Moments of Generalized Motzkin Paths

Robert A. Sulanke
Boise State University
Boise, ID 83725 USA
Email address: sulanke@math.idbsu.edu

Abstract: Consider lattice paths in the plane allowing the steps $(1,1)$, $(1,-1)$, and $(w,0)$, for some nonnegative integer w. For $n > 1$, let $E(n,0)$ denote the set of paths from $(0,0)$ to $(n,0)$ running strictly above the x-axis except initially and finally. Generating functions are given for sums of moments of the ordinates of the lattice points on the paths in $E(n,0)$. In particular, recurrencess are derived for the cardinality, the sum of the first moments (essentially the area), and the sum of the second moments for paths in $E(n,0)$. These recurrences unify known results for $w = 0, 1, 2$, i.e. those for the Dyck (or Catalan), Motzkin, and Schröder paths, respectively. The sum of the second moments is seen to equal the number of unrestricted paths running from $(0,0)$ to $(0,n-2)$.

Contents:

1. Introduction: the paths and their moments

Let w be a fixed nonnegative integer. We will consider those lattice paths in the Cartesian plane whose permitted step types are the *up diagonal step* $(1,1)$ denoted by U, the *down diagonal step* $(1,-1)$ denoted by D, and the *horizontal step* $(w,0)$ denoted by H. When $w = 0$, only U-steps and D-steps are permitted. We weight the steps by assigning 1 to each U-step, 1 to each D-step, and an indeterminate t to each H-step. The t-weight of a path P, denoted by $|P|$, is the product of the weights of its steps; and the t-weight of a set of paths S, denoted by $|S|$, is the sum of the t-weights of the paths in S.

Often we will suppress the parameter w and the indeterminate t in our notation. Let $U(x,y)$ denote the set of

all *unrestricted* lattice paths using the permitted step types and running from $(0,0)$ to (x,y). We define the set of *generalized Motzkin paths*, denoted by $M(x,y)$, to be the set of paths in $U(x,y)$ that never run below the x-axis except initially and perhaps finally. Of particular interest is the set of *elevated paths*, denoted by $E(x,y)$, consisting of those paths in $M(x,y)$ that never touch the x-axis except initially and perhaps finally. For an example, see Figure 1 and the left column of Table 1 which give the four paths in $E(5,0)$ when $w = 1$.

Figure 1: The 4 elevated Motzkin paths of $E(5,0)$, for $w = 1$, bound a total area of 20 units. Equivalently the sum of their path ordinates is 20.
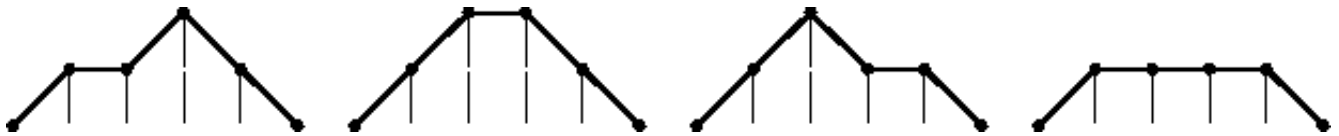
Let $f_n(w)$ denote $|E(n,0)|$ for $n >= 2$, with $f_0(w) = f_1(w) = 0$. For $t = 1$, there are three classical sequences covered by this notation, specifically for $w = 0$, 1 or 2. When $w = 0$, there are no horizontal steps and the paths of $E(n,0)$ are the so-called elevated Dyck (or Catalan) paths; the corresponding sequence $(f_n(0))_{n >= 2} = (1, 0, 1, 0, 2, 0, 5, 0, 14, \dots)$ is the sequence of (aerated) Catalan numbers. For $w = 1$ and $t = 1$, $(f_n(1))_{n >= 2}$ is the sequence of Motzkin numbers. For $w = 2$ and $t = 1$, $(f_n(2))_{n >= 2}$ is the sequence of (aerated) large Schröder numbers. See Table 2 in Section 7. In 1948, using different indexing, Motzkin [12] introduced the sequence $(f_n(1))_{n >= 2}$, where $f_n(1)$ denotes the number of ways to join $n-2$ points on a circle by nonintersecting chords. Donaghey and Shapiro [4] made an early study of this sequence which included lattice paths equivalent to those of $M(n,0)$ with $w = 1$.

