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A symbolic operator approach to several summation formulas for power series

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Abstract

This paper deals with the summation problem of power series of the form $S_a^b(f;x)=\sum_{a\leqslant k\leqslant b} f(k)x^k$, where $0\leqslant a<b\leqslant\infty$, and $\{f(k)\}$ is a given sequence of numbers with $k\in[a,b)$ or $f(t)$ is a differentiable function defined on $[a,b)$. We present a symbolic summation operator with its various expansions, and construct several summation formulas with estimable remainders for $S_a^b(f;x)$, by the aid of some classical interpolation series due to Newton, Gauss and Everett, respectively.

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1. Introduction

It is known that the symbolic operations Δ (difference), E (displacement) and D (derivative) play an important role in the calculus of finite differences as well as in certain topics of computational methods.

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For various classical results, see, e.g., [7,8], etc. Certainly, the theoretical basis of the symbolic methods could be found within the theory of formal power series, in as much as all the symbolic expressions treated are expressible as power series in Δ, E or D, and all the operations employed are just the same as those applied to formal power series. For some easily accessible references on formal series, we may recommend [2,3,11].

Recall that the operators Δ, E and D may be defined via the following relations:

$$\Delta f(t) = f(t+1) - f(t), \quad Ef(t) = f(t+1), \quad Df(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t).$$

Using the number 1 as an identity operator, viz. $1f(t) = f(t)$, one can observe that these operators satisfy the formal relations

$$E = 1 + \Delta = e^D, \quad \Delta = E - 1 = e^D - 1, \quad D = \log(1 + \Delta).$$

Powers of these operators are defined in the usual way. In particular, one may define for any real number x, $E^x f(t) = f(t + x)$.

Note that $E^k f(0) = \left[E^k f(t)\right]_{t=0} = f(k)$, so that any power series of the form $\sum_{k=0}^{\infty} f(k)x^k$ could be written symbolically as

$$\sum_{k\geqslant 0} f(k)x^k = \sum_{k\geqslant 0} x^k E^k f(0) = \sum_{k\geqslant 0} (xE)^k f(0) = (1 - xE)^{-1} f(0).$$

This shows that the symbolic operator $(1 - xE)^{-1}$ with parameter x can be applied to $f(t)$ (at $t = 0$) to yield a power series or a generating function for $\{f(k)\}$.

We shall show in Section 3 that $(1 - xE)^{-1}$ could be expanded into series in various ways to derive various symbolic operational formulas as well as summation formulas for $\sum_{k\geqslant 0} f(k)x^k$. Note that the closed form representation of series has been studied extensively. See, for example, [9] which presents a unified treatment of summation of series using function theoretic method. Some consequences of the summation formulas as well as the examples will be shown in Section 4, can be useful for computational purpose, accelerating the series convergence. In Section 5, we shall give the remainders of the summation formulas.

2. Preliminaries

We shall need several definitions as follows.

Definition 2.1. The expression $f(t) \in C^m_{[a,b)}$ $(m \geqslant 1)$ means that $f(t)$ is a real function continuous together with its mth derivative on $[a, b)$.

Definition 2.2. $\langle x, x_0, x_1, \ldots, x_n \rangle$ represents a least interval containing x and the numbers $x_0, x_2, \ldots, x_n$.

Definition 2.3. $\alpha_k(x)$ is called an Eulerian fraction and may be expressed in the form (cf. [3])

$$\alpha_k(x) = \frac{A_k(x)}{(1-x)^{k+1}}, \quad (x \neq 1),$$

where $A_k(x)$ is the kth degree Eulerian polynomial having the expression

$$A_k(x) = \sum_{j=1}^{k} A(k,j)x^j, \quad A_0(x) \equiv 1$$

with the $A(k,j)$ being known as Eulerian numbers, expressible as

$$A(k,j) = \sum_{i=0}^{j} (-1)^i \binom{k+1}{i} (j-i)^k, \quad (1 \leqslant j \leqslant k).$$

Definition 2.4. δ is Sheppard central difference operator defined by the relation $\delta f(t) = f\left(t+\frac{1}{2}\right) - f\left(t-\frac{1}{2}\right)$, so that (cf. [7])

$$\delta = \Delta E^{-1/2} = \Delta/E^{1/2}, \quad \delta^{2k} = \Delta^{2k}E^{-k}.$$

Moreover, in Sections 3 and 4, we will make use of several simple and well-known propositions which may be stated as lemmas as follows.

Lemma 2.5. *There is a simple binomial identity*

$$\sum_{m=k}^{\infty} \binom{m}{k} x^m = \frac{x^k}{(1-x)^{k+1}}, \quad (|x|<1).$$

Lemma 2.6. *Newton's symbolic expression for E^x is given by*

$$E^x = (1+\Delta)^x = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k.$$

For $f \in C^{n+1}_{[0,\infty)}$ we have Newton's interpolation formula

$$f(x) = E^x f(0) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k f(0) + \binom{x}{n+1} f^{(n+1)}(\xi),$$

where $x \in (0,\infty)$ and $\xi \in \langle x, 0, 1, \ldots, n\rangle$.

