identification takes place

$$p_n(x +_q y) \equiv E^y(\partial_q)\, p_n(x) = \sum_{k\geq 0} \binom{n}{k}_q p_k(x)\, p_{n-k}(y) = p_n\left(\hat{x} + y\hat{Q}\right)\mathbf{1}.$$

q-Sheffer polynomials $\{s_n(x)\}_{n=0}^\infty$ are defined correspondingly by (see: 2.1.1. in [10]) $s_n(A+B) \equiv \sum_{k\geq 0} \binom{n}{k}_q s_k(A)\, p_{n-k}(B)$ where $[\mathrm{B,A}]_q \equiv BA - qAB = 0$ and the polynomial sequence $\{p_n(x)\}_{n=0}^\infty$ is of q-binomial type. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. Then the following identification takes place:

$$s_n(x +_q y) \equiv E^y(\partial_q)\, s_n(x) = \sum_{k\geq 0} \binom{n}{k}_q s_k(x)\, p_{n-k}(y) = s_n\left(\hat{x} + y\hat{Q}\right)\mathbf{1}$$

This means that one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables $A, B : \quad AB - qBA \equiv [A,B]_q = 0$.

The above identifications of polynomial sequences $\{p_n\}_o^\infty$ of q-binomial type and Sheffer q-polynomials $\{s_n(x)\}_{n=0}^\infty$ fail to be extended to the more general ψ-case. This means that we can not formulate that way the ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : AB - \hat{q}_{\psi,Q}BA \equiv [A,B]_{\hat{q}_{\psi,Q}} = 0$. We shall explain this in the sequel. For that to do we introduce the following notions - important on their own.

Definition 5.1. *Let $\{p_n\}_{n\geq 0}$ be the ∂_q-basic polynomial sequence of the ∂_q-delta operator$Q(\partial_q)$. A linear map $x_{Q(\partial_q)}$: $P \to P$, $x_{Q(\partial_q)} p_n = p_{n+1}, \quad n \geq 0$ is called the operator dual to $Q(\partial_q)$.*

For $Q = id$ we have: $x_{Q(\partial_q)} \equiv \quad x_{\partial_q} \quad \equiv \quad \hat{x}$.
Comment: The addjective dual in the above sense corresponds to the addjective adjoint in q-umbral calculus language of linear functionals' umbral algebra (see : Proposition 1.1.21 in [10]). In this connection we note that in the formulation of undeformed umbral calculus [11] in terms of algebra P^* of linear functionals on P - the map $x_{Q(\partial_\psi)}$ is called an "umbral shift operator" and it is *adjoint* to a derivation of P^* ; see: Theorem 5 in [11]. See also 1.1.16 in [10].

Definition 5.2. *Let $\{p_n\}_{n\geq 0}$ be the ∂_ψ-basic polynomial sequence of the ∂_ψ-delta operator $Q(\partial_\psi) = Q$. Then the $\hat{q}_{\psi,Q}$-operator is a liner map*
$\hat{q}_{\psi,Q} : P \to P; \quad \hat{q}_{\psi,Q} p_n = \frac{(n+1)_\psi - 1}{n_\psi} p_n, \quad n \geq 0.$

We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$-*mutator operator.*
Example: For $Q = id$ $Q(\partial_\psi) = \partial_\psi$ the natural notation is $\hat{q}_{\psi,id} \equiv \hat{q}_\psi$. For $Q = id$ and $\psi_n(q) = \frac{1}{R(q^n)!}$ and $R(x) = \frac{1-x}{1-q}$ $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = q^n x^n$.

Definition 5.3. *Let A and B be linear operators acting on P; $A : P \to P$, $B : P \to P$. Then $AB - \hat{q}_{\psi,Q} BA \equiv [A,B]_{\hat{q}_{\psi,Q}}$ is called $\hat{q}_{\psi,Q}$-mutator of A and B operators.*

Observation 5.1. $Q(\partial_\psi) x_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} x_{Q(\partial_\psi)} Q(\partial_\psi) \equiv [Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = id.$
This is easily verified in the ∂_ψ-basic $\{p_n\}_{n\geq 0}$ of the ∂_ψ-delta operator $Q(\partial_\psi)$.

Equipped with pair of operators ($Q(\partial_\psi)$, $x_{Q(\partial_\psi)}$) and $\hat{q}_{\psi,Q}$-mutator we have at our disposal all possible representants of "canonical pairs" of differential operators on the P algebra. The meaning of the adjective: "canonical" includes also the content of the Remark 5.2.

For important historical reasons at first here is the Remark 5.1.

