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On a Generalization of the Classical Moment Problem

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Let $\mathscr{M}^*(\mathbb{R}^n)$, $n \in \mathbb{N}$, denote the set of all positive Borel measures on $\mathbb{R}^n$ having moments of all orders. We study the following generalization of the classical moment problem: Given a multisequence $\{S_\alpha = (s_{ij}(\alpha));\ \alpha \in \mathbb{N}_0^n\}$ of (k, k) matrices with complex entries $s_{ij}(\alpha)$, when does there exist a nonnegative (k, k) matrix $\Lambda = (\lambda_{ij})$ of complex Borel measures λ_{ij} on $\mathbb{R}^n$ such that $|\lambda_{ij}| \in \mathscr{M}^*(\mathbb{R}^n)$ and $s_{ij}(\alpha) = \int x^\alpha \, d\lambda_{ij}(x)$ for all $\alpha \in \mathbb{N}_0^n$ and $i, j = 1,..., k$? © 1987 Academic Press, Inc.

INTRODUCTION

The problem formulated above will be called the k-moment problem. Moment problems of this kind or of a more general type in case $n = 1$ occur (for instance) in [12, 9, and 8]. Obviously, in case $k = 1$ the problem reduces to the classical n-dimensional moment problem.

Section 1 contains some preliminaries and some basic definitions needed in the sequel. In Section 2 we give necessary and sufficient conditions for the existence of a solution. In Section 3 we define and discuss two concepts of determinacy for the k-moment problem.

Notation. $\mathbb{N}_0$ are the nonnegative integers. For $\alpha = (\alpha_1,..., \alpha_n) \in N_0^n$ and $x = (x_1,..., x_n) \in \mathbb{R}^n$, we let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $x_j^0 := 1$. $\mathscr{B}_n$ are the Borel sets on $\mathbb{R}^n$. Let δ_x denote the unit mass concentrated at x.

The inner product of $\mathbb{C}^k$ is denoted by $(\cdot, \cdot)$. Let $e_j = (\delta_{ij})$, $j = 1,..., k$, be the standard basis of $\mathbb{C}^k$. δ_{ij} is the Kronecker symbol. We shall identify the vector space $M(k, \mathbb{C})$ of all (k, k) matrices with complex entries and the vector space $L(\mathbb{C}^k)$ of all endomorphisms of $\mathbb{C}^k$ via the basis e_j, $j = 1,..., k$. Let $\mathscr{P}_n$ be the vector space of all polynomials in $x_1,..., x_n$ with complex coefficients, considered as functions from $\mathbb{R}^n$ into $\mathbb{C}$. We denote by $\mathscr{F}_n$ the vector space of all Borel functions f on $\mathbb{R}^n$ which grow at most like polynomials (i.e., there exists a $p_f \in \mathscr{P}_n$ such that $|f(x)| \leqslant P_f(x)$ for $x \in \mathbb{R}^n$). $\mathscr{P}_n \otimes M_k$ and $\mathscr{F}_n \otimes M_k$ are the vector spaces of al (k, k) matrices with

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entries in $\mathscr{P}_n$ and $\mathscr{F}_n$, respectively. We denote by $\mathscr{P}_n^+$, $\mathscr{F}_n^+$, $(\mathscr{P}_n \otimes M_k)^+$ and $(\mathscr{F}_n \otimes M_k)^+$ the sets of all $p \in \mathscr{P}_n$, $f \in \mathscr{F}_n$, $(p_{ij}) \in \mathscr{P}_n \otimes M_k$ and $(f_{ij}) \in \mathscr{F}_n \otimes M_k$, respectively, which are nonnegative for all $x \in \mathbb{R}^n$.

We refer to [1, 10] for the notation and the results concerning the classical moment problem we use.

1. THE k-MOMENT PROBLEM

(1.1)

Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. We denote by $\mathscr{M}_k^*(\mathbb{R}^n)$ the set of all (k, k) matrices $\Lambda = (\lambda_{ij})$ of complex Borel measures λ_{ij} on $\mathbb{R}^n$ such that:

(a) $\lambda(\mathfrak{M}) = (\lambda_{ij}(\mathfrak{M}))$ is a nonnegative matrix for each $\mathfrak{M} \in \mathscr{B}_n$.

(b) $|\lambda_{ij}| \in \mathscr{M}^*(\mathbb{R}^n)$ for all $i, j = 1,..., k$.