Lemma 2.7. *Euler's summation formula for the arithmetic–geometric series is given by*

$$\sum_{j=0}^{\infty} j^k x^j = \frac{A_k(x)}{(1-x)^{k+1}} = \alpha_k(x), \quad (|x|<1),$$

where k is a positive integer, and $\alpha_k(x)$ is the Eulerian fraction.

Lemma 2.8. *For $n \geqslant 1$ we have Everett's symbolic expression* (*cf.* [7, *Section* 129]).

$$E^x = \sum_{k=0}^{\infty} \left(\binom{x+k}{2k+1} \frac{\Delta^{2k}}{E^{k-1}} - \binom{x+k-1}{2k+1} \frac{\Delta^{2k}}{E^k} \right).$$

For $f \in C^{2m}_{(-\infty,\infty)}$ we have Everett's interpolation formula

$$f(x) = \sum_{k=0}^{m-1} \left(\binom{x+k}{2k+1} \delta^{2k} f(1) - \binom{x+k-1}{2k+1} \delta^{2k} f(0) \right) + \binom{x+m-1}{2m} f^{(2m)}(\xi),$$

where $x \in (-\infty, \infty)$ and $\xi \in \langle x, 0, \pm 1, \ldots, \pm m, m+1 \rangle$.

Lemma 2.9. *Gauss's symbolic expression for E^x is given by*

$$E^x = \sum_{k=0}^{\infty} \left(\binom{x+k}{2k} \frac{\Delta^{2k}}{E^k} + \binom{x+k}{2k+1} \frac{\Delta^{2k+1}}{E^{k+1}} \right).$$

For $f \in C^{2m}_{(-\infty,\infty)}$ we have Gauss interpolation formula (*cf.* [7, *Section* 129]).

$$f(x) = \sum_{k=0}^{m-1} \left(\binom{x+k}{2k} \Delta^{2k} f(-k) + \binom{x+k}{2k+1} \Delta^{2k+1} f(-k-1) \right) + \binom{x+m}{2m} f^{(2m)}(\xi),$$

where $x \in (-\infty, \infty)$ and $\xi \in (-m, m-1)$.

Lemma 2.10 (*Mean value theorem*). *Let $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \geqslant 0$ be a convergent series for $x \in (0, 1)$. Suppose that $\phi(t)$ is a bounded continuous function of t on $(-\infty, \infty)$, and $\{t_n\}$ is a sequence of real numbers. Then there is a number $\xi \in (-\infty, \infty)$ such that*

$$\sum_{n=0}^{\infty} a_n \phi(t_n) x^n = \phi(\xi) \sum_{n=0}^{\infty} a_n x^n.$$

3. Main results

We now state and prove the following proposition of various expansions of $(1 - xE)^{-1}$.

Proposition 3.1. *The operator* $(1 - xE)^{-1}$ *has four symbolic expansions, as follows.*

$$(1 - xE)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)^{k+1}} \Delta^k, \tag{3.1}$$

$$(1 - xE)^{-1} = \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{k!} D^k, \tag{3.2}$$

$$(1 - xE)^{-1} = 1 + \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(\frac{\Delta^{2k}}{E^{k-1}} - x\frac{\Delta^{2k}}{E^k}\right), \tag{3.3}$$

$$(1 - xE)^{-1} = 1 + \sum_{k=0}^{\infty} \left(\frac{x}{(1-x)^2}\right)^{k+1} \left(x^{-1}\frac{\Delta^{2k}}{E^k} - \frac{\Delta^{2k}}{E^{k+1}}\right), \tag{3.4}$$

where the condition $x \neq 1$ *is assumed, and moreover,* $x \neq 0$ *for* (3.4).

Proof. Here we present a proof in the sense of symbolic calculus, viz., every series expansion is considered as a formal series.

Clearly (3.1) may be derived as follows:

$$\begin{aligned}(1 - xE)^{-1} &= (1 - x(1+\Delta))^{-1} = (1 - x - x\Delta)^{-1} \\ &= (1-x)^{-1}(1 - x\Delta/(1-x))^{-1} = \sum_{k=0}^{\infty} \frac{x^k \Delta^k}{(1-x)^{k+1}}.\end{aligned}$$

For proving (3.2) it suffices to make use of $E = e^D$ and Lemma 2.7. Indeed we have

$$\begin{aligned}(1 - xE)^{-1} &= (1 - xe^D)^{-1} = \sum_{k=0}^{\infty} x^k e^{kD} \\ &= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{\infty} \frac{(kD)^j}{j!} = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} x^k k^j\right) \frac{D^j}{j!} = \sum_{j=0}^{\infty} \alpha_j(x) \frac{D^j}{j!}.\end{aligned}$$

Eqs. (3.3) and (3.4) can be justified in an entirely similar manner by using Lemma 2.5, Lemma 2.8 and Lemma 2.9, respectively. Indeed, (3.4) may be derived as follows.

