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The Euler operator for basic hypergeometric series

Research Article

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Abstract: In this paper, we introduce the Euler operator and give some of it's propositionerties. So we define the trivariate Rogers-Szegö polynomials $h_n(x, y, z/q)$ as a general form of four polynomials: the classical Rogers-Szegö polynomials $h_n(x/q)$, the generalized Rogers-Szegö polynomials $r_n(x, z)$, the homogeneous (bivariate) Rogers-Szegö polynomials $h_n(x, y/q)$ and the Cauchy polynomials $p_n(x, y)$. We represent the trivariate Rogers-Szegö polynomials by special case of Euler operator and derive the generating function, Mehler's formula and the Rogers formula with it's applications for the trivariate Rogers-Szegö polynomials, where Mehler's formula for $h_n(x, y, z|q)$ involves a ${}_3\phi_2$ sum and the Rogers formula involves a ${}_2\phi_1$ sum. Also we give new Mehler's and Rogers formulas for the Cauchy polynomials $p_n(x, y)$. Then, we introduce a transformation from ${}_1\phi_1$ sum to ${}_2\phi_1$ sum.

MSC: 05A30 • 33D45

Keywords: Rogers-Szegö • Polynomials • Generating function • Mehler's formula • Rogers formula

1. Introduction

The Rogers-Szegö polynomials play an important role in the theory of the orthogonal polynomials, particularly in the study of the Askey-Wilson polynomials [1, 5, 9, 11, 24, 26, 30, 37], there are three kinds of the Rogers-Szegö polynomials: classical $h_n(x|q)$, generalized $r_n(x, z)$, and homogeneous $h_n(x, y|q)$. In this paper we introduce the fourth form, it's the trivariate Rogers-Szegö polynomials $h_n(x, y, z|q)$ and derive it's generating function, Mehler's formula, the Rogers formula, and another identities, these identities have some special cases lead us to the corresponding identities for the classical, generalized, homogeneous Rogers-Szegö polynomials and for the Cauchy polynomials $p_n(x, y)$.

Firstly, let us review some common notation and terminology for basic hypergeometric series in [4, 19], where we assume that $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_0 = 1, \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ n \in \mathbb{Z}.$$

The multiple q-shifted factorials can be given as:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_m;q)_n,$$
$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The q-binomial coefficients, or the Gauss polynomials, are given as:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

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The basic hypergeometric series ${}_{r+1}\phi_r$ are defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_{r+1};q)_n}{(q,b_1,\dots,b_r;q)_n}x^n. $$

The Cauchy identity is defined as:

$$ \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}x^k=\frac{(ax;q)_\infty}{(x;q)_\infty},\quad |x|<1, \tag{1} $$

putting $a=0$, (1) becomes Euler's identity:

$$ \sum_{n=0}^{\infty}\frac{x^k}{(q;q)_k}=\frac{1}{(x;q)_\infty},\quad |x|<1, \tag{2} $$

and its inverse relation:

$$ \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}x^k}{(q;q)_k}=(x;q)_\infty. \tag{3} $$

The classical form of the Rogers-Szegö polynomials [1, 5, 9, 11, 24, 26, 30, 37] is defined in 1926 by Szegö, as:

$$ h_n(x|q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^k, $$

after that the generalized Rogers-Szegö polynomials [10, 16, 17] is defined as:

$$ r_n(x,z)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^k z^{n-k}, $$

then in 2003 Chen, Fu and Zhang [12] defined the bivariate (homogeneous) Rogers-Szegö polynomials as:

$$ h_n(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}P_k(x,y), $$

where $P_k(x,y)=(x-y)(x-qy)\cdots(x-q^{k-1}y)$ is the Cauchy polynomials.
The q-differential operator is defined as:

$$ D_q f(a)=\frac{f(a)-f(aq)}{a}, $$

with the Leibniz rule for D_q [34]:

$$ D_q^n\{f(a)g(a)\}=\sum_{k=0}^{n}q^{k(k-n)}\begin{bmatrix} n \\ k \end{bmatrix}D_q^k\{f(a)\}D_q^{n-k}\{g(q^k a)\}, $$

and the q-exponential operator is given by [13]:

$$ T(bD_q)=\sum_{n=0}^{\infty}\frac{(bD_q)^n}{(q;q)_n} $$

where

$$ T(D_q)\{x^n\}=h_n(x|q). \tag{4} $$

Chen, Saad and Sun [15] gave the following operator identity:

$$ T(bD_q)\left\{\frac{(av;q)_\infty}{(as,at;q)_\infty}\right\}=\frac{(bv;q)_\infty}{(as,bs,bt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} v/t,bs \\ bv \end{matrix};q,at\right), \tag{5} $$

where $\max\{|bs|,|bt|\}<1$.
Chen, Fu and Zhang [12] introduced the homogeneous q-difference operator:

$$ D_{xy}f(x,y)=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y} $$

and the homogeneous q-shift operator:

$$ \mathbb{E}(D_{xy})=\sum_{k=0}^{\infty}\frac{D_{xy}^k}{(q;q)_k}, $$

where

$$ D_{xy}\{P_n(x,y)\}=(1-q^n)P_{n-1}(x,y), \tag{6} $$

$$ D_{xy}\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=t\frac{(yt;q)_\infty}{(xt;q)_\infty}. \tag{7} $$