We denote by $\lambda_\Lambda = \sum_{j=1}^k \lambda_{jj}$ the tracial measure of $\Lambda = (\lambda_{ij})$. Obviously, (a) implies that λ_{jj}, $j = 1,..., k$, and λ_Λ are positive Borel measures. Moreover, if (a) is true, then (b) is equivalent to $\lambda_\Lambda \in \mathscr{M}^*(\mathbb{R}^n)$. This is an immediate consequence of Lemma 1.1. For later use we collect some simple and well-known properties (see [2], V.2) of $\Lambda = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ in

LEMMA 1.1. *Let $\Lambda = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ and let $m, r \in \{1,..., k\}$.*

(i) *λ_{mr} is absolutely continuous with respect to Λ_{mm}, λ_{rr}, and λ_Λ. Let g_{mr} denote the Radon–Nikodym derivative of λ_{mr} with respect to λ_Λ.*

(ii) *(g_{ij}) is a nonnegative (k, k) matrix λ_Λ-a.e. and $\sum_{j=1}^k g_{jj} = 1$ λ_Λ-a.e..*

(iii) *$|g_{mr}|^2 \leqslant g_{mm} g_{rr} \leqslant g_{rr} \leqslant 1$ λ_Λ-a.e..*

(iv) *$\operatorname{supp} \lambda_{mr} \subseteq \operatorname{supp} \lambda_{mm} \cap \operatorname{supp} \lambda_{rr}$.*

Proof. (i): Since $(\lambda_{ij}(\mathfrak{M}))$ is a nonnegative for each $\mathfrak{M} \in \mathscr{B}_n$, we have $|\lambda_{mr}(\mathfrak{M})| \leqslant \lambda_{mm}(\mathfrak{M})^{1/2} \lambda_{rr}(\mathfrak{M})^{1/2} \leqslant \lambda_\Lambda(\mathfrak{M})$. The absolute continuity follows.

(ii): From (a) we obtain

$$\sum_{i,j=1}^{k} \lambda_{ij}(\mathfrak{M})\, t_i \bar{t}_j = \int_{\mathfrak{M}} \sum_{i,j=1}^{k} g_{ij} t_i \bar{t}_j \, d\lambda_\Lambda \geqslant 0 \quad \text{for all}$$

$t = (t_1,..., t_k) \in \mathbb{C}^k$ and $\mathfrak{M} \in \mathscr{B}_n$. Taking t from a countable dense subset of $\mathbb{C}^k$ and using that $\mathfrak{M} \in \mathscr{B}_n$ is arbitrary, a simple measure-theoretic argument shows that (g_{ij}) is nonnegative λ_Λ-*a.e.*.

The other assertions follow immediately.

Some converse of the preceding is given in Lemma 1.2. We omit the easy proof.

LEMMA 1.2. *Suppose $\lambda \in \mathcal{M}^*(\mathbb{R}^n)$. Let g_{ij}, $i, j = 1,..., k$, be λ-almost everywhere defined measurable functions on $\mathbb{R}^n$. Suppose that the matrix (g_{ij}) is nonnegative λ-a.e. and $\sum_{j=1}^k g_{jj} = 1$ λ-a.e. on $\mathbb{R}^n$. We define Borel measures λ_{ij} on $\mathbb{R}^n$ by $d\lambda_{ij} = g_{ij}\, d\lambda$ for $i, j = 1,..., k$. Then $\Lambda := (\lambda_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$ and $\lambda_\Lambda = \lambda$.*

(1.2)

A multisequence $\{S_\alpha = (s_{ij}(\alpha)); \alpha \in \mathbb{N}_0^n\}$ of matrices $S_\alpha \in M(k, \mathbb{C})$ is called a *k-moment sequence* if there exists a $\Lambda = (\lambda_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$ such that $s_{ij}(\alpha) = \int x^\alpha\, d\lambda_{ij}(x)$ for all $\alpha \in \mathbb{N}_0^n$ and $i, j = 1,..., k$. (All integrals in this paper are over $\mathbb{R}^n$.) The latter can also be written as $S_\alpha = \int x^\alpha\, d\Lambda(x)$ for $\alpha \in \mathbb{N}_0^n$. In this case $\Lambda = (\lambda_{ij})$ is called a *representing matrix of measures* for $\{S_\alpha; \alpha \in \mathbb{N}_0^n\}$.

For $\Lambda = (\lambda_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$, we denote by V_Λ the set of all $\tilde{\Lambda} = (\tilde{\lambda}_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$ which represents the same sequence of matrices as Λ, that is, $\int x^\alpha\, d\lambda_{ij}(x) = \int x^\alpha\, d\tilde{\lambda}_{ij}(x)$ for $\alpha \in \mathbb{N}_0^n$ and $i, j = 1,..., k$.