$$\begin{aligned}(1-xE)^{-1}-1 &= \sum_{j=1}^{\infty}(xE)^j\\ &= \sum_{k=0}^{\infty}\left\{\left(\sum_{j=1}^{\infty}\binom{j+k}{2k}x^j\right)\frac{\Delta^{2k}}{E^k}+\left(\sum_{j=1}^{\infty}\binom{j+k}{2k+1}x^j\right)\frac{\Delta^{2k+1}}{E^{k+1}}\right\}\\ &= \sum_{k=0}^{\infty}\left\{\frac{x^k}{(1-x)^{2k+1}}\frac{\Delta^{2k}}{E^k}+\frac{x^{k+1}}{(1-x)^{2k+2}}\frac{\Delta^{2k+1}}{E^{k+1}}\right\}\\ &= \sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}\left(\frac{1-x}{x}\frac{\Delta^{2k}}{E^k}+\frac{\Delta^{2k+1}}{E^{k+1}}\right)\\ &= \sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}\left(x^{-1}\frac{\Delta^{2k}}{E^k}-\frac{\Delta^{2k}}{E^{k+1}}\right).\end{aligned}$$

Once (3.3) is derived by the aid of Lemmas 2.5 and 2.8, it can also be verified by symbolic computations. In fact we have

$$\begin{aligned}RHS\ of\ (3.3) &= 1+\frac{x}{(1-x)^2}\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^k\left(\frac{\Delta^2}{E}\right)^k(E-x)\\ &= 1+\frac{x}{(1-x)^2}\frac{E-x}{1-\frac{x}{(1-x)^2}\frac{\Delta^2}{E}} = 1+\frac{x(E-x)}{(1-x)^2-x\frac{\Delta^2}{E}}\\ &= 1+\frac{Ex(E-x)}{(1-x)^2E-x(E-1)^2} = 1+\frac{Ex}{1-xE}\\ &= (1-xE)^{-1} = LHS\ of\ (3.3).\end{aligned}$$

Certainly (3.4) could also be verified in the like manner as above. □

Remark 3.1. Note that all the operators displayed on the right-hand sides of (3.1)–(3.4) involve Δ or D, so that they will yield finite expressions when they are applied to any polynomial $f(t)$ at $t=0$. In particular, we see that for the pth degree polynomial $f(t)$, (3.1) gives a generating function (GF) in the form

$$\sum_{k=0}^{\infty}f(k)x^k=\sum_{k=0}^{p}\frac{x^k}{(1-x)^{k+1}}\Delta^k f(0). \tag{3.5}$$

Actually, this is a well-known formula and was mentioned in [7, Section 11]. Moreover, an exact formula parallel to (3.5) may be obtained from (3.2), namely

$$\sum_{k=0}^{\infty}f(k)x^k=\sum_{k=0}^{p}\frac{\alpha_k(x)}{k!}D^k f(0). \tag{3.6}$$

Certainly, both (3.5) and (3.6) may be used either as summation formulas for the power series $\sum_{k=0}^{\infty} f(k)x^k$ with $|x|<1$, or as a tool for getting GFs for the sequence $\{f(k)\}$.

Remark 3.2. Observe that Euler's formula as given by Lemma 2.7 is a particular case of (3.6) with $f(t)=t^p$ $(p\geqslant 1)$. Obviously, Euler's formula may also be deduced from (3.5) by recalling the fact that (cf. [5])

$$\alpha_k(x)=\sum_{j=0}^{k} j!S(k,j)\frac{x^j}{(1-x)^{j+1}},$$

where $S(k,j)$ are Stirling numbers of the second kind.

Proposition 3.2. *Let $\{f(k)\}$ be a given sequence of numbers (real or complex), and let $h(t)$ be infinitely differentiable at $t=0$. Then we have formally*

$$\sum_{k=0}^{\infty} f(k)x^k=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}\Delta^k f(0), \tag{3.7}$$

$$\sum_{k=0}^{\infty} h(k)x^k=\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{k!}D^k h(0), \tag{3.8}$$

$$\sum_{k=1}^{\infty} f(k)x^k=\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}(\delta^{2k}f(1)-x\delta^{2k}f(0)), \tag{3.9}$$

$$\sum_{k=1}^{\infty} f(k)x^k=\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}(x^{-1}\delta^{2k}f(0)-\delta^{2k}f(-1)), \tag{3.10}$$

where we always assume that $x\neq 0$ and $x\neq 1$.

Proof. Clearly (3.7)–(3.10) are merely consequences of (3.1)–(3.4) by applying the operators to $f(t)$ or $h(t)$ at $t=0$. □

As in the case of (3.5)–(3.6), we have a corollary form (3.9)–(3.10), namely

Corollary 3.3. *If $f(t)$ is a polynomial in t of degree p, then*

$$\sum_{k=1}^{\infty} f(k)x^k=\sum_{k=0}^{[p/2]}\left(\frac{x}{(1-x)^2}\right)^{k+1}(\delta^{2k}f(1)-x\delta^{2k}f(0)), \tag{3.11}$$

$$\sum_{k=1}^{\infty} f(k)x^k=\sum_{k=0}^{[p/2]}\left(\frac{x}{(1-x)^2}\right)^{k+1}(x^{-1}\delta^{2k}f(0)-\delta^{2k}f(-1)). \tag{3.12}$$

Certainly, (3.11)–(3.12) may also be used as a rule for obtaining GFs of $\{f(k)\}$.