2. The Euler operator

Based on the homogeneous q-difference operator D_{xy} we can give our operator as:

$$\mathbb{J}(bD_{xy})=\sum_{k=0}^{\infty}\frac{(bD_{xy})^k}{(q;q)_k},$$

which is reminiscent of the Euler's identity (2) so we call it as the Euler operator. Compared with the homogeneous q-shift operator $\mathbb{E}(D_{xy})$, our operator can be considered a general form of $\mathbb{E}(D_{xy})$, where the homogeneous q-shift operator $\mathbb{E}(D_{xy})$ is a special case of the Euler operator $\mathbb{J}(bD_{xy})$ for $b=1$. Let us give the following two operator identities of the Euler operator.

Theorem 2.1.
Let D_{xy} and $\mathbb{J}(bD_{xy})$ be defined as above, we have

$$\mathbb{J}(bD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=\frac{(yt;q)_\infty}{(bt,xt;q)_\infty},\tag{8}$$

where $|bt|<1$.

Proof.

$$\begin{aligned}\mathbb{J}(bD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}&=\sum_{k=0}^{\infty}\frac{b^k}{(q;q)_k}D_{xy}^k\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}\\&=\frac{(yt;q)_\infty}{(xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(bt)^k}{(q;q)_k}\\&=\frac{(yt;q)_\infty}{(bt,xt;q)_\infty}.\end{aligned}$$

□

Theorem 2.2.
We have

$$\mathbb{J}(bD_{xy})\{p_n(x,y)\}=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}p_k(x,y)b^{n-k}.\tag{9}$$

Proof.

$$\begin{aligned}\mathbb{J}(bD_{xy})\{p_n(x,y)\}&=\sum_{k=0}^{\infty}\frac{b^k}{(q;q)_k}D_{xy}^k\{p_n(x,y)\}\\&=\sum_{k=0}^{\infty}\frac{b^k}{(q;q)_k}\frac{(q;q)_n}{(q;q)_{n-k}}p_{n-k}(x,y)\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}p_{n-k}(x,y)b^k.\end{aligned}$$

By setting $k\to n-k$, we get the required identity. □

3. The trivariate Rogers-Szegö polynomials

Here we define the trivariate Rogers-Szegö polynomials $h_n(x,y,z/q)$ as a polynomials with three variables x, y and z as follows:

Definition 3.1.

$$h_n(x,y,z/q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}p_k(x,y)z^{n-k}.\tag{10}$$

The trivariate Rogers-Szegö polynomials (10) can be considered a general form for four kinds of polynomials, when setting $z = 1$ in definition (10), get the bivariate Rogers-Szegö polynomials $h_n(x, y/q)$, also if we set $y = 0$ we will get the generalized Rogers-Szegö polynomials $r_n(x,z)$, so that if we set $y = 0$ and $z = 1$ we will get the classical Rogers-Szegö polynomials $h_n(x/q)$, and finally setting $x = 0$ and $z = x$ to get the Cauchy polynomials $p_n(x, y)$. Therefore all the identities of $h_n(x, y, z/q)$ which is deriving in this paper are a generalization of the corresponding identities of $h_n(x, y/q)$, $r_n(x,z)$, $h_n(x/q)$ and $p_n(x, y)$.

In the following propositionosition, we represent the trivariate Rogers-Szegö polynomials $h_n(x, y, z/q)$ by the Euler operator.

Proposition 3.1.

$$\mathbb{J}(zD_{xy}) = h_n(x, y, z/q). \tag{11}$$

Proof. By Lemma 2.2 and definition 3.1. □

Depending on the operator representation (11) for $h_n(x, y, z/q)$, we derive the generating function, Mehler's formula and the Rogers formula.

Theorem 3.1 (The generating function for $h_n(x, y, z|q)$).

