JOURNAL OF APPROXIMATION THEORY 6, 242–248 (1972)

Approximation of Analytic Functions by Bernstein-Type Operators*

SHELDON EISENBERG

Department of Mathematics, University of Hartford, West Hartford, Connecticut 06117

AND

BRUCE WOOD

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

Communicated by Oved Shisha

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

Let $\{h_j(z)\}$ denote a sequence of complex-valued functions defined on $\bar{\Delta} = \{z : |z| \leqslant 1\}$. Define a matrix $(a_{nk}(z))$ for each $z \in \bar{\Delta}$ by the relations

$$a_{00}(z) = 1, \qquad a_{0k}(z) = 0, \qquad k > 0, \tag{1.1}$$
$$\prod_{j=1}^{n} (wh_j(z) + 1 - h_j(z)) = \sum_{k=0}^{n} a_{nk}(z) w^k.$$

The matrix (a_{nk}) is a generalization of the Lototsky matrix [1, 2]. The substitution $h_j = (1 + d_j)^{-1}$ gives the usual form when $\{h_j\}$ is a bounded sequence of complex constants.

The linear operator L_n associated with the transform (1.1) is defined, for each function f whose domain includes [0, 1], by

$$L_n(f; z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{nk}(z). \tag{1.2}$$

A recent paper of King [4] discussed conditions on a sequence of realvalued functions $\{h_j(x)\}$ which ensure the uniform convergence of $\{L_n(f; x)\}$ to

---

* The first author gratefully acknowledges financial support from the National Science Foundation under Grant No. NSF-GY-5349. The second author was supported by the National Science Foundation under Grant No. NSF-GU-1534.

Copyright © 1972 by Academic Press, Inc.
All rights of reproduction in any form reserved.

$f(x)$, for each $f \in C[0, 1]$. King also pointed out that, when $h_j(x) = x$ $(j = 1, 2, \ldots)$, L_n becomes the classical n-th order Bernstein polynomial [6]. Henceforth, we shall refer to (1.2) as the Lototsky–Bernstein operator.

The present paper concerns uniform approximation of analytic functions by means of Lototsky–Bernstein operators. In Section 2 we obtain very general conditions on $\{h_j(z)\}$ which ensure that $\{L_n(f; z)\}$ converges uniformly to $f(z)$ on the closed unit disk when $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\sum_{k=0}^{\infty} | a_k | < \infty$. Also, uniform convergence of the operators to f, for f analytic in an elliptical region, is discussed.

In Section 3, similar results are given for a class of polynomial operators recently introduced by Stancu [7].

In the sequel, let $e_k(x) = x^k$, $k = 0, 1, \ldots$.

2. The Lototsky–Bernstein Operator

The central result of this section is the following;

THEOREM 2.1. *Let $\{h_i(z)\}$ be a sequence of complex-valued functions having the following properties*:

$$h_i \text{ is analytic in } | z | < r, \qquad r > 1, \qquad i = 1, 2, \ldots; \tag{2.1}$$

$$h_i(1) = 1,\ i = 1, 2, \ldots; \tag{2.2}$$

$$h_i^{(v)}(0) \geqslant 0, \qquad v = 0, 1, 2, \ldots, \qquad i = 1, 2, \ldots; \tag{2.3}$$

$$\sum_{i=1}^{n} h_i'(1) = O(n) \tag{2.4}$$

and

the (C,1) transform of $\{h_i(z)\}$ converges to z on a set of points having a limit point in the open unit disk. (2.5)

If L_n denotes the n-th Lototsky–Bernstein operator generated by $\{h_i(z)\}$ and if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with $\sum_{k=0}^{\infty} | a_k | < \infty$, then $\| L_n(f;) - f \| \to 0$ as $n \to \infty$, where $\|f\| = \max\{ | f(z) | : z \in \bar{\Delta}\}$.

Proof. A function f satisfying the hypotheses is of the form $f = f_1 - f_2 + if_3 - if_4$, where each f_j has positive Taylor coefficients. Therefore it suffices to prove the theorem in the case $a_k \geqslant 0$ for all k.