Similarly as in the theory of the classical moment problem, it is convenient to replace $\{S_\alpha; \alpha \in \mathbb{N}_0^n\}$ by the associated linear mapping $\mathcal{S}$ from $\mathcal{P}_n$ into $M(k, \mathbb{C})$. $\mathcal{S}(p) \equiv (\mathfrak{s}_{ij}(p))$ is the (k, k) matrix defined by $\mathcal{S}(p) = \sum a_\alpha S_\alpha$ for $p(x) = \sum a_\alpha x^\alpha \in \mathcal{P}_n$. As already mentioned, we want to identify $M(k, \mathbb{C})$ and $L(\mathbb{C}^k)$.

Let $k \in \mathbb{N}$ and let $\mathcal{S}$ be a linear mapping from $\mathcal{P}_n$ into $L(\mathbb{C}^k)$. $\mathcal{S}$ is called *k-positive* if $\sum_{i,j=1}^k (\mathcal{S}(p_{ij})\, c_i, c_j) \geq 0$ for all vectors $c_1,..., c_k \in \mathbb{C}^k$ and all matrices $(p_{ij}) \in (\mathcal{P}_n \otimes M_k)^+$. $\mathcal{S}$ is called *positive* if $\sum_{i,j=1}^k (\mathcal{S}(p_i\overline{p_j})\, c_i, c_j) \geq 0$ for all vectors $c_1,..., c_k \in \mathbb{C}^k$ and all $p_1,..., p_k \in \mathcal{P}_n$.

2. EXISTENCE OF A SOLUTION

(2.1)

PROPOSITION 2.1. *Suppose that $\{S_\alpha = (s_{ij}(\alpha)); \alpha \in \mathbb{N}_0^n\}$ is a multisequence of matrices $S_\alpha \in M(k, \mathbb{C})$. Let $\mathcal{S} = (\mathfrak{s}_{ij})$ be the associated linear mapping of $\mathcal{P}_n$ into $L(\mathbb{C}^k)$. The following statements are equivalent*:

(i) $\{S_\alpha, \alpha \in \mathbb{N}_0^n\}$ *is a k-moment sequence.*

(ii) $\mathcal{S}$ *is k-positive.*

(iii) $\sum_{i,j=1}^k \mathfrak{s}_{ij}(p_{ij}) \geq 0$ *for all* $(p_{ij}) \in (\mathcal{P}_n \otimes M_k)^+$.

In the proof of Proposition 2.1 we need

LEMMA 2.2. *Each matrix $(p_{ij}) \in (\mathcal{P}_n \otimes M_k)^+$ is a finite sum of (k, k) matrices of the form $(f_i\overline{f_j})$, where $f_j \in \mathcal{F}_n$ for $j = 1,..., k$.*

Proof. Fix $x \in \mathbb{R}^n$. Let $h_r(x)$, $r=1,\ldots,k$, denote the eigenvalues of the nonnegative (k,k) matrix $(p_{ij}(x))$. By the finite dimensional version of the spectral theorem, there is a unitary (k,k) matrix $(u_{ij}(x))$ such that $p_{ij}(x)=\sum_{r=1}^{k} h_r(x)^{1/2} u_{ri}(x) \cdot \overline{h_r(x)^{1/2} u_{rj}(x)}$. It is easily seen that the functions $h_i^{1/2} u_{ij}$ are in $\mathscr{F}_n$ for $i,j=1,\ldots,k$.

Proof of Proposition 2.1. (i) $\rightarrow$ (ii): Assume that there is a $\Lambda=(\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ such that $s_{ij}(\alpha)=\int x^\alpha \, d\lambda_{ij}(x)$ for $\alpha \in \mathbb{N}_0^n$ and $i,j=1,\ldots,k$. Let $c_j=(c_{j1},\ldots c_{jk}) \in \mathbb{C}^k$ for $j=1,\ldots,k$. Suppose $(p_{ij}) \in (\mathscr{P}_n \otimes M_k)^+$. As in Lemma 1.1, we let g_{ij} be the Radon–Nikodym derivative $d\lambda_{ij}/d\lambda_\Lambda$ for $i,j=1,\ldots,k$. Then

$$\sum_{i,j=1}^{k} (\mathscr{S}(p_{ij})\, c_i, c_j)$$

$$= \sum_{i,j,m,r=1}^{k} \mathscr{s}_{mr}(p_{ij})\, c_{ir} \overline{c_{jm}}$$

$$= \sum_{i,j=1}^{k} \int \sum_{m,r=1}^{k} p_{ij}(x)\, c_{ir} \overline{c_{jm}}\, g_{mr}(x)\, d\lambda_\Lambda.$$

By Lemma 2.2, it suffices to replace (p_{ij}) by a (k,k) matrix $(f_i \overline{f_j})$, where $f_1,\ldots,f_k \in \mathscr{F}_n$. In this case the above expression is clearly nonnegative, because (g_{mr}) is a nonnegative matrix λ_Λ-a.e. by Lemma 1.1, (ii).