We have

$$\sum_{n=0}^{\infty} h_n(x, y, z/q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt, zt;q)_\infty}, \tag{12}$$

where $\max\{|xt|, |zt|\} < 1$.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(x, y, z/q)\frac{t^n}{(q;q)_n} =& \sum_{n=0}^{\infty} \mathbb{J}(zD_{xy})\{p_n(x, y)\}\frac{t^n}{(q;q)_n}\\ =&\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty} p_n(x, y)\frac{t^n}{(q;q)_n}\right\}; \quad |xt| < 1\\ =&\mathbb{J}(zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}; \qquad |zt| < 1\\ =&\frac{(yt;q)_\infty}{(xt, zt;q)_\infty}.\end{aligned}$$

□

- Setting $z = 1$ in Theorem 3.1 to get the generating function of the bivariate Rogers-Szegö polynomials $h_n(x, y/q)$ [10, 12, 15, 36]:

$$\sum_{n=0}^{\infty} h_n(x, y|q)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t, xt;q)_\infty}, \tag{13}$$

where $\max\{|t|, |xt|\} < 1$.

- Setting $y = 0$ in Theorem 3.1 to get the generating function of the generalized Rogers-Szegö polynomials $r_n(x,z)$ [10, 16, 17]:

$$\sum_{n=0}^{\infty} r_n(x,z)\frac{t^n}{(q;q)_n} = \frac{1}{(xt, zt;q)_\infty}, \tag{14}$$

where $\max\{|xt|, |zt|\} < 1$.

- Setting $z = 1$ and $y = 0$ in Theorem 3.1 to get the generating function of the classical Rogers-Szegö polynomials $h_n(x/q)$ [1, 5, 9, 11, 15]:

$$\sum_{n=0}^{\infty} h_n(x|q)\frac{t^n}{(q;q)_n} = \frac{1}{(t, xt;q)_\infty}, \tag{15}$$

where $\max\{|t|, |xt|\} < 1$.

- Setting $x = 0$ and $z = x$ in Theorem 3.1 to get the generating function of the Cauchy polynomials $p_n(x, y)$ [10, 12, 15, 36]:

$$\sum_{n=0}^{\infty} P_n(x, y)\frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \qquad |xt| < 1. \tag{16}$$

4. Mehler's formula for $h_n(x,y,z|q)$

In this section, we introduce Mehler's formula for the trivariate Rogers-Szegö polynomials with it's special cases, after we give Mehler's formula, the following lemma will be derive to get a new identity for the Euler operator approaching to Mehler's formula for $h_n(x,y,z|q)$ polynomials.

Lemma 4.1.
We have

$$\mathbb{J}(zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_n(x,y)}{(yt;q)_n}\right\}=\frac{(yt;q)_\infty}{(xt,zt;q)_\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(y/z,xt;q)_k}{(yt;q)_k}z^kx^{n-k}, \tag{17}$$

where $\max\{|xt|,|zt|\}<1$.

Proof. Let us solve the following sum in two ways:

$$\sum_{n=0}^{\infty}h_n(x,y,z|q)h_n(w|q)\frac{t^n}{(q;q)_n}. \tag{18}$$

In the first way we express $h_n(w|q)$ as $T(D_q)\{w^n\}$ by (4), the sum (18) equals

$$\begin{aligned}&\sum_{n=0}^{\infty}h_n(x,y,z|q)T(D_q)\{w^n\}\frac{t^n}{(q;q)_n}\\&=T(D_q)\left\{\sum_{n=0}^{\infty}h_n(x,y,z|q)\frac{(wt)^n}{(q;q)_n}\right\};(|xwt|<1,\ |zwt|<1)\\&=T(D_q)\left\{\frac{(ywt;q)_\infty}{(xwt,zwt;q)_\infty}\right\};\quad(|xt|<1,\ |zt|<1).\end{aligned}$$

According to (5), (18) equals

$$\frac{(yt;q)_\infty}{(xwt,xt,zt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}y/z,xt\\yt\end{matrix};q,zwt\right).$$

On the second way, we express $h_n(x,y,z/q)$ as $\mathbb{J}(zD_{xy})\{P_n(x,y)\}$, (18) equals

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathbb{J}(zD_{xy})\{P_n(x,y)\}h_n(w|q)\frac{t^n}{(q;q)_n}\\&=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty}P_n(x,y)h_n(w|q)\frac{t^n}{(q;q)_n}\right\}\\&=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty}P_n(x,y)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}w^k\frac{t^n}{(q;q)_n}\right\}\\&=\mathbb{J}(zD_{xy})\left\{\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}P_n(x,q^ky)\frac{t^n}{(q;q)_n}\right)P_k(x,y)\frac{(wt)^k}{(q;q)_k}\right\}\\&=\sum_{k=0}^{\infty}\frac{(wt)^k}{(q;q)_k}\mathbb{J}(zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_k(x,y)}{(yt;q)_k}\right\},\end{aligned}$$

where $|t|,|xt|,|zt|,|zxt|<1$. Now by equate the two results, we get

$$\sum_{k=0}^{\infty}\frac{(wt)^k}{(q;q)_k}\mathbb{J}(zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_k(x,y)}{(yt;q)_k}\right\}=\frac{(yt;q)_\infty}{(xwt,xt,zt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}y/z,xt\\yt\end{matrix};q,zwt\right).$$