4. Consequences of Proposition 3.2 and examples

As observed in Section 3, any of formulas (3.5), (3.6), (3.11) and (3.12) solves generally the summation problem of power series $\sum_{k=0}^{\infty} f(k)x^k$ in the case $f(t)$ is a polynomial. Thus for instance, a few summation formulas of the forms

$$\sum_{k=0}^{\infty}(k+\lambda|\theta)_p x^k = \sum_{k=0}^{p} \frac{k!S(p,k,\lambda|\theta)x^k}{(1-x)^{k+1}}, \tag{4.1}$$

$$\sum_{k=0}^{\infty} D_p(k,\alpha)x^k = \sum_{k=0}^{p} \frac{x^{\alpha}k!S(p,k,\alpha|\theta)x^k}{(1-x)^{k+1}} \tag{4.2}$$

as given in [5] are just particular cases of (3.5) in which $f(t)=(t+\lambda|\theta)_p$ and $f(t)=D_p(t,\alpha)$ are known as the generalized falling factorial and the Dickson polynomial, respectively, or more precisely

$$(t+\lambda|\theta)_p = \prod_{j=0}^{p-1}(t+\lambda-j\theta), \ \ (p\geqslant 1), \quad (t+\lambda|\theta)_0 = 1,$$

and

$$D_p(t,\alpha) = \sum_{j=0}^{[p/2]} \frac{p}{p-j}\binom{p-j}{j}(-\alpha)^j t^{p-2j}, \quad D_0(t,\alpha)=2.$$

Moreover, $S(p,k,\lambda|\theta)$ denotes Howard's degenerate weighted Stirling numbers. (For more in details, cf. [5] loc. cit.)

Another important consequence of Proposition 3.2 is that (3.7), (3.9) and (3.10) with $x=-1$ yield three series transforms, respectively,

$$\sum_{k=0}^{\infty}(-1)^k f(k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1}} \Delta^k f(0), \tag{4.3}$$

$$\sum_{k=1}^{\infty}(-1)^{k-1} f(k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} (\delta^{2k} f(1) + \delta^{2k} f(0)), \tag{4.4}$$

$$\sum_{k=1}^{\infty}(-1)^k f(k) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}} (\delta^{2k} f(0) + \delta^{2k} f(-1)). \tag{4.5}$$

Note that (4.3) is the well-known Euler series transform that can be used to convert a slowly convergent alternating series $\sum_{k=0}^{\infty}(-1)^k f(k)$ with $f(k)\downarrow 0$ (as $k\to\infty$) into rapidly convergent series. For instance, the series

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots \tag{4.6}$$

can be converted using (4.3) with $f(k)=\frac{1}{k+1}$ $(k=0,1,2,\ldots)$ into a quickly convergent series of the form

$$\ln 2=\frac{1}{2}+\frac{1}{2^2\cdot 2}+\frac{1}{2^3\cdot 3}+\frac{1}{2^4\cdot 4}+\cdots \tag{4.7}$$

Actually, the above expression can be derived by substituting

$$\Delta^k f(0)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j+1)^{-1} \tag{4.8}$$

into (4.3). Thus,

$$\begin{aligned}\ln 2&=\sum_{k=0}^{\infty}(-1)^k f(k)=\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j+1)^{-1}\\&=\sum_{k=0}^{\infty}\frac{1}{(k+1)2^{k+1}}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\\&=\sum_{k=0}^{\infty}\frac{1}{(k+1)2^{k+1}}=\sum_{k=1}^{\infty}\frac{1}{k2^k}.\end{aligned} \tag{4.9}$$

Remark 4.1. Obviously, the convergence of the series shown in (4.7) with a rate of $O(1/2^n)$ is much faster than the convergence of the series in (4.6), which has the rate of $O(1/n)$. For instance, to arrive the accuracy of the five digits of $\ln 2=0.69315$, we only need to sum the first 15 terms of the series in (4.7), while the partial sum of the first 40,000 terms of the series in (4.6) is 0.69313. Eqs. (4.4)–(4.5) appear to be novel, and they could also be used to convert slowly convergent alternating series $\sum_{k=1}^{\infty}(-1)^k f(k)$ into quickly convergent ones if a definition for $f(k)=0$ $(k=0,-1,-2,\ldots)$ is introduced. A later work will give the comparison on the rate of the convergence of series (4.3)–(4.5) for the positive decreasing functions.

We now give some examples of the summations shown in Proposition 3.2. Our first example is for function $f(x)=1/(x+1)^2$. Similar to expression (4.8) we obtain

$$\Delta^k f(0)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(j+1)^{-2}.$$

Substituting the above expression into (4.3) and noting the well-known identity (cf. [4])

$$\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\frac{1}{j}=\sum_{\ell=1}^{k}\frac{1}{\ell},$$

(see [6]), we use the process similar to that in (4.9) and have

$$\sum_{k=0}^{\infty}(-1)^k f(k) = \sum_{k=0}^{\infty}\frac{1}{2^{k+1}}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{(j+1)^2}$$
$$= \sum_{k=0}^{\infty}\frac{1}{(k+1)2^{k+1}}\sum_{j=0}^{k}\binom{k+1}{j+1}\frac{(-1)^j}{j+1} = \sum_{k=1}^{\infty}\frac{1}{k2^k}\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j-1}}{j}$$
$$= \sum_{k=1}^{\infty}\frac{1}{k2^k}\sum_{\ell=1}^{k}\frac{1}{\ell} = \sum_{j=0}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{\ell(j+\ell)2^{j+\ell}} = \sum_{\ell=1}^{\infty}\frac{1}{\ell^2 2^\ell} + \sigma,$$

where summation 360(c) in [6] gives the first sum as $\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2$ and

$$\sigma = \sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{\ell(j+\ell)2^{j+\ell}}$$

easily seen to equal

$$\frac{1}{2}\sum_{j=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{j\ell 2^{j+\ell}} = \frac{1}{2}\ln^2\frac{1}{2}$$

yielding

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2} = \frac{\pi^2}{12}. \tag{4.10}$$