Consider a path P in $E(n,0)$ as a rectilinear curve. Let $(0, P(0))$, $(1, P(1))$, ..., $(n, P(n))$ be the list of *all* lattice points (points with integer coordinates) on the path P. We will refer to the values $P(0)$, $P(1)$, $P(2)$, ..., $P(n)$ as the *path ordinates* of P. Define the rth moment of P to be

$$\frac{1}{n-1} \sum_{j=1}^{n-1} P(j)^r.$$

The zeroth moment of P equals 1. By the elementary formula for the area of a trapezoid, we see that $\sum_{0<j<n} P(j)$ equals the area bounded by the path P and the x-axis.

path	contribution to $f_5(1)$	contribution to $g_5(1)$	contribution to total area	contribution to $h_5(1)$
UHUDD	t	$5t/4$	5	$7t/4$
UUHDD	t	$6t/4$	6	$10t/4$

UUDHD		t		$5t/4$		5		$7t/4$
UHHHD		t^3		$4t^3/4$		4		$4t^3/4$

Table 1: This table gives the contributions to $f_5(1) = 3t+t^3$, $g_5(1) = 4t+t^3$, $h_5(1) = 6t+t^3$, and the total area by the the four paths of $E(5,0)$ for $w = 1$.

For fixed $w >= 0$ and for $n >= 2$, we define the following *sums of t-weighted moments* for the path set $E(n,0)$:

$$f_n(w) = \sum_{P \in E(n,0)} |P| \tag{1}$$

$$a_n(w) = \sum_{P \in E(n,0)} |P| \sum_{j=1}^{n-1} P(j) \tag{2}$$

$$g_n(w) = \sum_{P \in E(n,0)} \frac{|P|}{n-1} \sum_{j=1}^{n-1} P(j) \tag{3}$$

$$h_n(w) = \sum_{P \in E(n,0)} \frac{|P|}{n-1} \sum_{j=1}^{n-1} P(j)^2. \tag{4}$$

The main results of this paper are the three recurrences for these sequences, given uniformly in equations (5), (6), and (7), and the generating function for the factorial moments given in Proposition 5. In Section 2 we state these recurrences, which we then establsh by generating function methods in Sections 3, 4, and 5. In Section 6 we prove a surprising result relating second moments to central unrestricted numbers. In Section 7 we will give some examples of these sequences.

2. The recurrences

For any $w >= 0$, consider the following unified set of recurrences for the sequences, $(f_n)_{n >= 2}$, $(g_n)_{n >= 2}$, and $(h_n)_{n >= 2}$, which we have defined above in terms of t-weighted elevated paths:

$$n f_n = 4(n-3) f_{n-2} + (2n - 3w) t f_{n-w} - (n - 3w) t^2 f_{n-2w}, \tag{5}$$

$$(n-1)g_n = 4(n-3) g_{n-2} + (2n - 2w-2)tg_{n-w} - (n - 2w-1)t^2 g_{n-2w}, \tag{6}$$

$$(n-2)h_n = 4(n-3) h_{n-2} + (2n - w - 4)th_{n-w} - (n - w - 2)t^2 h_{n-2w}. \tag{7}$$

These recurrences are valid except for certain initial values, as specified in the propositions below. We first state these recurrences for the known case of elevated Dyck (or Catalan) paths.

Proposition 1. For $w = 0$, the sequences $(f_n(0))_{n \,>= \,2}$, $(g_n(0))_{n \,>= \,2}$, and $(h_n(0))_{n \,>= \,2}$ satisfy

$$n f_n(0) = 4(n\text{-}3) f_{n\text{-}2}(0) \tag{8}$$
$$(n\text{-}1) g_n(0) = 4(n\text{-}3) g_{n\text{-}2}(0) \tag{9}$$
$$(n\text{-}2) h_n(0) = 4(n\text{-}3) h_{n\text{-}2}(0) \tag{10}$$

for $n >= 3$, subject to the initial conditions that $f_n(0) = g_n(0) = h_n(0) = 0$ for $n < 2$ and $f_2(0) = g_2(0) = h_2(0) = 1$.