Remark 5.1. The ψ-derivative is a particular example of a linear operator that reduces by one the degree of any polynomial. In 1901 it was proved [59] that every linear operator T mapping P into P may be represented as infinite series in operators $\hat{x}$ and D. In 1986 the authors of [60] supplied us with the explicit expression for such series in most general case of polynomials in one variable. We quote here the Proposition 1 from [60] one has:

Let L be a linear operator that reduces by one each polynomial. Let $\{q_n(\hat{x})\}_{n\geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}$. Then $T = \sum_{n\geq 0} q_n(\hat{x})$ L^n defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n\geq 0} q_n(\hat{x}) L^n.$$

In 1996 this was extended to algebra of many variables polynomials [61].

Remark 5.2. The importance of the pair of dual operators : $Q(\partial_\psi)$ and $x_{Q(\partial_\psi)}$ is reflected by the facts:

a) $Q(\partial_\psi)\, x_{Q(\partial_\psi)}$- $\hat{q}_{\psi,Q} x_{Q(\partial_\psi)} Q(\partial_\psi) \equiv [Q(\partial_\psi),\, x_{Q(\partial_\psi)}]_{\hat{q}_{R,Q}} = id.$

b) *Let* $\{q_n\left(x_{Q(\partial_\psi)}\right)\}_{n\geq 0}$ *be an arbitrary sequence of polynomials in the operator* $\hat{x}_{Q(\partial_\psi)}$. *Then* $T = \sum_{n\geq 0} q_n\left(x_{Q(\partial_\psi)}\right) Q(\partial_\psi)^n$ *defines a linear operator that maps polynomials into polynomials. Conversely , if* T *is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form*

$T = \sum_{n\geq 0} q_n\left(x_{Q(\partial_\psi)}\right) Q(\partial_\psi)^n$.

One may now be tempted to formulate the basic notions of ψ-extended finite operator calculus with help of the "quantum ψ-plane" $\hat{q}_{\psi,Q}$-commuting variables $A, B : [A, B]_{\hat{q}_{\psi,Q}} = 0$ exactly in the same way as in [9, 10]. For that to try consider appropriate generalization of $A = \hat{x}$ and $B = y\hat{Q}$ where this time the action of $\hat{Q}$ on $\{x^n\}_0^\infty$ is to be found from the condition $AB - \hat{q}_\psi BA \equiv [A, B]_{\hat{q}_\psi} = 0$. Acting with $[A, B]_{\hat{q}_\psi}$ on $\{x^n\}_0^\infty$ due to $\hat{q}_\psi x^n = \frac{(n+1)_\psi - 1}{n_\psi} x^n$, $n \geq 0$ one easily sees that now $\hat{Q}x^n = b_n x^n$ where $b_0 = 0$ and $b_n = \prod_{k=1}^{n} \frac{(k+1)_\psi - 1}{k_\psi}$ for $n > 0$ is the solution of the difference equation:

$$b_n - b_{n-1}\frac{(n+1)_\psi - 1}{n_\psi} = 0 \quad , \quad n > 0.$$

With all above taken into account one immediately verifies that for our A and B $\hat{q}_\psi$-commuting variables

$$(A+B)^n \neq \sum_{k\geq 0} \binom{n}{k}_\psi A^k B^{n-k}$$

unless $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ hence $\hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = q^n x^n$. Therefore in conclusion one affirms that the case of q-deformed finite operator calculus is fairly enough distinguished by the Kirchenhofer approach and the quantum plane notion.

Remark 5.3. It is therefore not a supprise that the q-deformations are now a daily bread for q-theoretician physicists. We have mentioned also an "intermediate" R-case because it is of primary importance for advanced theory of coherent states [21] and it seams to be so in connection with extended binomial theorem recently proved in [21].

Our observation on this occasion is that unless the condition: $\exists\, \varphi$; $R(q^{n+1}) - \varphi(q)\, R(q^n) = 1$, where $R(1) = 1$ is satisfied neither the method from [62] nor the method from [63] used in proof of q-binomial theorem is successful. The extension of iterative method works fine for general R-case [21].

As for the eventual ψ-binomial theorem however - the present author feels helpless with all the methods mentioned.

Remark on references: Of course references are not complete. Nevertheless we would like to indicate at the end some of them as the source of further references.

Among them the papers [64] by Andrews (1971) and [65] by Goldman and Rota (1970) are to be quoted as the ones in which q-umbral calculus is being started and applied at first - to our knowledge.

The very recent source of sources for references on umbral calculus and its generalisations developed by Loeb and others is "*The World of Generating Functions and Umbral Calculus*" (1999) [51]. See also paper [66] by Loeb and Rota.

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