Remark 4.2. Although formula (4.10) can be easily derived by using Fourier cosine expansion of x^2, we give a different approach here by using formula (4.3) because it converts the series in (4.10) into the following quickly convergent series:

$$\frac{\pi^2}{12} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{k2^k}\sum_{\ell=1}^{k}\frac{1}{\ell}.$$

Hence, we can use the last series shown above to evaluate $\zeta(2)$ as

$$\zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k2^{k-1}}\sum_{\ell=1}^{k}\frac{1}{\ell}.$$

The sum of the first 13 terms of the last series gives 1.6449, the first 5 digits of $\pi^2/6$, while the sum of the first 5000 terms of the series in (4.10) is only 1.6447. This example shows that formula (4.3) can be used to convert an alternating series into quickly convergent ones.

We now consider another example generated by function $f(x)=(g(t))^x$, where $g:\mathbb{R}\mapsto\mathbb{R}$ and f is defined on $\mathbb{N}\cup\{0\}$. Obviously, we have

$$\Delta^k f(0)=\sum_{j=0}^{k}\binom{k}{j}(g(t))^j(-1)^{k-j}=(g(t)-1)^k \tag{4.11}$$

and for $i=0,1$

$$\begin{aligned}\delta^{2k}f(i)&=\Delta^{2k}E^{-k}f(i)=\Delta^{2k}(g(t))^{i-k}\\&=\sum_{j=0}^{2k}\binom{2k}{j}(g(t))^{i-k+j}(-1)^{2k-j}=(g(t)-1)^{2k}(g(t))^{i-k}.\end{aligned} \tag{4.12}$$

Hence, substituting (4.11) into (3.7) yields

$$\begin{aligned}\sum_{k=0}^{\infty}(g(t))^k x^k&=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}(g(t)-1)^k\\&=\frac{1}{1-x}\frac{1}{1-\frac{x(g(t)-1)}{1-x}}=\frac{1}{1-xg(t)}.\end{aligned} \tag{4.13}$$

Similarly, substituting (4.12) into (3.9), we obtain the following summation formula:

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}(g(t)-1)^{2k}\{(g(t))^{1-k}-x(g(t))^{-k}\}\\&\quad=\frac{g(t)(g(t)-x)}{(g(t)-1)^2}\sum_{k=0}^{\infty}\left(\frac{x(g(t)-1)^2}{g(t)(1-x)^2}\right)^{k+1}\\&\quad=\frac{g(t)(g(t)-x)}{(g(t)-1)^2}\frac{x(g(t)-1)^2}{g(t)(1-x)^2-x(g(t)-1)^2}\\&\quad=\frac{xg(t)(g(t)-x)}{g(t)(1+x^2)-x((g(t))^2+1)}=\frac{xg(t)}{1-xg(t)}.\end{aligned} \tag{4.14}$$

As examples, we take $g(t)=\mathrm{e}^{\mathrm{i}t}$, with $i=\sqrt{-1}$, and $g(t)=t$. Thus, from (4.13) we have, respectively,

$$\sum_{k=0}^{\infty}\mathrm{e}^{\mathrm{i}tk}x^k=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}(\mathrm{e}^{\mathrm{i}t}-1)^k=\frac{1}{1-x\mathrm{e}^{\mathrm{i}t}} \tag{4.15}$$

and

$$\sum_{k=0}^{\infty}(tx)^k=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)^{k+1}}(g(t)-1)^k=\frac{1}{1-xt}. \tag{4.16}$$

By applying (4.14) for $g(t)=\mathrm{e}^{\mathrm{i}t}$ and t we obtain

$$\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}(\mathrm{e}^{\mathrm{i}t}-1)^{2k}\{\mathrm{e}^{-\mathrm{i}(k-1)t}-x\mathrm{e}^{-\mathrm{i}kt}\}=\frac{x\mathrm{e}^{\mathrm{i}t}}{1-x\mathrm{e}^{\mathrm{i}t}} \tag{4.17}$$

and

$$\sum_{k=0}^{\infty}\left(\frac{x}{(1-x)^2}\right)^{k+1}(t-1)^{2k}\{t^{1-k}-xt^{-k}\}=\frac{tx}{1-tx}, \tag{4.18}$$

respectively.