The proof of this Proposition is covered by the proofs of Propositions 2, 3, and 4. Recurrence (8) dates from about 1758, when Euler [5] recorded it, slightly re-indexed, when he and Segner [14] were considering counting triangulations of convex polygons. See Section 8. It follows immediately that, for $k >= 0$,

$$f_{2k+2}(0) = \frac{1}{k+1}\binom{2k}{k}, \quad g_{2k+2}(0) = \frac{4^k}{2k+1}, \quad \text{and} \quad h_{2k+2}(0) = \binom{2k}{k}. \tag{11}$$

For $w >= 1$, we have the following more general result, which in proved in the next section:

Proposition 2. For $w >= 1$, the sequence $(f_n)_{n \,>= \,2}$ satisfies recurrence (5) for $n > 2w$, with initial values satisfying $f_n = f_n(0) + (n\text{-}w\text{-}1)t f_{n\text{-}w}(0)$ for $n <= 2w$.

With $a_n = a_n(w)$ denoting the t-weighted area, $\sum_{P \in E(n,0)} |P| \cdot [\text{area under } P]$, the trapezoidal area formula shows that $a_n = (n\text{-}1)g_n$ for all $n >= 2$. The following result, proved in Section 5, generalizes one of Kreweras [9] for $w = 2$ and $t = 1$.

Proposition 3. For $w >= 1$, $(g_n)_{n \,>= \,2}$ satisfies recurrence (6) for $n > w + 2$, with initial values $g_n = g_n(0)$ for $n <= w + 2$, and $g_{w+2} = g_{w+2}(0) + t$. Equivalently, for $w >= 1$, the sequence $(a_n)_{n \,>= \,2}$ satisfies the recurrence

$$a_n = 4 a_{n\text{-}2} + 2 t a_{n\text{-}w} - t^2 a_{n\text{-}2w} \tag{12}$$

for $n > w + 2$, with initial values $a_n = (n\text{-}1)g_n(0)$ for $n <= w + 2$, and $a_{w+2} = (w+1)g_{w+2}(0) + (w+1)t$.

We remark that (10) is a well-known recurrence for the central binomial coefficients. In the case when $w = 1$ with $t = 1$, recurrence (7) dates from 1764, as Euler [6] proved that the central trinomial coefficients satisfy this recurrence when appropriately re-indexed. Our knowledge that these central coefficients are solutions to the recurrences (7) and (10) led to an interesting relationship between second moments and central numbers of the form $|U(n,0)|$ for arbitrary w. Specifically, in Section 6 we will see that $|U(n-2, 0)|$ satisfies a recurrence that is also satisfied by $h_n(t)$; thus we have a proof of identity (13) below. The proof of the first part of the following appears in Section 5.

Proposition 4. For $w >= 1$, the sequence $(h_n)_{n >= 2}$ satisfies recurrence (7) for $n >= 3$, with the initial values $h_n = 0$ for $n < 2$ and $h_2 = 1$. Moreover, for any w and for $n >= 2$,

$$h_n = |U(n\text{-}2,0)|. \tag{13}$$

3. Enumerating restricted paths

Consider the generating function $M(z) = \sum_{n \geq 0} |M(n, 0)| z^n$. With the exception of the point path, each path in $M(n,0)$ either begins with an H-step or immediately leaves the x-axis and then later returns for a first time. Consequently, with L denoting $\bigcup_{n \geq 0} M(n, 0)$ and with juxtaposition indicating concatenation, we have a decomposition that defines L recursively:

$$L = M(0, 0) \ \cup \ HL \ \cup \ ULDL.$$

With z marking a horizontal unit and with t marking each H-step, the decomposition yields

$$M(z) = 1 + tz^w M(z) + z^2 M(z)^2, \tag{14}$$

and hence

$$M(z) = (1 - tz^w - \sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}})/2z^2. \tag{15}$$

Let $F(z) = \sum_{n \geq 2} f_n z^n$. Since $f_{n+2} = |M(n,0)|$,

$$F(z) = (1 - tz^w - \sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}})/2. \tag{16}$$

Note that the coefficient of z^n both in the power series for $F(z)$ and in the power series for

$$\Phi(z) = -(\sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}})/2$$

must agree for all coefficients f_n, except for $n = 0$ or $n = w$. Logarithmic differentiation yields

$$(1 - 4z^2 - 2tz^w + t^2 z^{2w})\Phi'(z) + (4z + wtz^{w-1} - wt^2 z^{2w-1})\Phi(z) = 0. {}^{j+w,k)} \text{ to } (n,0).$$

By symmetry, R can be matched with a path in $E(n-j,k)$. Let $m(j,k)$ denote $|E(j,k)|$.