Express $1/(xzt;q)_\infty$ by Euler's identity (2) to get

$$\sum_{k=0}^{\infty}\frac{(wt)^k}{(q;q)_k}\mathbb{J}(zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\frac{P_k(x,y)}{(yt;q)_k}\right\}=\frac{(yt;q)_\infty}{(xt,zt;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(y/z,xt;q)_n}{(q,yt;q)_n}\frac{(wt)^{n+k}x^kz^n}{(q;q)_k}.$$

Equating the coefficients of w^n then set $n\to n-k$, get the desired identity. □

Theorem 4.1 (Mehler's formula for $h_n(x,y,z|q)$).
We have
$$\sum_{n=0}^{\infty} h_n(x,y,z|q)h_n(u,v,w|q)\frac{t^n}{(q;q)_n}=\frac{(ywt,xvt;q)_\infty}{(xwt,zwt,xut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} xwt,y/z,v/u \\ ywt,xvt\end{matrix};q,uzt\right), \tag{19}$$
where $\max\{|xwt|,|zwt|,|xut|,|uzt|\}<1$.

Proof. By (11)
$$\sum_{n=0}^{\infty} h_n(x,y,z|q)h_n(u,v,w|q)\frac{t^n}{(q;q)_n}$$
$$=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)h_n(u,v,w|q)\frac{t^n}{(q;q)_n}\right\}$$
$$=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}P_k(u,v)w^{n-k}\right\}$$
$$=\mathbb{J}(zD_{xy})\left\{\sum_{k=0}^{\infty} P_k(u,v)P_k(x,y)\frac{t^k}{(q;q)_k}\left(\sum_{n=0}^{\infty}P_n(x,q^ky)\frac{(wt)^n}{(q;q)_n}\right)\right\};\quad (|xwt|<1)$$
$$=\mathbb{J}(zD_{xy})\left\{\sum_{k=0}^{\infty} P_k(u,v)P_k(x,y)\frac{t^k}{(q;q)_k}\frac{(q^kywt;q)_\infty}{(xwt;q)_\infty}\right\}$$
$$=\sum_{k=0}^{\infty} P_k(u,v)\frac{t^k}{(q;q)_k}\mathbb{J}(zD_{xy})\left\{\frac{(ywt;q)_\infty}{(xwt;q)_\infty}\frac{P_k(x,y)}{(ywt;q)_k}\right\};\quad (|xwt|,|zt|<1).$$

By setting $t\to wt$ in Lemma 4.1, the above summation equals
$$\frac{(ywt;q)_\infty}{(xwt,zwt;q)_\infty}\sum_{k=0}^{\infty}P_k(u,v)\frac{t^k}{(q;q)_k}\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}\frac{(y/z,xwt;q)_j}{(ywt;q)_j}z^jx^{k-j}.$$
Exchanging the order of summations, get
$$\frac{(ywt;q)_\infty}{(xwt,zwt;q)_\infty}\sum_{j=0}^{\infty}P_j(u,v)\frac{(y/z,xwt;q)_j}{(q,ywt;q)_j}(zt)^j\sum_{k=0}^{\infty}P_k(u,q^jv)\frac{(xt)^k}{(q;q)_k};\quad (|xut|<1)$$
$$=\frac{(ywt,xvt;q)_\infty}{(xwt,zwt,xut;q)_\infty}\sum_{j=0}^{\infty}P_j(u,v)\frac{(y/z,xwt;q)_j}{(q,ywt,vxt;q)_j}(zt)^j$$
$$=\frac{(ywt,xvt;q)_\infty}{(xwt,zwt,xut;q)_\infty}\sum_{j=0}^{\infty}\frac{(v/u,y/z,xwt;q)_j}{(q,ywt,vxt;q)_j}(uzt)^j$$
$$=\frac{(ywt,xvt;q)_\infty}{(xwt,zwt,xut;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} xwt,y/z,v/u \\ ywt,xvt\end{matrix};q,uzt\right);\quad (|uzt|<1).$$
The proof is complete. □