We now illustrate (3.8) with $h(x)=(g(t))^x$ with $g:\mathbb{R}\mapsto\mathbb{R}$ and $g(t)>0$. Hence, $D^k h(0)=(\ln\ g(t))^k$ and from (3.8) and Definition 2.3

$$\sum_{k=0}^{\infty}(g(t))^k x^k=\frac{1}{1-xg(t)}=\sum_{k=0}^{\infty}\frac{\alpha_k(x)}{k!}(\ln\ g(t))^k$$
$$=\sum_{k=0}^{\infty}\frac{A_k(x)}{k!(1-x)^{k+1}}(\ln g(t))^k. \tag{4.19}$$

Replacing $\ln\ g(t)$ by t and $t(1-x)$, respectively, in Eq. (4.19) yields GFs (cf. [3, Section 6.5], [10])

$$\sum_{k=0}^{\infty}\alpha_k(x)\frac{t^k}{k!}=\frac{1}{1-xe^t} \tag{4.20}$$

and

$$\sum_{k=0}^{\infty}A_k(x)\frac{t^k}{k!}=\frac{1-x}{1-xe^{t(1-x)}}. \tag{4.21}$$

Some other GFs such as $(5i)-(5k)$ shown in [3, Section 6.5] can be derived from Eq. (4.19). In addition, from Eq. (4.20), we can establish the recurrence relation for $\alpha_k(x)$ by multiplying both sides of the equation by $(1-xe^t)$. The details can be found in [10].

Finally, we consider a special case of (4.19) by letting $h(x)=e^{itx}$, i.e., $g(t)=e^{it}$ with $t\in\mathbb{R}$, and $x=-1$ in (4.19), we obtain

$$\sum_{k=0}^{\infty}e^{ikt}(-1)^k=\frac{1}{1+e^{it}}=\frac{1}{2}\left\{1-\mathrm{i}\quad\tan\ \frac{t}{2}\right\}=\sum_{k=0}^{\infty}\frac{A_k(-1)}{k!2^{k+1}}(\mathrm{i}t)^k.$$

Therefore, direct verification of the rightmost equality would be effected by the identity

$$\sum_{k=1}^{\infty}\frac{A_k(-1)z^k}{k!}=-\tanh\ z, \tag{4.22}$$

implying $A_k(-1)$ is the negative of the respective tangent coefficient [7]. Note that

$$A_k(-1)=\sum_{j=1}^{k}A(k,\ j)(-1)^j=-\sum_{j=1}^{k}j!S(k,\ j)(-2)^{k-j}, \tag{4.23}$$

with $S(k,\ j)$ denoting the Stirling number of the second kind (cf. [3, Formula [5l]in 6.5], [2,5]). Therefore, implementing exponential GFs, in z, on both sides of (4.23), (4.22) follows from $j!\sum_{k=j}^{\infty}S(k,\ j)z^k/k!=(e^z-1)^j$ (cf. [1]).

The numerical results in Remarks 4.1 and 4.2 are obtained by using mathematical package Mathematica. All of the sums, except for the few discussed in the examples, can also be done with any mathematical package, for examples Mathematica and Maple.

5. Summation formulas with remainders

In this section we will establish four summation formulas with remainders whose forms are suggested by Lemmas 2.6, 2.8 and 2.9.

Theorem 5.1. *Let* $f(t) \in C^m_{[0,\infty)}$, $(m \geqslant 1)$, *with bounded derivative* $f^{(m)}(t)$ *in* $[0,\infty)$, *and let* $\sum_{k=0}^{\infty} f(k)x^k$ *be convergent for* $|x|<1$. *Then for* $x \in (0,1)$ *we have*

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1} (x^M \Delta^k f(M) - x^N \Delta^k f(N)) \frac{x^k}{(1-x)^{k+1}} + \rho_m, \tag{5.1}$$

where the remainder ρ_m *has a form with* $\xi \in [0,\infty)$ *as follows*:

$$\rho_m = (x^M f^{(m)}(M+\xi) - x^N f^{(m)}(N+\xi)) \frac{x^m}{(1-x)^{m+1}}. \tag{5.2}$$

Proof. Let $\phi(t) = \phi(t,x) = x^M f(t+M) - x^N f(t+N)$, so that $\phi(t) \in C^m_{[0,\infty)}$. Then by Lemma 2.6 and using the mean-value theorem (Lemma 2.10) with $a_n = \binom{n}{m}$, we obtain

$$\begin{aligned}
\sum_{k=M}^{N-1} f(k)x^k &= \sum_{n=0}^{\infty} \phi(n)x^n \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{m-1} \Delta^k \phi(0) \binom{n}{k} \right\} x^n + \sum_{n=0}^{\infty} \phi^{(m)}(\xi_n) \binom{n}{m} x^n \ \ (\xi_n \in \langle n, 0, 1, 2, \ldots, m-1 \rangle) \\
&= \sum_{k=0}^{m-1} \Delta^k \phi(0) \left(\sum_{n=0}^{\infty} \binom{n}{k} x^n \right) + \phi^{(m)}(\xi) \sum_{n=0}^{\infty} \binom{n}{m} x^n \\
&= \sum_{k=0}^{m-1} \Delta^k \phi(0) \frac{x^k}{(1-x)^{k+1}} + \phi^{(m)}(\xi) \frac{x^m}{(1-x)^{m+1}} \\
&= \text{RHS of (5.1)}
\end{aligned}$$

with ρ_m being given by (5.2). □

Note that the RHS of (5.1) without ρ_m may be regarded as a rational approximation to the series on the LHS. In particular, if $x^N \Delta^k f(N) \to 0$ $(N \to \infty, 0 \leqslant k \leqslant m-1)$, then (5.1)–(5.2) reduce to

$$\sum_{k=M}^{\infty} f(k)x^k = \sum_{k=0}^{m-1} x^M \Delta^k f(M) \frac{x^k}{(1-x)^{k+1}} + x^M f^{(m)}(M+\xi) \frac{x^m}{(1-x)^{m+1}}. \tag{5.3}$$