For $n >= 2$,

$$\mu(n,r) = \sum_{P \in E(n,0)} |P| \sum_{0<i<n} (P(i))_r$$

$$= \sum_{P} \sum_{k>0} \sum_{0<i<n} (k)_r |P| \, B(P(i) = k)$$

$$= \sum_{k>0} \sum_{0<i<n} \sum_{P} (k)_r |P| \, B(P(i) = k)$$

$$= \sum_{k>0} (k)_r \sum_{0<i<n} \sum_{P} |P| \, (B((i,k) \text{ step end of } P) + B((i,k) \text{ interior of } P))$$

$$= \sum_{k>0} (k)_r \sum [\sum_{j>0} \sum_{Q,R} |Q| \cdot |R| + \sum_{Q,R'} (w-1)t|Q| \cdot |R'|]$$

$$= \sum_{k} (k)_r [\sum_{j} m(j,k)m(n-j,k) + (w-1)t \sum_{j} m(j,k)m(n-j-w,k)]$$

With M denoting $M(z)$, we claim that the following string of equations holds:

$$\sum_{n>=2} \mu(n,r)z^n = \sum_{k} (k)_r [\sum_{n} \sum_{j} m(j,k)m(n-j,k)z^n$$

$$+ (w-1)tz^w \sum_{n} \sum_{j} m(j,k)m(n-j-w,k)z^{n-w}]$$

$$= [1 + (w-1)tz^w] \sum_{k} (k)_r (zM)^{2k} \tag{17}$$

$$= (1 + (w-1)tz^w)\frac{r!(z^2M^2)^r}{(1-z^2M^2)^{r+1}} \tag{18}$$

$$= r!(1 + (w-1)tz^w) \cdot \frac{(z^2M^2)}{(1-z^2M^2)^2} \cdot \frac{(z^2M^2)^{r-1}}{(1-z^2M^2)^{r-1}} \tag{19}$$

$$= \frac{r!z^2(1 + (w-1)tz^w)(1 - tz^w - \sqrt{(1-tz^w)^2 - 4z^2}\,)^{r-1}}{2^{r-1}(\sqrt{(1-tz^w)^2 - 4z^2}\,)^{r+1}}.$$

To establish this string we first note that each path in $E(j,k)$ must depart from each line $y = c$, for integer c, $0 <= c < k$, for a last time. Hence a simple convolution argument shows that the generating function for $m(j,k)$ satisfies

$$\sum_j m(j,k)z^j = (zM(z))^k. \tag{20}$$

This implies (17). Line (18) is a consequence of binomial theorem in the form
$$r!y^r(1-y)^{-r-1} = \sum_{k>=r}(k)_r y^k.$$

To handle the middle fraction in (19) we use (14) twice:

$$\frac{z^2M^2}{(1-z^2M^2)^2} = \frac{z^2M^2}{((1-tz^w)M - 2)^2}$$

$$= \frac{z^2M^2}{(1-tz^w)^2M^2 + 4(1 - (1-tz^w)M)}$$

$$= \frac{z^2M^2}{(1-tz^w)^2M^2 - 4z^2M^2}$$

$$= \frac{z^2}{(1-tz^w)^2 - 4z^2}.$$

To handle the last fraction in (19) use the following result derived from formula (15), with
$$\Delta = (1-tz^w)^2 - 4z^2:$$

$$\frac{z^2 M^2}{1 - z^2 M^2} = \frac{(2z^2 M)^2}{4z^2 - (2z^2 M)^2}$$

$$= \frac{(1 - tz^w - \sqrt{\Delta})^2}{4z^2 - (1 - tz^w)^2 + 2(1 - tz^w)\sqrt{\Delta} - \Delta}$$

$$= \frac{(1 - tz^w - \sqrt{\Delta})^2}{-2\Delta + 2(1 - tz^w)\sqrt{\Delta}}$$

$$= \frac{1 - tz^w - \sqrt{\Delta}}{2\sqrt{\Delta}}.$$

5. Area and second moments

Setting $r = 1$ in Proposition 5, we obtain a generating function for sums of the t-weighted areas:

$$\sum_{n>=2} a_n z^n = \frac{z^2 (1 + (w - 1)tz^w)}{(1 - tz^w)^2 - 4z^2}. \tag{21}$$