(ii) $\rightarrow$ (ii)$'$ is obvious, since

$$(\mathscr{S}(\cdot)\, e_j, e_i) = \mathscr{s}_{ij}(\cdot).$$

(ii$'$) $\rightarrow$ (i):

Suppose that the above condition is fulfilled. Then, $F((p_{ij})) := \sum_{i,j=1}^{k} \mathscr{s}_{ij}(p_{ij})$ defines a positive linear functional on $\mathscr{P}_n \otimes M_k$. Since $(\mathscr{P}_n \otimes M_k)^+$ is cofinal in $(\mathscr{F}_n \otimes M_k)^+$, F can be extended to a positive linear functional on $\mathscr{F}_n \otimes M_k$ which will be again denoted by F. Put $\mathscr{s}_{mr}(f) := F((f\delta_{im}\delta_{jr}))$ for $f \in \mathscr{F}_n$ and $m, r=1,\ldots,k$.

We now argue as in the proof of Proposition 4 in [11].

Since the (k,k) matrix $(f_i \overline{f_j})$ is in $(\mathscr{F}_n \otimes M_k)^+$ for $f \in \mathscr{F}_n^+$, $G(f) := F((f\delta_{ij})) = \sum_{j=1}^{k} \mathscr{s}_{ij}(f)$ is a positive linear functional on $\mathscr{F}_n$. Hence there is a measure $\lambda \in \mathscr{M}^*(\mathbb{R}^n)$ such that $G(f)=\int f(x)\, d\lambda(x)$ for all $f \in \mathscr{F}_n$ (see, e.g. [4], Chap. 8). Take $f \in \mathscr{F}_n^+$. For $t_1,\ldots,t_k \in \mathbb{C}$, the (k,k) matrix $(f\, t_i \overline{t_j})$ is in $(\mathscr{F}_n \otimes M_k)$ and hence

$$F((f\, t_i \overline{t_j})) = \sum_{i,j=1}^{k} \mathscr{s}_{ij}(f)\, t_i \overline{t_j} \geqslant 0. \qquad (*)$$

Consequently, for $i, j = 1, \ldots, k$,

$$|\mathfrak{s}_{ij}(f)| \leqslant \mathfrak{s}_{ii}(f)^{1/2} \mathfrak{s}_{jj}(f)^{1/2} \leqslant \mathfrak{s}_{ii}(f) + \mathfrak{s}_{jj}(f) \leqslant \int f \, d\lambda.$$

Writing $f \in \mathscr{F}_n$ as $f = (f_1 - f_2) + i(f_3 - f_4)$ with $f_1, \ldots, f_4 \in \mathscr{F}_n^+$, we get $|\mathfrak{s}_{ij}(f)| \leqslant 2 \int |f(x)| \, d\lambda(x)$ for all $f \in \mathscr{F}_n$. This implies the existence of a function $g_{ij} \in L^\infty(\lambda)$ such that $\mathfrak{s}_{ij}(f) = \int f g_{ij} \, d\lambda$ for $f \in \mathscr{F}_n$ and $i, j = 1, \ldots, k$.

If $f \in \mathscr{F}_n^+$, then (*) yields $\int (\sum_{i,j=1}^k g_{ij} t_i \bar{t}_j) f \, d\lambda \geqslant 0$ for arbitrary $t_1, \ldots, t_k \in \mathbb{C}$. A routine measure-theoretic argument (as in the proof of Lemma 1.1, (ii)) shows that (g_{ij}) is nonnegative λ-a.e. on $\mathbb{R}^n$. Defining λ_{ij} by $\lambda_{ij} = g_{ij} \, d\lambda$ for $i, j = 1, \ldots, k$, we have $\Lambda := (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$. By construction, $S_\alpha = \int x^\alpha \, d\Lambda(x)$ for all $\alpha \in \mathbb{N}_0^n$ which completes the proof.

(2.2)

In case $n = 1$ the conditions in Proposition 2.1 can be weakened.

