Orthogonal polynomials and the moment problem

Henry Roland Steere

School of Mathematics

University of the Witwatersrand

Johannesburg

South Africa

Under the supervision of

Dr S. J. Johnston.

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 2012.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

________________________ ________________________

Name Student Number

______ day of ____________________ 2012

Abstract

The classical moment problem concerns distribution functions on the real line. The central feature is the connection between distribution functions and the moment sequences which they generate via a Stieltjes integral. The solution of the classical moment problem leads to the well known theorem of Favard which connects orthogonal polynomial sequences with distribution functions on the real line. Orthogonal polynomials in their turn arise in the computation of measures via continued fractions and the Nevanlinna parametrisation. In this dissertation classical orthogonal polynomials are investigated first and their connection with hypergeometric series is exhibited. Results from the moment problem allow the study of a more general class of orthogonal polynomials. q-Hypergeometric series are presented in analogy with the ordinary hypergeometric series and some results on q-Laguerre polynomials are given. Finally recent research will be discussed.

Acknowledgements

I acknowledge the support of the National Research Foundation of South Africa in funding my MSc.
I am grateful for the help and support of Dr Sarah Jane Johnston while researching the topic and compiling my dissertation.
Lastly I thank my family for their companionship.

Contents

Bibliography

[1] N. I. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, 1965.

[2] N. I. Akhiezer, I. M. Glazman *Theory of Linear Operators in Hilbert Space*, volume 1, Frederick Ungar Publishing Company, New York, 1961, (Dover reprint 1993).

[3] N. I. Akhiezer, I. M. Glazman *Theory of Linear Operators in Hilbert Space*, volume 2, Frederick Ungar Publishing Company, New York, 1963, (Dover reprint 1993).

[4] N. I. Akhiezer, M. G. Krein, *Some Questions in the Theory of Moments*, American Mathematical Society, Providence, Rhode Island, 1962.

[5] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[6] H. Anton, C. Rorres, *Elementary Linear Algebra*, John Wiley and Sons, 9th edition, 2005.

[7] R. Askey, Ramanujan's extension of the gamma and beta functions, *Amer. Math. Monthly*, 87(5) (1980), 346-359.

[8] C. Berg, J. Christensen, Density questions in the classical theory of moments, *Ann. Inst. Fourier (Grenoble)*, 31(3), (1981), 99-114.

[9] M. Barnsley, J. Geronimo, A. Harrington, Condensed Julia sets with an application to a fractal lattice model hamiltonian, *Trans. Amer. Math. Soc.*, 288(2), (1985), 537-561.

[10] K. Case, J. Geronimo, Scattering theory and orthogonal polynomials on the real line, *Trans. Amer. Math. Soc.*, 258(2), (1980), 467-494.

[11] P. Chebyshev, *Ouvres*, Chelsea Publishing Company, New York (two volumes), 1961.

[12] T. Chihara, Chain sequences and orthogonal polynomials, *Trans. Amer. Math. Soc.*, 104(1), (1962), 1-16.

[13] T. Chihara, Hamburger moment problems and orthogonal polynomials, *Trans. Amer. Math. Soc.*, 315(1), (1989), 189-203.

[14] T. Chihara, Indeterminate symmetric moment problems, *J. Math. Anal. Appl.*, 85, (1982) 331-346.

[15] T. Chihara, *Introduction to Orthogonal Polynomials*, Gordon and Breach, 1978.

[16] T. Chihara, On co-recursive orthogonal polynomials, *Proc. Amer. Math. Soc.*, 8(5), (1957), 899-905.

[17] T. Chihara, Non-linear recurrence relations for classical orthogonal polynomials, *Amer. Math. Monthly*, 65(3), (1958), 195-197.

[18] T. Chihara, The parameters of a chain sequence, *Proc. Amer. Math. Soc.*, 108(3), (1990), 775-780.

[19] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, American Mathematical Society, Providence, Rhode Island, 1999 (Reprint 2000).

[20] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge, 1997.

[21] J. Geronimo, A relation between the coefficients in the recurrence formula and the spectral function for orthogonal polynomials, *Trans. Amer. Math. Soc.*, 260(1), (1980) 65-82.

[22] J. Geronimus, *Orthogonal Polynomials: Estimates, Asymptotic Formulae and Series of Polynomials Orthogonal on the Unit Circle and on an Interval*, Consultants Bureau, New York, 1961.