Then the first part of Proposition 3 follows upon comparing coefficients in (21), rewritten as

$$(1 - 4z^2 - 2tz^w + t^2 z^{2w}) \sum_{n>=2} a_n z^n = z^2 (1 + (w - 1)tz^w),$$

and checking the obvious initial conditions. Recurrence (6) is then immediately derived from (12) by (2) and (3).

There is an interesting corollary when $w = 1$. Using partial fractions decomposition, the generating function (21) yields

$$A(z) = \frac{z}{4} \left(\frac{1}{1 - (2z + tz)} - \frac{1}{1 + (2z - tz)} \right),$$

and so, for $w = 1$ and $n >= 2$,

$$a_n = \frac{1}{4} ((t + 2)^{n-1} - (t - 2)^{n-1}).$$

To obtain the generating function for the second moments, $H(z) = \sum_{n>=2} h_n z^n$, we use the following,

where the constant of integration is checked to be 0:

$$H(z) = \sum_{n \geq 2} \sum_{P \in E(n,0)} \frac{|P|}{n-1} \sum_{j}((P(j))_1 + ((P(j))_2) z^n$$

$$= z \sum_{n \geq 2} \frac{1}{n-1}(\mu(n,1) + \mu(n,2)) z^{n-1}$$

$$= z \int \sum_{n \geq 2} z^{-2}(\mu(n,1) + \mu(n,2)) z^n \, dz$$

$$= z \int \sum_{n \geq 2} \frac{(1 + (w-1)tz^w)(1 - tz^w)}{(1 - 4z^2 - 2tz^w + t^2 z^{2w})^{3/2}} \, dz$$

$$= \frac{z^2}{\sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}}}. \tag{22}$$

Let $\Psi(z) := \sum_{n \geq 0} h_{n+2} z^n = H(z)/z^2$. From (22), differentiation with respect to z yields

$$(1 - 4z^2 - 2tz^w + t^2 z^{2w})\Psi'(z) - (4z + wtz^{w-1} - wt^2 z^{2w-1})\Psi(z) = 0.$$ runs from
(0,-2) to (n,0) }   (23) <-> $U(n,0)$ - $U(n,2)$.   (24) To obtain (23), observe that each path P in the set $\{P$ in $U(n,0) : P$ intersects the line $y=-1\}$ can be decomposed as $P = QR$, where Q terminates at the first intercept of the line $y = -1$ by the path P. Let Q' denote the reflection of the path Q about the line $y = -1$. The matching $P = QR$ with $P' = Q'R$ now defines the bijection indicated in (23). A simple translation yields (24).

Let $u(x,y)$ denote $|U(x,y)|$. Since any path to the point $(n+1,k)$ must end with a U, D, or H step, we have

$$u(n+1,k) = u(n, k-1) + u(n, k+1) + t\, u(n-w+1)$$

and $u(n, -1) = u(n, 1)$. Using (24) and these identities, we obtain

$$2f_{n+2} = 2u(n,0) - 2u(n,2)$$
$$= 4\, u(n,0) - 2u(n+1,1) + 2\, t\, u(n-w+1,1)$$
$$= 4\, u(n,0) + t\, u(n-w+2,0) - u(n+2,0) - (t^2 u(n-2w+2,0) - t\, u(n-w+2,0))$$
$$= -u(n+2,0) + 4\, u(n,0) + 2\, t\, u(n-w+2,0) - t^2 u(n-2w+2,0). \tag{25}$$

Returning to results (16) and (22), we observe that they imply

$$1 - tz^w - 2 \sum_{n \geq 2} f_n z^n = \frac{1 - 4z^2 - 2tz^w + t^2 z^{2w}}{\sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}}} \qquad \text{and}$$

$$z^{-2} \sum_{n \geq 2} h_n z^n = \frac{1}{\sqrt{1 - 4z^2 - 2tz^w + t^2 z^{2w}}}.$$