PROPOSITION 2.3. *Suppose that $n = 1$. Let $\{S_\alpha;\ \alpha \in \mathbb{N}_0\}$ and $\mathscr{S} = (\mathfrak{s}_{ij})$ be an in Proposition* 2.1. *The following statements are equivalent*:

(i) $\{S_\alpha;\ \alpha \in \mathbb{N}\}$ *is a k-moment sequence.*

(ii) $\mathscr{S}$ *is positive.*

(iii) $\sum_{i,j=1}^k \mathfrak{s}_{ij}(p_i \bar{p}_j) \geqslant 0$ *for all* $p_1, \ldots, p_k \in \mathscr{P}_n$.

Proof. Since $(p_i \bar{p}_j) \in (\mathscr{P}_n \otimes M_k)^+$ for arbitrary $p_1, \ldots, p_k \in \mathscr{P}_n$, we obviously have that (ii) → (iii). Combined with Proposition 2.1. this yields (i) → (iii). From

$$(\mathscr{S}(\cdot)\, e_j,\ e_i) = \mathfrak{s}_{ij}(\cdot)$$

we see that (iii) → (iii)′.

To prove that (iii)′ → (i), it is sufficient to show that (iii)′ → (ii)′ because of Proposition 2.1. Let (p_{ij}) be a (k, k) matrix from $(\mathscr{P}_n \otimes M_k)^+$. Then there exists a (k, k) matrix $(q_{ij}) \in \mathscr{P}_n \otimes M_k$ such that $(p_{ij}) = (q_{ij})(q_{ij})^*$ (see, e.g. [5]), that is, $p_{ij} = \sum_{r=1}^k q_{ir} \overline{q_{jr}}$ for $i, j = 1, \ldots, k$.

Therefore, the matrix (p_{ij}) is a finite sum of matrices of the form $(p_i \bar{p}_j)$, where $p_1, \ldots, p_k \in \mathscr{P}_n$. Thus (iii)′ → (ii)′ and the proof is complete.

Remarks. (1) The proof of Proposition 2.1 gives the following generalization of Haviland's theorem as well. Let $\mathfrak{K}$ be a closed subset of $\mathbb{R}^n$ and let $(\mathscr{P}_n \otimes M_k)_{\mathfrak{K}}^+$ denote the set of all $(p_{ij}) \in \mathscr{P}_n \otimes M_k$ which are nonnegative for each $x \in \mathfrak{K}$. If $\{S_\alpha; \alpha \in \mathbb{N}_0^n\}$ and $\mathscr{S} = (\mathfrak{s}_{ij})$ are as in Proposition 2.1, then the following are equivalent:

(i) There exists a representing matrix $\Lambda \in \mathscr{M}_k^*(\mathbb{R}^n)$ for $\{S_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ such that supp $\lambda_\Lambda \subseteq \mathfrak{K}$.

(ii) $\sum_{i,j=1}^{k} (\mathscr{S}(p_{ij})\, c_i, c_j) \geq 0$ for all vectors $c_i, \ldots, c_k \in \mathbb{C}^n$ and every $(p_{ij}) \in (\mathscr{P}_n \otimes M_k)_{\mathfrak{K}}^{+}$.

(iii) $\sum_{i,j=1}^{k} s_{ij}(p_{ij}) \geq 0$ for every $(p_{ij}) \in (\mathscr{P}_n \otimes M_k)_{\mathfrak{K}}^{+}$.

(In the above proof of (ii)′ → (i) we use Haviland's theorem ([7]) to infer that supp $\lambda_A \subseteq \mathfrak{K}$.)

(2) It is clear the k-moment problem could be reformulated and studied in terms of operator theory (that is, simultaneous spectral resolution of commuting self-adjoint operators in dilation spaces). We do not consider these aspects in the present paper.

3. Determinacy

(3.1)

Definition 3.1. Let $\{S_\alpha = (s_{ij}(\alpha));\ \alpha \in \mathbb{N}_0^n\}$ be a k-moment sequence and let $A = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ be a representing matrix of measures for $\{S_\alpha;\ \alpha \in \mathbb{N}_0^n\}$. Let $m, r \in \{1, \ldots, k\}$. We say that $s_{mr}(\cdot)$ (or the measure λ_{mr}) is k-determinate if $\lambda_{mr} = \tilde{\lambda}_{mr}$ for each $\tilde{A} = (\tilde{\lambda}_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ with $\tilde{A} \in V_A$. $\{S_\alpha;\ \alpha \in \mathbb{N}_0^n\}$ (or A) is called k-determinate if V_A is a singleton.