[23] J. Geronimus, On the trigonometric moment problem, *Ann. of Math. (2)*, 47(4), (1946) 742-761.

[24] F. Gesztesy, B. Simon, m-Functions and inverse speactral analysis for finite and semi-infinite Jacobi matrices, *J. Anal. Math.*, 73 (1997) 267-297.

[25] R. Gilman, A class of functions continuous but not absolutely continuous, *Ann. of Math. (2)*, 33(3), (1932), 433-442.

[26] H. Hamburger, Ueber eine erweiterung des Stieltjes'schen momentenproblems, *Math. Ann.*, 81, (1920), 235-319, 82, (1921), 120-164, 168-187.

[27] J. Hanna, J. Rowland, *Fourier Series, Transforms, and Boundary Value Problems*, Dover, 2008.

[28] F. Hausdorff, Momentprobleme für ein endliches intervall, *Math. Z.*, 16, (1923), 220-248.

[29] H. Horst, Riemann-Stieltjes and Lebesgue-Stieltjes integrability, *Amer. Math. Monthly*, 91(9), (1984), 551-559.

[30] M. Ismail, G. Rahman, The q-Laguerre polynomials and related moment problems, *J. Math. Anal. Appl.*, 218, (1998) 155-174.

[31] R. Kannan, C. Krueger, *Advanced Analysis on the Real Line*, Springer, New York, 1996.

[32] S. Karlin, J. McGregor, The classification of birth and death processes, *Trans. Amer. Math. Soc.*, 86(2) (1957) 366-400.

[33] S. Karlin, J. McGregor, The differential equations of birth and death processes and the Stieltjes moment problem, *Trans. Amer. Math. Soc.*, 85(2), (1957) 489-546.

[34] S. Khrushchev, Continued fractions and orthogonal polynomials on the unit circle, *J. Comput. Appl. Math.*, 178, (2005) 267-303.

[35] S. Khrushchev, *Orthogonal Polynomials and Continued Fractions*, Cambridge University Press, Cambridge, 2008.

[36] S. Khrushchev, Schur's algorithm, orthogonal polynomials and convergence of Wall's continued fractions in $L^2(\mathbb{T})$, *J. Approx. Theory*, 108, (2001), 161-248.

[37] D. Lubinsky, Singularly continuous measures in Nevai's class M, *Proc. Amer. Math. Soc.*, 111(2), (1991), 413-420.

[38] D. Moak, The q-analogue of the Lageurre polynomials, *J. Math. Anal. Appl.* 81, (1981) 20-47.

[39] F. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1976.

[40] J. Oxtoby, *Measure and Category*, Springer-Verlag, 1971.

[41] G. Pólya, G. Szegö, *Problems and Theorems in Analysis*, volume 2, Berlin, 1976.

[42] E. D. Rainville, *Special Functions*, MacMillan, 1960.

[43] E. D. Rainville, *Intermediate Differential Equations*, MacMillan, 1964.

[44] F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Company, New York, 1955 (Dover reprint 1990).

[45] J. Shohat, J. Tamarkin *The Problem of Moments*, American Mathematical Society, Providence, Rhode Island, 1943.

[46] G. Shilov, B. Gurevich, *Integral Measure and Derivative*, Prentice-Hall, 1966, Englewood Cliffs, New Jersey.

[47] B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math.*, 137, (1998), 82-203.

[48] T. Stieltjes, Reserches sur les fractions continues, *Anns. Fac. Sci. Univ. Toulouse*, 8, (1894), J1-J122, 9, (1895), A5-A47.

[49] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, 2003 (reprint).

[50] N. Temme, *Special Functions*, John Wiley and Sons, 1996.

[51] D. Varberg, On absolutely continuous functions, *Amer. Math. Monthly*, 72(8) (1965), 831-841.

[52] H. Wall, *Continued Fractions*, D. Van Nostrand Company, 1948.

[53] H. Wall, Continued fractions and totally monotone sequences, *Trans. Amer. Math. Soc.*, 48(2), (1940), 165-184.

[54] H. Wall, Contributions to the analytic theory of J-fractions, *Trans. Amer. Math. Soc.*, 55(3), (1944), 373-392.

[55] E. Whitakker, G. Watson, *Modern Analysis*, Merchant Books, 2008 (reprint of 1915 2nd edition).