Write

$$P_n(x; z) = \prod_{i=1}^{n} (1 - h_i(x) + zh_i(x)).$$

Easy computations show that

$$\begin{aligned} L_n(e_0\,; x) &= P_n(x; 1) = 1; \\ L_n(e_1\,; x) &= \frac{1}{n}\frac{\partial P_n(x; 1)}{\partial z} = \frac{1}{n}\sum_{i=1}^{n} h_i(x); \\ L_n(e_2\,; x) &= \frac{1}{n^2}\left(\frac{\partial^2 P_n(x; 1)}{\partial z^2} + \frac{\partial P_n(x; 1)}{\partial z}\right) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} h_i(x)\right)^2 - \frac{1}{n^2}\sum_{i=1}^{n} (h_i(x))^2 + \frac{1}{n^2}\sum_{i=1}^{n} h_i(x). \end{aligned}$$

In fact, for $k \geqslant 1$,

$$\begin{aligned} n^k L_n(e_k\,; x) &= \sum_{m=0}^{n} m^k a_{nm}(x) \\ &= \sum_{m=0}^{n}\sum_{t=1}^{k} \sigma_k{}^t\, m(m-1)\cdots(m-t+1)\, a_{nm}(x) \\ &= \sum_{t=1}^{k} \sigma_k{}^t \frac{\partial^t P_n(x; 1)}{\partial z^t}, \qquad (2.6) \end{aligned}$$

where $\sigma_k{}^t$ denotes a Stirling number of the second kind [3]. But $\sigma_k{}^t$ is a positive integer for $1 \leqslant t \leqslant k$ and $\sigma_k{}^1 = \sigma_k{}^k = 1$. Also (2.3) implies that

$$\frac{\partial^{v+s} P_n(0; 1)}{\partial z^v \partial x^s} \geqslant 0, \qquad v = 1, 2,\ldots, \qquad s = 0, 1,\ldots, \qquad n = 1, 2,\ldots .$$

Therefore, $L_n^{(s)}(e_k\,; 0) \geqslant 0$, $n = 1, 2,\ldots$, $k = 1, 2,\ldots$, $s = 0, 1,\ldots$. This fact with (2.1) and (2.6) yield the inequalities

$$| L_n(e_k\,; z) | \leqslant L_n(e_k\,; | z |) \leqslant L_n(e_k\,; 1), \qquad \text{for} \quad | z | \leqslant 1,$$

$n = 1, 2,\ldots$, $k = 0, 1,\ldots$, Using the definition of $L_n(e_k\,; x)$ and (2.2) it is easy to see that $L_n(e_k\,; 1) = 1$ for all n and k. Clearly, for $| z | \leqslant 1$ and $n = 1, 2,\ldots$,

$$L_n(f; z) = \sum_{k=0}^{\infty} a_k L_n(e_k\,; z)$$

and therefore the sequence $\{L_n(f; z)\}$ is uniformly bounded on $| z | \leqslant 1$. Now hypotheses (2.1)–(2.3) and (2.5) together with Vitali's theorem imply that the $(C, 1)$ transform of the sequence $\{h_i(z)\}$ is uniformly convergent to z on closed subsets of the open unit disk. In addition, since $0 \leqslant h_i(x) \leqslant 1$

for $0 \leqslant x \leqslant 1$ and $i = 1, 2, \ldots$, the operators are positive on $[0, 1]$ (see [4]). It now follows that $L_n(f; x) \to f(x)$ for $0 \leqslant x \leqslant 1$ [4]. Therefore the functions $L_n(f; z)$ converge uniformly to $f(z)$ on each disk $|z| \leqslant p < 1$. Since the series

$$\sum_{k=0}^{\infty} a_k \sum_{v=0}^{\infty} \frac{L_n^{(v)}(e_k\,; 0)}{v!} z^v$$

converges uniformly on $|z| \leqslant 1$, $|L_n'(f; z)| \leqslant L_n'(f; p)$ for $|z| \leqslant p \leqslant 1$. Next, for any $|z| \leqslant 1$, $p \leqslant |z| \leqslant 1$, $z = te^{i\alpha}$,

$$\begin{aligned} |L_n(f; z) - L_n(f; pe^{i\alpha})| &\leqslant \int_p^t |L_n'(f; xe^{i\alpha})|\, dx \\ &\leqslant L_n(f; t) - L_n(f; p) \\ &\leqslant (t-p)\, L_n'(f; 1). \end{aligned}$$

Thus the functions $L_n(f; z)$ will be equicontinuous in $|z| \leqslant 1$ if the sequence $\{L_n'(f; 1)\}$ is bounded. But (2.2) and easy computations show that

$$\begin{aligned} L_n'(f; 1) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) a_{nk}'(1) \\ &= \left(f(1) - f\left(\frac{n-1}{n}\right)\right) \sum_{j=1}^{n} h_j'(1), \end{aligned}$$

and the boundedness of $\{L_n'(f; 1)\}$ follows from (2.4). Finally, since the $L_n(f; z)$ converge uniformly to $f(z)$ on each disk $|z| \leqslant p < 1$ and are continuous on $|z| \leqslant 1$, they converge uniformly on $|z| \leqslant 1$. This completes the proof.

