On a Conjecture of Phadke and Thakare

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ABSTRACT

We prove the connectedness of the set of all nonzero bounded linear operators on a complex Hilbert space having a generalized inverse.

In a recent paper [3] S. V. Phadke and N. K. Thakare conjectured that in a complex Hilbert space H the set of operators having a generalized inverse is not connected. The purpose of this note is to disprove this conjecture. We recall that a bounded linear operator $A \neq 0$ on H is said to have a generalized inverse if there is a bounded linear operator B on H such that

$$ABA = A. \tag{1}$$

As usual we write $|A| := (A^*A)^{1/2}$ and denote by $s(|A|)$ the support of $|A|$. Then (1) is easily seen to be equivalent to the following condition: there is $C > 0$ such that

$$A^*A \geqslant Cs(|A|). \tag{2}$$

The set of all operators with generalized inverse will be denoted by $\mathrm{GI}(H)$.

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THEOREM. $\mathrm{GI}(H)$ *is pathwise connected.*

Proof. Let $A \neq 0$ be a bounded linear operator on H with generalized inverse, and let $U|A| = A$ be the polar decomposition of A. Then

$$t \mapsto U((1-t)|A| + ts(|A|)), \qquad t \in [0,1],$$

is a path in $\mathrm{GI}(H)$ in view of (2), connecting A and U. The operators $P := UU^*$ and $Q := U^*U$ are orthogonal projections on H, and we may assume that $\dim(1_H - P)(H) \leqslant \dim(1_H - Q)(H)$. Now if P is finite, then these dimensions are equal. Consequently, there exists a partial isometry V on H with $VV^* = 1_H - P$, $V^*V = 1_H - Q$. But then $U + V$ is unitary and can be connected with U through a path in $\mathrm{GI}(H)$, namely

$$t \mapsto U + tV, \qquad t \in [0,1].$$

Next we assume that P is infinite. Then we can find a partial isometry V on H with $VV^* = 1_H - P$ and $V^*V \leqslant 1_H - Q$. As before, U can be connected with $U + V$ in $\mathrm{GI}(H)$, so we may assume $P = 1_H$ from now on. We pick projections P_1, P_2 on H with $P_1P_2 = 0$, $P_1 + P_2 = 1_H$, and $\dim P_1(H) = \dim P_2(H) = \dim H$. Then the operators $Q_i := U^*P_iU$, $i = 1,2$, are orthogonal projections, too, satisfying $Q_1Q_2 = 0$, $Q_1 + Q_2 = Q$, and $\dim Q_i(H) = \dim P_i(H) = \dim H$, $i = 1,2$. But then also $\dim(1_H - Q_1)(H) = \dim H$, implying that there is a partial isometry W on H with $WW^* = P_2$ and $W^*W = 1_H - Q_1$. We now define

$$U(t) := \begin{cases} UQ_1 + (1-t)UQ_2, & t \in [0,1], \\ UQ_1 + (t-1)W, & t \in [1,2]. \end{cases}$$

Then $U(0) = U$, and $U(2)$ is again unitary. Moreover, using (2), it follows that $U(t) \in \mathrm{GI}(H)$ for $t \in [0,2]$. Since the set of all invertible bounded linear operators on H is connected [2, p. 70], U can be connected with 1_H and the theorem is proved. ■

We remark that (1) makes sense in an arbitrary W^*-algebra. The above statement holds also in this more general case; the details of the proof can be found in [1].

REFERENCES

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2 P. R. Halmos, *A Hilbert Space Problem Book.* Princeton, Van Nostrand, 1967.

3 S. V. Phadke and N. K. Thakare, Generalized inverses and operator equations, *Linear Algebra and Appl.* 23:191–199 (1979).

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