It is clear from the above definition that a k-moment sequence $\{S_\alpha = (s_{ij}(\alpha));\ \alpha \in \mathbb{N}_0^n\}$ (or $A = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$) is k-determinate if $s_{ij}(\cdot)$ (or λ_{ij}) is k-determinate for all $i, j = 1, \ldots, k$.

Proposition 3.1. *Suppose that $A = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$. If λ_A is determinate, then A is k-determinate.*

Proof. Let $\tilde{A} = (\tilde{\lambda}_{ij}) \in V_A$. That is, we have by definition $\int p(x)\, d\lambda_{ij}(x) = \int p(x)\, d\tilde{\lambda}_{ij}(x)$ for all $p \in \mathscr{P}_n$ and $i, j = 1, \ldots, k$. In particular, $\int p(x)\, d\lambda_A(x) = \int p(x)\, d\lambda_{\tilde{A}}(x)$ for $p \in \mathscr{P}_n$. Since λ_A is assumed to be determinate, this implies $\lambda_A = \lambda_{\tilde{A}}$.

Fix $m, r \in \{1, \ldots, k\}$. As in Lemma 1.1, we denote by g_{mr} and $\tilde{g}_{mr}$ the Radon–Nikodym derivatives $d\lambda_{mr}/d\lambda_A$ and $d\tilde{\lambda}_{mr}/d\lambda_{\tilde{A}}$, respectively. From $\tilde{A} \in V_A$ and $\lambda_A = \lambda_{\tilde{A}}$ we conclude that $\int p(x)(g_{mr}(x) - \tilde{g}_{mr}(x))\, d\lambda_A = 0$ for all polynomials $p \in \mathscr{P}_n$. Since λ_A is determinate, $\mathscr{P}_n$ is dense in $L^1(\lambda_A)$ [1, p. 47]. (Though the result is stated in [1] only in case $n = 1$, the proof applies to any dimension $n \in \mathbb{N}$.) By Lemma 1.1, (iii), $g_{mr} - \tilde{g}_{mr} \in L^\infty(\lambda_A)$. Therefore, we get $g_{mr} - \tilde{g}_{mr} = 0$ λ_A-a.e. on $\mathbb{R}^n$. This clearly implies $\lambda_{mr} = \tilde{\lambda}_{mr}$, i.e., λ_{mr} is k-determinate. This completes the proof.

The converse of Proposition 3.1 is not valid. The following examples shows that $A = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ can be k-determinate if λ_A is indeterminate.

EXAMPLE 3.1. Let λ_{11} and λ_{22} be determinate measures from $\mathcal{M}^*(\mathbb{R}^n)$ with disjoint supports such that $\lambda := \lambda_{11} + \lambda_{22}$ is indeterminate. (The simplest way to fulfill these assumptions is as follows. Let $n = 1$. Take an indeterminate $\mathcal{N}$-extremal measure $\lambda = \sum_{n=1}^{\infty} a_n \delta_{x_n}$ on $\mathbb{R}$ and set $\lambda_{11} = a_1 \delta_{x_1}$ and $\lambda_{22} = \lambda - \lambda_{11}$. By [1, p. 115], or [3, Theorem 7], λ_{11} and λ_{22} are determinate.) Put $\lambda_{12} = \lambda_{21} = 0$. Then $\Lambda := (\lambda_{ij}) \in \mathcal{M}^*(\mathbb{R}^n)$ is 2-determinate, but $\lambda_\Lambda = \lambda$ is indeterminate. λ_{11} and λ_{22} are obviously 2-determinate. For $\lambda_{12} \cap \operatorname{supp} \lambda_{22} = \varnothing$ and Lemma 1.1, (iv).

(3.2)

Now we discuss the relation between the k-determinacy of $\Lambda = (\lambda_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$ and the determinacy of $\lambda_{11}, ..., \lambda_{kk}$.

PROPOSITION 3.2. *Suppose $\Lambda = (\lambda_{ij}) \in \mathcal{M}_k^*(\mathbb{R}^n)$. Let $r \in \{1, ..., k\}$. Suppose that λ_{rr} is determinate. In case $n \geqslant 2$ we assume in addition that $\mathcal{P}_n$ is dense in $L^2(\lambda_{rr})$. Then λ_{mr} and λ_{rm} are k-determinate for each $m \in \{1, ..., k\}$.*

If these assumptions are satisfied for all $r = 1, ..., k$, then Λ is k-determinate.