LEMMA 2.2. *Let $h_j(z) = a_j z + b_j$ $(j = 1, 2, \ldots)$, where a_j and b_j are complex constants. If g is a polynomial of degree k, then $L_n(g; z)$ is a polynomial of degree $\leqslant k$.*

Proof. Let

$$r_i(w, z) = h_i(w)(zh_i(w) + 1 - h_i(w))^{-1}$$

and it follows that

$$\frac{\partial P_n(w; z)}{\partial z} = P_n(w; z) \sum_{i=1}^{n} r_i(w, z). \tag{2.7}$$

Hence

$$\frac{\partial P_n(w; 1)}{\partial z} = ns_n(w),$$

where $s_n(w)$ denotes the $(C, 1)$ transform of the sequence $\{h_i(w)\}$.

After differentiating (2.7) j times with respect to z, we obtain

$$\begin{aligned}\frac{1}{n^{j+1}}\frac{\partial^{j+1}P_n(w;1)}{\partial z^{j+1}} &= \frac{1}{n^{j+1}}\sum_{v=0}^{j}\binom{j}{v}\frac{\partial^{j-v}P_n(w;1)}{\partial z^{j-v}}\sum_{i=1}^{n}\frac{\partial^{v}r_i(w,1)}{\partial z^{v}} \\ &= n^{-j}\frac{\partial^{j}P_n(w;1)}{\partial z^{j}}s_n(w)+R_n(w)\end{aligned} \tag{2.8}$$

with

$$R_n(w) = n^{-j-1}\sum_{v=1}^{j}\binom{j}{v}\frac{\partial^{j-v}P_n(w;1)}{\partial z^{j-v}}\sum_{i=1}^{n}\frac{\partial^{v}r_i(w;1)}{\partial z^{v}}.$$

Using (2.7) and (2.8) it is easy to see that $\partial^j P_n(w;1)/\partial z^j$ is a polynomial in w of degree j. The conclusion follows from the linearity of L_n and (2.6) by induction.

We remark that if the sequence $\{h_j(w)\}$ does not consist only of linear factors, the operator $L_n(f;z)$ will not necessarily take polynomials of degree k into polynomials of degree $\leqslant k$.

With the aid of the above lemma, we can obtain, in a manner similar to that used for the Bernstein polynomials [6, p. 90], an analog of Kantorovitch's theorem.

THEOREM 2.3. *Let $\{L_n\}$ be the sequence of Lototsky–Bernstein operators generated by $\{h_j(w)\}$, where*

$$0 \leqslant h_j(x) \leqslant 1 \quad \textit{for} \quad 0 \leqslant x \leqslant 1, \qquad j = 1, 2, \ldots; \tag{2.9}$$

$$\frac{1}{n}\sum_{j=1}^{n} h_j(x) \to x \ \textit{at two points of}\ [0, 1];\ \textit{and} \tag{2.10}$$

$$h_j(x) = a_j x + b_j, \qquad j = 1, 2, \ldots. \tag{2.11}$$

Let f be analytic on the interior of an ellipse with foci 0 *and* 1. *Then*

$$\lim_{n\to\infty} L_n(f;z) = f(z)$$

uniformly on any closed subset interior to the ellipse.

3. THE POLYNOMIAL OPERATOR $P_m^{(\alpha)}$

In a recent paper, Stancu [7] introduced a general class of positive, polynomial linear operators $P_m^{(\alpha)}$, where

$$P_m^{(\alpha)}(f;x) = \sum_{k=0}^{m} w_{m,k}(x;\alpha)f\left(\frac{k}{m}\right), \tag{3.1}$$

and

$$w_{m,k}(x;\alpha) = \binom{m}{k} \frac{\prod_{v=0}^{k-1}(x+v\alpha)\prod_{\beta=0}^{m-k-1}(1-x+\beta\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+[m-1]\alpha)}, \tag{3.2}$$

α being a parameter which may depend only on the natural number m. Clearly $P_m^{(\alpha)}(f;x)$ is a polynomial of degree m.

For $\alpha = -1/m$, (3.1) becomes the Lagrange interpolation polynomial corresponding to the function f and the equally spaced points k/m ($k = 0, 1,..., m$), while $\alpha = 0$ yields the classical Bernstein polynomial. It is also shown in [7] that the well-known Szasz–Mirakyan operator may be obtained as a limiting case of (3.1).