Comparing coefficients yields the mixed recurrence

$$2f_{n+2} = -h_{n+4} + 4\,h_{n+2} + 2t\,h_{n-w+4} - t^2\,h_{n-2w+4}. \tag{26}$$

But this recurrence for f_n and h_n has the same form as that for f_n and $u(n,0)$ given in (25). Since the initial conditions agree, we have the second statement of Proposition 4 by induction. Moreover, we have that the generating function for $u(n,0) = |U(n,0)|$ satisfies

$$\sum_{n \geq 0} |U(n,0)| z^n = \Psi(z).$$

We have omitted the explicit formulas for f_n and h_n, which are weighted sums of Catalan and binomial coefficients, respectively. In light of (13), we can find these sums by counting the ways to insert horizontal steps into the respective paths.

The first formula of (11) yields the following known relation between the Catalan numbers and the central binomial numbers. Upon replacing $2k+2$ by n in that formula, we find for $h = 0$, $n \geq 2$ and n even,

$$f_n(0) = \frac{1}{(n-2)/2 + 1} \binom{n-2}{(n-2)/2} = \frac{1}{2(n-1)} \binom{2n}{n} = \frac{|U(n,0)|}{2(n-1)}.$$

The following gives an analogous result for general w:

Proposition 6. For $n > 2w$,

$$f_n = \frac{|U(n,0)| + (w-2)t|U(n-h,0)| + t^2(1-w)|U(n-2w,0)|}{2(n-1)}.$$

Proof: One can substitute expressions given by recurrence (7) and (26) into (5), which is valid for n > 2w. Our substitutions were facilitated using a computer algebra program. Equation (13) then is applied to complete the proof.

7. Examples

In Table 2, we record the previously studied, and named, examples satisfying the recurrences in Propositions 1 to 4, along with their reference number from Sloane's *On-Line Encyclopedia of Integer Sequences* [16]. These examples correspond to sets of elevated paths, $(E(n,0))_{n \geq 2}$, so in the table $n = 2, 3, 4 \ldots$ and $k = 1, 2, 3 \ldots$.

t	Sequence		Name	Sloane
1	$f_n(0)$	1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42	aerated Catalan nos.	
1	$f_{2k}(0)$	1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862	Catalan nos.	A000108
1	$a_{2k}(0)$	1, 4, 16, 64, 256, 1024, 4096, 16384	powers of 4	A000302
1	$h_{2k}(0)$	1, 2, 6, 20, 70, 252, 924, 3432, 12870	central binomial nos.	A000984
1	$f_n(1)$	1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188	Motzkin nos.	A001006
2	$f_n(1)$	1, 2, 5, 14, 42, 132, 429, 1430, 4862	(lacking initial 1)	A000108
3	$f_n(1)$	1, 3, 10, 36, 137, 543, 2219, 9285, 39587	(tree-like polyhexes)	A002212
4	$f_n(1)$	1, 4, 17, 76, 354, 1704, 8421, 42508	(walks on cubic lattice)	A005572
1	$a_n(1)$	1, 2, 7, 20, 61, 182, 547, 1640, 4921		A014983
1	$h_n(1)$	1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139	central trinomial nos.	A002426
1	$f_{2k}(2)$	1, 2, 6, 22, 90, 394, 1806, 8558, 41586	large Schröder	A006318
1	$a_{2k}(2)$	1, 7, 41, 239, 1393, 8119, 47321, 275807		A002315
1	$h_{2k}(2)$	1, 3, 13, 63, 321, 1683, 8989, 48639	central Delannoy nos.	A001850
1	$f_n(3)$	1, 0, 1, 1, 2, 3, 6, 10, 20, 36, 72, 136		A005418
1	$a_n(3)$	1, 0, 4, 4, 16, 24, 71, 128, 328, 650		A053441
1	$h_n(3)$	1, 0, 2, 1, 6, 6, 21, 30, 82, 141, 342, 650		A053442