Theorem 5.2. *Under the same condition of Theorem* 5.1, *we have*

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1} (x^M f^{(k)}(M) - x^N f^{(k)}(N)) \frac{\alpha_k(x)}{k!} + \rho_m, \tag{5.4}$$

where the remainder is given by

$$\rho_m = (x^M f^{(m)}(M+\xi) - x^N f^{(m)}(N+\xi)) \frac{\alpha_m(x)}{m!}. \tag{5.5}$$

Proof. Denote $\phi(t) = x^M f(t+M) - x^N f(t+N)$ so that $\phi(t) \in C^m_{[0,\infty)}$. Clearly, by using Taylor's expansion with Lagrange's remainder, we have

$$\begin{aligned}\sum_{k=M}^{N-1} f(k)x^k &= \sum_{n=0}^{\infty} \phi(n)x^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{m-1} \frac{1}{k!} \phi^{(k)}(0) n^k \right) x^n + \sum_{n=0}^{\infty} \frac{1}{m!} \phi^{(m)}(\xi_n) n^m x^n, \quad (0 < \xi_n < n) \\ &= \sum_{k=0}^{m-1} \frac{1}{k!} \phi^{(k)}(0) \left(\sum_{n=0}^{\infty} n^k x^n \right) + S_2.\end{aligned}$$

Here we can apply Lemma 2.10 to the series S_2, thus obtaining

$$\begin{aligned}S_2 &= \frac{1}{m!} \phi^{(m)}(\xi) \left(\sum_{n=0}^{\infty} n^m x^n \right) \quad (0 < \xi < \infty) \\ &= \frac{1}{m!} \phi^{(m)}(\xi) \alpha_m(x) = \rho_m.\end{aligned}$$

Hence, in accordance with Lemma 2.7, we have (5.4) and (5.5). □

Remark 5.1. Theorem 5.2 with expressions (5.4)–(5.5) is of similar nature as that of [10, Theorems 1 and 2]. However, our present result appears to be a little more restrictive since we have assumed here the condition $0 < x < 1$ and the convergence of $\sum_{k=0}^{\infty} f(k)x^k$ for $|x| < 1$. In what follows we shall give formulas using the central difference operators $\delta^{2k} = \Delta^{2k}/E^k$ which appear to be more available for numerical computations.

Theorem 5.3. *Let* $f(t) \in C^{2m}_{(-\infty,\infty)}$ *with bounded derivative* $f^{(2m)}(t)$ *in* $(-\infty, \infty)$ *and let* $\sum_{k=0}^{\infty} f(k)x^k$ *be convergent for* $|x| < 1$. *Then for* $x \in (0, 1)$, *we have*

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1} (\delta^{2k}\phi(1) - x\delta^{2k}\phi(0)) \left(\frac{x}{(1-x)^2} \right)^{k+1} + \rho_m, \tag{5.6}$$

where $\delta^{2k}\phi(t) = x^M \delta^{2k} f(t+M) - x^N \delta^{2k} f(t+N)$ *and* ρ_m *is given by the following with* $\xi \in [-m, \infty)$

$$\rho_m = (x^M f^{(2m)}(M+\xi) - x^N f^{(2m)}(N+\xi))\frac{x^{m+1}}{(1-x)^{2m+1}}. \tag{5.7}$$

Proof. Denote $\phi(t) = x^M f(t+M) - x^N f(t+N)$, so that $\phi(t) \in C^{2m}_{(-\infty,\infty)}$. Let us now make use of Everett's formula (Lemma 2.8) for $\phi(t)$ at $t = n$,

$$\begin{aligned}\phi(n) = &\sum_{k=0}^{m-1}\left[\binom{n+k}{2k+1}\delta^{2k}\phi(1) - \binom{n+k-1}{2k+1}\delta^{2k}\phi(0)\right]\\ &+ \binom{n+m-1}{2m}\phi^{(2m)}(\xi_n),\end{aligned}$$

where $\xi_n \in \langle n, 0, \pm 1, \pm 2, \ldots, \pm m, m+1\rangle$. Clearly we have

$$\begin{aligned}\sum_{k=M}^{N-1} f(k)x^k &= \sum_{n=0}^{\infty}\phi(n)x^n, \quad (0 < x < 1)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{m-1}\left[\binom{n+k}{2k+1}\delta^{2k}\phi(1) - \binom{n+k-1}{2k+1}\delta^{2k}\phi(0)\right]x^n\\ &\quad + \sum_{n=0}^{\infty}\binom{n+m-1}{2m}\phi^{(2m)}(\xi_n)x^n\\ &= \sum_{k=0}^{m-1}\delta^{2k}\phi(1)\left(\sum_{n=0}^{\infty}\binom{n+k}{2k+1}x^n\right)\\ &\quad - \sum_{k=0}^{m-1}\delta^{2k}\phi(0)\left(\sum_{n=0}^{\infty}\binom{n+k-1}{2k+1}x^n\right) + \rho_m\\ &= \sum_{k=0}^{m-1}\delta^{2k}\phi(1)\frac{x^{k+1}}{(1-x)^{2k+2}} - \sum_{k=0}^{m-1}\delta^{2k}\phi(0)\frac{x^{k+2}}{(1-x)^{2k+2}} + \rho_m\\ &= \sum_{k=0}^{m-1}(\delta^{2k}\phi(1) - x\delta^{2k}\phi(0))\left(\frac{x}{(1-x)^2}\right)^{k+1} + \rho_m.\end{aligned}$$