THEOREM 3.1. *Let* $0 \leqslant \alpha = \alpha(m) \to 0$ $(m \to \infty)$. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *with* $\sum_{k=0}^{\infty} |a_k| < \infty$. *Then* $\| P_m^{(\alpha)}(f;) - f \| \to 0$ *and, for* $|z| < 1$,

$$\left(\frac{m(1+\alpha)}{1+m\alpha}\right)(P_m^{(\alpha)}(f;z) - f(z)) = O(1)\ (m \to \infty). \tag{3.3}$$

Proof: As in the proof of Theorem 2.1, we may let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k \geqslant 0$ for all k. Theorem 3.1 of [7] implies

$$D_v P_m^{(\alpha)}(e_k\,; 0) \geqslant 0, \quad k = 0, 1,..., \quad v = 0, 1,..., \quad m = 1, 2,..., \tag{3.4}$$

where D_v denotes the operation of taking the v-th derivative. Next (3.4) and [7, p. 1182] yield

$$|P_m^{(\alpha)}(e_k\,; z)| \leqslant P_m^{(\alpha)}(e_k\,; |z|) \leqslant P_m^{(\alpha)}(e_k\,; 1) = 1, \tag{3.5}$$

for $k = 0, 1,..., m = 1, 2,..., |z| \leqslant 1$. According to Theorem 4.1 of [7],

$$\lim_{m\to\infty} P_m^{(\alpha)}(f;x) = f(x), \qquad 0 \leqslant x \leqslant 1. \tag{3.6}$$

Using Theorem 3.1 of [7] and the assumption $a_k \geqslant 0$, $k = 0, 1,...$, we obtain

$$\begin{aligned} |D_1 P_m^{(\alpha)}(f;1)| &= \sum_{j=1}^{m} \binom{m}{j} \sum_{v=0}^{j-1} (1+\alpha v)^{-1} \Delta_{1/m}^{j} f(0) \\ &\leqslant \sum_{j=1}^{m} \binom{m}{j} j \Delta_{1/m}^{j} f(0) \\ &= D_1 B_m(f;1) \to f'(1), \end{aligned}$$

where B_m is the m-th order Bernstein polynomial. Thus

$$\{D_1 P_m^{(\alpha)}(f;1)\} \text{ is bounded.} \tag{3.7}$$

The first part of Theorem 3.1 now follows from (3.4)–(3.7) just as in the proof of Theorem 2.1.

Let $0 < |z| = x < 1$. Then

$$\begin{aligned}\left|\frac{P_m^{(\alpha)}(f; z) - f(z)}{1 - z}\right| &\leqslant \sum_{k=0}^{\infty} a_k \sum_{v=0}^{k} \frac{D_v P_m^{(\alpha)}(e_k\,; 0)}{v!} \left|\frac{z^v - z^k}{1 - z}\right| \\ &\leqslant \sum_{k=0}^{\infty} a_k \sum_{v=0}^{k} \frac{D_v P_m^{(\alpha)}(e_k\,; 0)}{v!} \left(\frac{x^v - x^k}{1 - x}\right) \\ &= \frac{P_m^{(\alpha)}(f; x) - f(x)}{1 - x},\end{aligned}$$

where we have used Theorem 3.1 of [7] to assert that $P_m^{(\alpha)}(e_k\,; z)$ is a polynomial of degree $\leqslant k$. The above and Theorem 7.1 of [7] yield (3.3).

We note that Theorem 3.1 of [7] implies $P_m^{(\alpha)}$ maps polynomials of degree k into polynomials of degree $\leqslant k$ and this fact may be used to obtain the analog of Theorem 2.3 for $P_m^{(\alpha)}$.

ACKNOWLEDGMENT

The authors are indebted to Professor G. G. Lorentz for a number of helpful suggestions concerning the proof of Theorem 2.1.

REFERENCES

1. V. F. COWLING AND C. L. MIRACLE, Some results for the generalized Lototsky transform, *Can. J. Math.* **14** (1962), 418–435.
2. A. JAKIMOVSKI, A generalization of the Lototsky method of summability, *Michigan Math. J.* **6** (1959), 277–290.
3. C. JORDAN, "Calculus of Finite Differences," Chelsea Press, New York, 1947.
4. J. P. KING, The Lototsky transform and Bernstein polynomials, *Can. J. Math.* **18** (1966), 89–91.
5. P. P. KOROVKIN, "Linear Operators and Approximation Theory," Hindustan, Delhi, 1960.
6. G. G. LORENTZ, "Bernstein Polynomials," Mathematical Expositons, No. 8, University of Toronto Press, Toronto, 1953.
7. D. D. STANCU, Approximation of functions by a new class of linear polynomial operators, *Rev. Roumaine Math. Pures Appl.* **13** (1968), 1173–1194.