Table 2

In Table 2, the ``walks on cubic lattice'' entry illustrates one role played by the indeterminate t. Corresponding to the case $w = 1$ and $t = 4$, Guy [7] found f_n (mildly re-indexed) to count the walks on a three-dimensional lattice that use unit steps in all six standard directions (i.e. the positive and negative unit steps parallel to the three axes), that start at $(0,0,0)$, end on the x-y plane, and never pass beneath that plane. More generally, for $m > 3$, $w = 1$ and $t = 2^{m-1}$, we find that f_n counts the walks of length n-2 on the m-dimensional integer lattice that use the unit steps in all 2^m standard directions, start at the origin, end on the hyperplane, $x_1 + ... + x_{m-1} = 0$, and never pass through a lattice point $(x_1, ... , x_m)$ for which $x_m < 0$. To see this we identify the unit step in the positive x_m direction with the U-step, the unit step in the negative x_m direction with the D-step, and the set of the other 2^{m-1} steps, none of which affects the distance from the hyperplane, $x_1 + ... + x_{m-1} = 0$, with a weighed H-step.

Another example utilizing t is the enumeration of the horizontal steps over all paths in $E(n,0)$. Let $f_{n,k}$ denote the number of paths in $E(n,0)$ having k horizontal steps. We find that the generating function for the total number of horizontal steps on paths having k horizontal steps satisfies

$$\sum_{n>=0}\sum_{k>=0} k f_{n,k} t^k z^n \;=\; \sum_{n>=0} \frac{d}{dt} f_n z^n$$

$$= \frac{d}{dt} F(z)$$

$$= z^w(-1 + (1 - tz^w)\Psi(z)).$$

Consequentially, the total number of horizontal steps is expressible in terms of t-weighted unrestricted paths as follows: for $n >= 0$,

$$\sum_{k>=0} k f_{n,k} t^k = (u(n - w, 0) - t u(n - 2w, 0))/2 = u(n - w - 1, 1).$$

8. Related Studies

As noted in [8], in the 1730's, Ming An-tu, a Mongolian mathematician, was aware of the Catalan numbers, $(c_n)_{n>=0} = (1, 1, 2, 5, 14, ...)$, in a non-combinatorial sitting. He discovered several recurrence for these numbers including the well-known convolution recurrence, $c_n = c_0 c_{n-1} + c_1 c_{n-2} + \ldots + c_{n-1} c_0$. In about 1758, Euler [5] and Segner [14] made the first European discovery of these numbers while counting the triangulations of a convex polygon. They observed that c_{n-2} is the number of ways to draw non-crossing diagonals between the vertices of a convex n-gon. Segner recorded and proved the above convolution recurrence in terms of triangulations of polygons. Euler observed, without giving a proof, that $c_n = (4n-2)/(n+1) \, c_{n-1}$, which is essentially (8), and then gave a closed form for c_n as a product of ratios that reduces to a ratio of product as in the first formula of (11).

There are several studies on lattice paths, in addition to those mentioned previously, that have influenced our results. Barcucci, Pinzani, and Sprugnoli [1] have made a systematic analysis containing recurrences - many mixed - for the Motzkin paths, i.e. $w = 1$ and $t = 1$, involving the sequences for count, central entries (central trinomial coefficients), and other related statistics. Recently the author [20] has established (5), (6), and (7) bijectively for Motzkin paths.

Besides Kreweras [9], Bonin, Shapiro, and Simion [2] have considered elevated Schröder paths and the recurrence (12) for $w = 2$. For $w = 2$ the author [18] has employed bijective schema to establish recurrences (5) and (12); in [19] he has considered (5), (6), and (7) in terms of parallelogram polyominoes. Most recently for $w = 2$, Merlini, Sprugnoli, and Verri [11] have given additional proofs for (5) with essentially an arbitrary t, while Pergola and Pinzani [13] have developed an encoding relating area to path count to obtain (12) bijectively.

Merlini, Sprugnoli, and Verri [10] have developed generating functions for the total area bounded by lattice paths where the permitted steps are more general than our U, D, and H. For Dyck paths, Chapman [3] has considered the generating functions for the sums of path moments and the relationship between the generating functions for elevated versus non-elevated paths.

Stanley [17] has given an extensive treatment of generalizations of the central entries, $|U(n,0)|$, under the

name ``diagonals''. In [17] his results for *differentiably finite power series* relate to our use of the generating functions Φ and Ψ of Sections 3 and 5. Correspondingly, he considered *polynomially recursive sequences,* for which our sequences (f_n), (g_n), and (h_n) are prime examples.

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