Here an application of Lemma 2.10 to the series representation of ρ_m yields

$$\rho_m = \phi^{(2m)}(\xi)\left(\sum_{n=0}^{\infty}\binom{n+m-1}{2m}x^n\right) = \phi^{(2m)}(\xi)\frac{x^{m+1}}{(1-x)^{2m+1}},$$

where $\xi \in [-m, \infty)$. Hence the theorem is proved. □

Theorem 5.4. *Under the same condition of Theorem* 5.3 *we have*

$$\sum_{k=M}^{N-1} f(k)x^k = \sum_{k=0}^{m-1}(x^{-1}\delta^{2k}\phi(0) - \delta^{2k}\phi(-1))\left(\frac{x}{(1-x)^2}\right)^{k+1} + \rho_m, \tag{5.8}$$

where $x \neq 0$ *and* $\delta^{2k}\phi(t) = x^M\delta^{2k}f(t+M) - x^N\delta^{2k}f(t+N)$, *and*

$$\rho_m = (x^M f^{(2m)}(M+\xi) - x^N f^{(2m)}(N+\xi))\frac{x^m}{(1-x)^{2m+1}}. \tag{5.9}$$

Proof. As before, denote $\phi(t)=x^M f(t+M)-x^N f(t+N)$ and let $x \in (0,1)$. Using Gauss interpolation formula with remainder (Lemma 2.9) for $\phi(t)$ at $t=n$, we get as in the case of proving Theorem 5.3 the following expressions

$$\begin{aligned}
\sum_{k=M}^{N-1} f(k)x^k &= \sum_{n=0}^{\infty}\phi(n)x^n \\
&= \sum_{k=0}^{m-1}\Delta^{2k}\phi(-k)\left(\sum_{n=0}^{\infty}\binom{n+k}{2k}x^n\right) \\
&\quad + \sum_{k=0}^{m-1}\Delta^{2k+1}\phi(-k-1)\left(\sum_{n=0}^{\infty}\binom{n+k}{2k+1}x^n\right) \\
&\quad + \sum_{n=0}^{\infty}\binom{n+m}{2m}\phi^{(2m)}(\xi_n)x^n \\
&= \sum_{k=0}^{m-1}\Delta^{2k}\phi(-k)\frac{x^k}{(1-x)^{2k+1}} + \sum_{k=0}^{m-1}\Delta^{2k+1}\phi(-k-1)\frac{x^{k+1}}{(1-x)^{2k+2}} + \rho_m \\
&= \sum_{k=0}^{m-1}\left(\frac{1-x}{x}\Delta^{2k}\phi(-k) + [\Delta^{2k}\phi(-k) - \Delta^{2k}\phi(-k-1)]\right) \\
&\quad \times \left(\frac{x}{(1-x)^2}\right)^{k+1} + \rho_m \\
&= \sum_{k=0}^{m-1}(x^{-1}\Delta^{2k}\phi(-k) - \Delta^{2k}\phi(-k-1))\left(\frac{x}{(1-x)^2}\right)^{k+1} + \rho_m.
\end{aligned}$$

Finally, an application of Lemma 2.10 to the series expression of ρ_m gives

$$\rho_m = \phi^{(2m)}(\xi)\sum_{n=0}^{\infty}\binom{n+m}{2m}x^n = \phi^{(2m)}(\xi)\frac{x^m}{(1-x)^{2m+1}},$$

where $\xi \in (-\infty, \infty)$. Hence Theorem 5.4 is proved. □

Remark 5.2. The uniform boundedness conditions for $f^{(m)}(t)$ in $[0, \infty)$ as well as for $f^{(2m)}(t)$ in $(-\infty, \infty)$ imply that $x^N f^{(m)}(N+\xi) \to 0$ and $x^N f^{(2m)}(N+\xi) \to 0$ as $N \to \infty$ and $0<x<1$. Thus, if in addition, $x^N f^{(k)}(N) = o(N)$, $(N \to \infty, 0 \leqslant k \leqslant m-1)$, then (5.4)–(5.5) yield

$$\sum_{k=M}^{\infty} f(k)x^k = \sum_{k=0}^{m-1} x^M f^{(k)}(M)\frac{\alpha_k(x)}{k!} + x^M f^{(m)}(M+\xi)\frac{\alpha_m(x)}{m!}. \tag{5.10}$$

Similar consequences from Theorems 5.3 and 5.4 may also be deduced by providing additional conditions such as $x^N \delta^{2k} f(N) \to 0$ $(N \to \infty, 0<x<1)$.

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