- Setting $z=1$ and $w=1$ in (19), we get Mehler's formula of $h_n(x,y/q)$ [10, 15, 36]:
$$\sum_{n=0}^{\infty} h_n(x,y|q)h_n(u,v|q)\frac{t^n}{(q;q)_n}=\frac{(yt,vxt;q)_\infty}{(t,xt,uxt;q)_\infty}\,{}_3\phi_2\left(\begin{matrix} y,xt,v/u \\ yt,vxt\end{matrix};q,ut\right), \tag{20}$$
where $\max\{|t|,|xt|,|xut|,|ut|\}<1$.
- Setting $y=0$ and $v=0$ in (19), we get Mehler's formula of $r_n(x,z)$ [10, 16, 17]:
$$\sum_{n=0}^{\infty} r_n(x,z)\,r_n(u,w)\frac{t^n}{(q;q)_n}=\frac{(xzt^2;q)_\infty}{(zt,xt,yt,xyt;q)_\infty}, \tag{21}$$
where $\max\{|xwt|,|zwt|,|xut|,|uzt|\}<1$.
- Setting $y=0,z=1,v=0$ and $w=1$ in (19), we get Mehler's formula of $h_n(x/q)$ [13, 24, 28, 30, 37, 38]:
$$\sum_{n=0}^{\infty} h_n(x|q)h_n(u|q)\frac{t^n}{(q;q)_n}=\frac{(xut^2;q)_\infty}{(t,xt,ut,xut;q)_\infty}, \tag{22}$$
where $\max\{|t|,|xt|,|xut|,|ut|\}<1$.
- Setting $x=0$ and $z=x$ in (19) then $u=0$ and $w=u$ to get Mehler's formula of the Cauchy polynomials $p_n(x,y)$ [36]:
$$\sum_{n=0}^{\infty} P_n(x,y)P_n(u,v)\frac{t^n}{(q;q)_n}=\frac{(yut;q)_\infty}{(xut;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} y/x \\ yut\end{matrix};q,xvt\right),\quad |xut|<1. \tag{23}$$

5. The Rogers formula of $h_n(x,y,z|q)$

In this section, we introduce the Rogers formula of the trivariate Rogers-Szegö polynomials $h_n(x,y,z|q)$ using the Euler operator and the technique of parameter augmentation. This Rogers formula implies a linearization formula for $h_n(x,y,z|q)$.

Theorem 5.1 (The Rogers formula for $h_n(x,y,z|q)$).
We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(zs,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y/z, xs \\ ys \end{matrix};q,zt\right), \tag{24}$$

where $\max\{|xs|,|xt|,|zs|,|zt|\}<1$.

Proof. By (11), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\
&=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\right\}\\
&=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\left(\sum_{m=0}^{\infty} P_m(x,q^ny)\frac{s^m}{(q;q)_m}\right)\right\}; \quad (|xs|<1)\\
&=\mathbb{J}(zD_{xy})\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{t^n}{(q;q)_n}\frac{(q^nys;q)_\infty}{(xs;q)_\infty}\right\}\\
&=\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\mathbb{J}(zD_{xy})\left\{\frac{(ys;q)_\infty}{(xs;q)_\infty}\frac{P_n(x,y)}{(ys;q)_n}\right\}; \quad (|zs|<1,\ |xs|<1).
\end{aligned}$$

From Lemma 4.1, we get

$$\begin{aligned}
&\frac{(ys;q)_\infty}{(zs,xs;q)_\infty}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(y/z,xs;q)_k}{(ys;q)_k}z^kx^{n-k}\\
&=\frac{(ys;q)_\infty}{(zs,xs;q)_\infty}\sum_{k=0}^{\infty}\frac{(y/z,xs;q)_k}{(q,ys;q)_k}(zt)^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}; \quad (|xt|<1)\\
&=\frac{(ys;q)_\infty}{(zs,xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(y/z,xs;q)_k}{(q,ys;q)_k}(zt)^k\\
&=\frac{(ys;q)_\infty}{(zs,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y/z, xs \\ ys \end{matrix};q,zt\right); \quad (|zt|<1).
\end{aligned}$$

The proof is complete. □

- Setting $z=1$ in Theorem 5.1 to get the Rogers formula of polynomials $h_n(x,y/q)$ [10, 15, 36]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(s,xs,xt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y, xs \\ ys \end{matrix};q,t\right), \tag{25}$$

where $\max\{|s|,|t|,|xs|,|xt|\}<1$.
- Setting $y=0$ in Theorem 5.1 to get the Rogers formula of polynomials $r_n(x,z)$ [10, 16, 17]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,z)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xzst;q)_\infty}{(xs,xt,zs,zt;q)_\infty}, \tag{26}$$

where $\max\{|xs|,|xt|,|zs|,|zt|\}<1$.
- Setting $y=0$ and $z=1$ in Theorem 5.1 to get the Rogers formula of polynomials $h_n(x/q)$ [13, 30, 31]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(xst;q)_\infty}{(s,t,xs,xt;q)_\infty}, \tag{27}$$

where $\max\{|s|,|t|,|xs|,|xt|\}<1$.
- Setting $x=0$ and $z=x$ in Theorem 5.1 to get another Rogers-type formula of the Cauchy polynomials $p_n(x,y)$ as:

Corollary 5.1 (Another Rogers-type formula of $p_n(x,y)$).
We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(xs;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y/x, 0 \\ ys \end{matrix}; q, xt\right), \tag{28}$$

where $\max\{|xs|, |xt|\} < 1$.