Proof. It clearly suffices to prove the first part. Fix $m \in \{1, ..., k\}$.

By Lemma 1.1, there are functions $g_{ij} \in L^\infty(\lambda_\Lambda)$ such that $d\lambda_{ij} = g_{ij}\, d\lambda_\Lambda$ for $i, j = 1, .., k$. Moreover, by Lemma 1.1, $|g_{mr}|^2 \leqslant g_{rr} g_{mm} \leqslant g_{rr}$ λ_Λ-a.e. on $\mathbb{R}^n$. The function $g_{mr} g_{rr}^{-1}$ is measurable and λ_{rr}-almost everywhere defined. Since $\int |g_{mr}|^2 g_{rr}^{-2}\, d\lambda_{rr} = \int |g_{mr}|^2 g_{rr}^{-1}\, d\lambda_\Lambda \leqslant d\lambda_\Lambda < \infty$ because of $\lambda_\Lambda \in \mathcal{M}^*(\mathbb{R}^n)$, we have $g_{mr} g_{rr}^{-1} \in L^2(\lambda_{rr})$.

Now let $\tilde{\Lambda} = (\tilde{\lambda}_{ij}) \in V_\Lambda$. Let $\tilde{g}_{ij}$ be the corresponding Radon–Nikodym derivatives $d\tilde{\lambda}_{ij}/d\lambda_{\tilde{\Lambda}}$, $i, j = 1, ..., k$. Since λ_{rr} is determinate, $\lambda_{rr} = \tilde{\lambda}_{rr}$. By the preceding we have $g_{mr} g_{rr}^{-1} - \tilde{g}_{mr} \tilde{g}_{rr}^{-1} \in L^2(\lambda_{rr}) \equiv L^2(\tilde{\lambda}_{rr})$. From $\tilde{\Lambda} \in V_\Lambda$ we get

$$\begin{aligned}\int p(x)\, d\lambda_{mr}(x) &= \int p(x)\, g_{mr}(x)\, d\lambda_\Lambda \\ &= \int p(x)\, g_{mr}(x)\, g_{rr}(x)^{-1}\, d\lambda_{rr}(x) \\ &= \int p(x)\, \tilde{g}_{mr}(x)\, \tilde{g}_{rr}(x)^{-1}\, d\tilde{\lambda}_{rr}(x) \\ &= \int p(x)\, d\tilde{\lambda}_{mr}(x)\end{aligned}$$

for $p \in \mathcal{P}_n$. Since $\lambda_{rr} = \tilde{\lambda}_{rr}$, this shows that $g_{mr} g_{rr}^{-1} - \tilde{g}_{mr} \tilde{g}_{rr}^{-1} =: h_{mr}$ is orthogonal to $\mathcal{P}_n$ in $L^2(\lambda_{rr})$. If $n = 1$, then $\mathcal{P}_n$ is dense in $L^2(\lambda_{rr})$, since λ_{rr} is determinate [1]. In case $n \geqslant 2$ this is true by assumption. Hence $h_{mr} = 0$ in

$L^2(\lambda_{rr})$, i.e., $g_{mr}g_{rr}^{-1}=\tilde{g}_{mr}\tilde{g}_{rr}^{-1}$ λ_{rr}-a.e. on $\mathbb{R}^n$. Therefore, $d\lambda_{mr}=g_{mr}\,d\lambda_\Lambda=g_{mr}g_{rr}^{-1}\,d\lambda_{rr}=\tilde{g}_{mr}\tilde{g}_{rr}^{-1}\,d\tilde{\lambda}_{rr}=d\tilde{\lambda}_{rr}$ and $\lambda_{mr}=\tilde{\lambda}_{mr}$. Similarly, $\lambda_{rm}=\tilde{\lambda}_{rm}$.

Remarks. (1) It seems to be unknown in case $n\geqslant 2$ whether the polynomials are dense in L^2 if the measure is determinate (see also [6]). I conjecture that this is not true.

(2) For $n=1$ Proposition 3.2 shows that $\Lambda=(\lambda_{ij})\in\mathcal{M}_k^*(\mathbb{R}^n)$ is k-determinate provided all diagonal measures λ_{rr}, $r=1,..,k$, are determinate. Again the converse is not valid as the next example shows.