Sukhi[36], give the Rogers formula of the Cauchy polynomials $p_n(x,y)$ in the following form:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m} = \frac{(yt;q)_\infty}{(xs,xt;q)_\infty}\,{}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix}; q, ys\right), \tag{29}$$

where $\max\{|xs|, |xt|\} < 1$.
By comparing the Rogers formulase (28) with (29), we can give the following important transformation from ${}_1\phi_1$ sum to ${}_2\phi_1$:

Corollary 5.2.
We have

$${}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix}; q, ys\right) = \frac{(xt, ys;q)_\infty}{(yt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} y/x, 0 \\ ys \end{matrix}; q, xt\right), \tag{30}$$

where $\max\{|xs|, |xt|\} < 1$.

As an application of the Rogers formula 5.1, we derive the linearization formula for the trivariate Rogers-Szegö polynomials $h_n(x,y,z|q)$ as a double summation identity.

Corollary 5.3.
For $n, m \geq 0$, we have

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ l \end{bmatrix}(y/z;q)_k(y/x;q)_l\, x^l z^k\, h_{n+m-k-l}(x,y,z|q)$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ l \end{bmatrix}(y/z;q)_k(y/x;q)_l\,(xq^k)^l\, h_{n-k}(x,y,z|q)\, h_{m-l}(x,y,z|q). \tag{31}$$

Proof. Multiply both sides of Theorem 5.1 by $\dfrac{(ys, yt;q)_\infty}{(xs, zt;q)_\infty}$ to get:

$$\frac{(ys;q)_\infty}{(xs;q)_\infty}\frac{(yt;q)_\infty}{(zt;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}$$
$$= \sum_{k=0}^{\infty}\frac{(y/z;q)_k}{(q;q)_k}\frac{(ysq^k;q)_\infty}{(xsq^k;q)_\infty}(zt)^k\sum_{n=0}^{\infty} h_n(x,y,z|q)\frac{t^n}{(q;q)_n}\sum_{m=0}^{\infty} h_m(x,y,z|q)\frac{s^m}{(q;q)_m}.$$

Verify $(ys;q)_\infty/(xs;q)_\infty, (yt;q)_\infty/(zt;q)_\infty$ and $(ysq^k;q)_\infty/(xsq^k;q)_\infty$ by Cauchy identity (1) and comparing the coefficients of $t^n s^m$, the proof will be completed. □

Corollary 5.4.
For $n, m \geq 0$, we have

$$\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} m \\ k \end{bmatrix}(q;q)_k\,(xz)^k\, r_{n+m-2k}(x,z)$$
$$= \left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} y^k\, h_{n-k}(x,y,z/q)\right)\left(\sum_{j=0}^{m}\begin{bmatrix} m \\ j \end{bmatrix} y^j\, h_{m-j}(x,y,z/q)\right). \tag{32}$$

Proof. Putting $y=0$ in the Rogers formula Theorem 5.1, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} & r_{n+m}(x,z)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\ &=\frac{1}{(zs,xs,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(xs;q)_k}{(q;q)_k}z^k t^k \quad ;|zt|<1\\ &=\frac{1}{(zs,xs,xt;q)_\infty}\frac{(xszt;q)_\infty}{(zt;q)_\infty}\\ &=\frac{(xszt;q)_\infty}{(ys,yt;q)_\infty}\frac{(ys;q)_\infty}{(zs,xs;q)_\infty}\frac{(yt;q)_\infty}{(zt,xt;q)_\infty} \quad ;\{|xt|,|xs|,|zt|,|zs|\}<1\\ &=\frac{(xszt;q)_\infty}{(ys,yt;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_n(x,y,z|q)h_m(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.\end{aligned} \tag{33}$$

Hence

$$\begin{aligned}\frac{1}{(xszt;q)_\infty}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}r_{n+m}(x,z)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\ &=\frac{1}{(ys,yt;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_n(x,y,z|q)h_m(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.\end{aligned}$$

Expand $1/(xszt;q)_\infty, 1/(yt;q)_\infty$ and $1/(ys;q)_\infty$ by the Euler's identity (2), then comparing the coefficients of $t^n s^m$ in both sides, the proof will be completed. □

- Setting $y=0$ in (32) to reduce the linearization formula of the generalized Rogers-Szegö polynomials $r_n(x,z)$ [17]:

$$r_n(x,z)\,r_m(x,z)=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(xz)^k\,r_{n+m-2k}(x,z). \tag{34}$$

- Setting $y=0$ and $z=1$ in (32) to reduce the linearization formula of the classical Rogers-Szegö polynomials $h_n(x/q)$ [11, 13, 24, 26, 32]:

$$h_n(x|q)\,h_m(x|q)=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,x^k\,h_{n+m-2k}(x|q). \tag{35}$$

- Setting $m=0$ in (32) to obtain the following relation between polynomials $r_n(x,z)$ and $h_n(x,y,z|q)$:

$$r_n(x,z)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}y^k\,h_{n-k}(x,y,z|q), \tag{36}$$

which has the inverse relation:

$$h_n(x,y,z|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}y^k\,r_{n-k}(x,z). \tag{37}$$

Lemma 5.1.
For $n,m\ge 0$, we have

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{m}&\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}q^{\binom{j}{2}+\binom{k}{2}}(-y)^{j+k}\,r_{n+m-j-k}(x,z)\\ &=\sum_{k=0}^{\min\{n,m\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,q^{\binom{k}{2}}(-xz)^k\,h_{n-k}(x,y,z|q)\,h_{m-k}(x,y,z|q).\end{aligned} \tag{38}$$

Proof. Rewrite (32) by multiplying $(yt,ys;q)_\infty$ on both sides:

$$\begin{aligned}(ys,yt;q)_\infty&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}r_{n+m}(x,z)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}\\ &=(xszt;q)_\infty\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}h_n(x,y,z|q)h_m(x,y,z|q)\frac{t^n}{(q;q)_n}\frac{s^m}{(q;q)_m}.\end{aligned}$$

Now expand $(ys;q)_\infty,(yt;q)_\infty$ and $(xszt;q)_\infty$ by Euler's identity (3), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}q^{\binom{j}{2}+\binom{k}{2}}(y)^{j+k}}{(q;q)_j(q;q)_k}r_{n+m}(x,z)\frac{t^{n+j}}{(q,q)_n}\frac{s^{m+k}}{(q,q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(xz)^k}{(q;q)_k}h_n(x,y,z|q)h_m(x,y,z|q)\frac{t^{n+k}}{(q,q)_n}\frac{s^{m+k}}{(q,q)_m}.$$

Comparing the coefficients of $t^n s^m$, we get the required identity. □

- Setting $y=0$ in 5.1 to get the inverse relation of the linearization formula of the generalized Rogers-Szegö polynomials $r_n(x,z)$ [16]:

$$r_{n+m}(x,z)=\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k q^{\binom{k}{2}}(-xz)^k r_{n-k}(x,z)r_{m-k}(x,z). \tag{39}$$

- Setting $y=0$ and $z=1$ in Lemma 5.1, to get (the Askey-Ismail formula) or the inverse relation of the linearization formula of the classical Rogers-Szegö polynomials $h_n(x/q)$ [5, 13]:

$$h_{m+n}(x|q)=\sum_{k=0}^{\min\{m,n\}}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k q^{\binom{k}{2}}(-x)^k h_{n-k}(x|q)h_{m-k}(x|q). \tag{40}$$

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References

[1] W. A. Al-Salam and M. E. H. Ismail, q-Beta integrals and the q-Hermite polynomials, Pacific J. Math. 135 (1988) 209–221.
[2] G. E. Andrews, On the foundations of combinatorial theory, V:Eulerian differential operators, Stud. Appl. Math. 50 (1971) 345–375.
[3] G. E. Andrews, Basic hypergeometric functions, SIAM Rev. 16 (1974) 441–484.
[4] G. E. Andrews, The Theory of Partitions, Cambridge University Press, Cambridge, 1985.
[5] R. Askey and M. E. H. Ismail, A generalization of ultraspherical polynomials, In: "Studies in Pure Mathematics", P. Erdös, Ed., Birkhäuser, Boston, MA, 1983, pp. 55–78.
[6] R. A. Askey, M. Rahman and S. K. Suslov, On a general q-Fourier transformation with nonsymmetric kernels, J. Comput. Appl. Math. 68 (1996) 25–55.
[7] R. Askey and J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, Mem. Amer. Math. Soc., 54 (1985) 319.
[8] M. K. Atakishiyeva and N. M. Atakishiyev, Fourier-Gauss transforms of the continuous big q-Hermite polynomials, J. Phys. A: Math. Gen. 30 (1997) L559–L565.
[9] N. M. Atkishiyev and S. M. Nagiyev, On Rogers-Szegö polynomials, J. Phys. A: Math. Gen. 27 (1994) L611–L615.
[10] M. A. Abdlhusein, The q-operators and Rogers-Szegö polynomials, M.SC. Thesis, Basrah University, Iraq, 2009.
[11] D. M. Bressoud, A simple proof of Mehler's formula for q-Hermite polynomials, Indiana Univ. Math. J. 29 (1980) 577–580.
[12] W. Y. C. Chen, A. M. Fu and B. Y. Zhang, The homogeneous q-difference operator, Adv. Appl. Math. 31 (2003) 659–668.
[13] W. Y. C. Chen and Z. G. Liu, Parameter augmenting for basic hypergeometric series, II, J. Combin. Theory, Ser. A 80 (1997) 175–195.
[14] W.Y.C. Chen and Z.G. Liu, Parameter augmentation for basic hypergeometric series, I, Mathematical Essays in Honor of Gian-Carlo Rota, Eds., B. E. Sagan and R. P. Stanley, Birkhäuser, Boston, 1998, pp. 111-129.
[15] W. Y. C. Chen, H. L. Saad and L. H. Sun, The bivariate Rogers-Szegö polynomials, J. Phys. A: Math. Theor. 40 (2007) 6071–6084.
[16] J. Cigler, Elementare q-identitäten, Publication de L'institute de recherche Mathématique avancé (1982) 23–57.
[17] J. Désarménien, Les q-analogues des polynômes d'Hermite, Sém. Lothar. Combin., B06b (1982) 12 pp.
[18] R. Floreanini, J. LeTourneux and L. Vinet, More on the q-oscillator algebra and q-orthogonal polynomials, J. Phys. A: Math. Gen. 28 (1995) L287–L293.

[19] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd Ed., Cambridge University Press, Cambridge, MA, 2004.

[20] J. Goldman and G.-C. Rota, On the foundations of combinatorial theory, IV: Finite vector spaces and Eulerian generating functions, Stud. Appl. Math. 49 (1970) 239–258.

[21] I. P. Goulden and D. M. Jackson, Combinatorial Enumeration, John Wiley, New York, 1983.

[22] Q.-H. Hou, A. Lascoux and Y. P. Mu, Continued fractions for Rogers-Szegö polynomials, Numer. Algoritheorems 35 (2004) 81–90.

[23] E. C. Ihrig and M. E. H. Ismail, A q-umbral calculus, J. Math. Anal. Appl. 84 (1981) 178–207.

[24] M. E. H. Ismail and D. Stanton, On the Askey-Wilson and Rogers polynomials, Canad. J. Math. 40 (1988) 1025–1045.

[25] M. E. H. Ismail and D. Stanton, Tribasic integrals and identities of Rogers-Ramanujan type, Trans. Amer. Math. Soc. 355 (2003) 4061–4091.

[26] M. E. H. Ismail, D. Stanton and G. Viennot, The combinatorics of q-Hermite polynomials and the Askey-Wilson integral, European J. Combin. 8 (1987) 379–392.

[27] W. P. Johnson, q-Extensions of identities of Abel-Rothe type, Discrete Math. 159 (1995) 161–177.

[28] B. K. Karande and N. K. Thakare, On certain q-orthogonal polynomials, Indian J. Pure Appl. Math. 7 (1976) 728–736.

[29] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Delft University of Technology, Report no. 98-17, 1998, http://aw.twi.tudelft.nl/˜koekoek/askey.html.

[30] L. J. Rogers, On a three-fold symmetry in the elements of Heine's series, Proc. London Math. Soc. 24 (1893) 171–179.

[31] L. J. Rogers, On the expansion of some infinite products, Proc. London Math. Soc. 24 (1893) 337–352.

[32] L. J. Rogers, Third memoir on the expansion of certain infinite products, Proc. London Math. Soc. 26 (1895) 15–32.

[33] S. Roman, The theory of the umbral calculus, I, J. Math. Anal. Appl. 87 (1982) 58–115.

[34] S. Roman, More on the umbral calculus, with emphasis on the q-umbral caculus, J. Math. Anal. Appl. 107 (1985) 222–254.

[35] H. L. Saad and M. A. Abdlhusein, The q-exponential operator and generalized Rogers-Szegö polynomials, Journal of Advances in Mathematics, 8 (2014) 1440–1455.

[36] A. A. Sukhi, The q-exponential operators and some of its properties, M.Sc. Thesis, Basrah University, Basrah, Iraq, 2009.

[37] D. Stanton, Orthogonal polynomials and combinatorics, In: "Special Functions 2000: Current Perspective and Future Directions", J. Bustoz, M. E. H. Ismail and S. K. Suslov, Eds., Kluwer, Dorchester, 2001, pp. 389–410.

[38] Z. Z. Zhang and M. Liu, Applications of operator identities to the multiple q-binomial theorem and q-Gauss summation theorem, Discrete Math. 306 (2006) 1424–1437.

[39] Z. Z. Zhang and J. Wang, Two operator identities and their applications to terminating basic hypergeometric series and q-integrals, J. Math. Anal. Appl. 312 (2005) 653–665.