EXAMPLE 3.2. Let $n=1$ and $k=2$. Let $\{s_n; n\in\mathbb{N}_0\}$ be an arbitrary indeterminate moment sequence of real numbers and let $\rho(z)$, $z\in\mathbb{C}$, be the corresponding function as defined in [10], p. 42. Since $\{s_n\}$ is indeterminate, $\rho(z)>0$ for all $z\in\mathbb{C}$ [10, Corollary 2.7, p. 50]. Fix a point $x_0\in\mathbb{R}$. There exists one and only one solution $\lambda_{11}\in\mathcal{M}^*(\mathbb{R})$ of the moment problem $\{s_n;\ n\in\mathbb{N}_0\}$ which has the mass $\rho(x_0)$ at x_0 [10], Corollary 2.4, p. 44]. Define $\lambda_{12}=\lambda_{21}=\lambda_{22}=\rho(x_0)\,\delta_{x_0}$. Clearly, $\Lambda=(\lambda_{ij})\in\mathcal{M}_2^*(\mathbb{R})$.

Recall that λ_{11} is indeterminate. We now check that Λ is 2-determinate. Take a $\tilde{\Lambda}=(\tilde{\lambda}_{ij})\in V_\Lambda$. Since $n=1$ and λ_{22} is determinate, Proposition 3.2 yields $\lambda_{12}=\tilde{\lambda}_{12}$, $\lambda_{21}=\tilde{\lambda}_{21}$ and $\lambda_{22}=\tilde{\lambda}_{22}$. It remains to prove that $\lambda_{11}=\tilde{\lambda}_{11}$. The measures $\tilde{\lambda}_{12},\tilde{\lambda}_{21}$ and $\tilde{\lambda}_{22}$ have the mass $\rho(x_0)>0$ at x_0. Since $\tilde{\Lambda}\in\mathcal{M}_2^*(\mathbb{R})$, this implies that $\tilde{\lambda}_{11}(\{x_0\})\geqslant\rho(x_0)$. Since $\tilde{\Lambda}\in V_\Lambda$, $\tilde{\lambda}_{11}$ is a solution of the moment problem $\{s_n;\ n\in\mathbb{N}_0\}$ as well. Since $\rho(x_0)$ is the largest mass concentrated at x_0 for all solutions of this moment problem [10, Corollary 2.4], $\tilde{\lambda}_{11}(\{x_0\})=\rho(x_0)$. By the uniqueness part of Corollary 2.4 in [10], $\lambda_{11}=\tilde{\lambda}_{11}$.

(3.3)

There is another concept of determinacy which might be useful.

DEFINITION 3.2. Suppose that $\{S_\alpha=(s_{ij}(\alpha));\ \alpha\in\mathbb{N}_0^n\}$ is a k-moment sequence and $\Lambda=(\lambda_{ij})\in\mathcal{M}_k^*(\mathbb{R}^n)$ is a representing matrix of measures for $\{S_\alpha;\ \alpha\in\mathbb{N}_0^n\}$. Let $m,r\in\{1,...,k\}$. We say that $s_{mr}(\cdot)$ or λ_{mr} is separately k-determinate (with respect to Λ) if $\lambda_{mr}=\tilde{\lambda}_{mr}$ for each $\tilde{\Lambda}=(\tilde{\lambda}_{ij})\in V_\Lambda$ satisfying $\lambda_{ij}=\tilde{\lambda}_{ij}$ for all i,j such that $(i,j)\neq(m,r)$ and $(i,j)\neq(r,m)$.

Obviously, if λ_{mr} is k-determinate, then λ_{mr} is separately k-determinate w.r.t Λ. That the converse is not true can be seen by the following examples. We set $n=1$ and $k=2$ in both examples.

EXAMPLE 3.3. Let μ be an arbitrary indeterminate measure from $\mathcal{M}^*(\mathbb{R})$ which is not V-extremal (that is, μ is not an extreme point of V_μ). Let λ_{11} and λ_{22} be measures from V_μ with disjoint supports. (For instance, we may take two different N-extremal measures from V_μ.) Put $\lambda_{12}=\lambda_{21}=0$

and $\Lambda = (\lambda_{ij})_{i,j=1,2}$. Then λ_{12} is separately k-determinate, since λ_{11} and λ_{22} have disjoint supports (Lemma 1.1, (iv)). Since μ is not V-extremal, $\mathscr{P}_n$ is not dense in $L^1(\mu)$ [1]. Hence there is a nonzero $f \in L^\infty(\mu)$ such that $\int p(x) f(x)\, d\mu(x) = 0$ for all $p \in \mathscr{P}_n$. We can assume that $|f(x)| \leqslant 1$ μ-a.e. on $\mathbb{R}$. Define $\tilde{\lambda}_{11} = \tilde{\lambda}_{22} = \mu$, $d\tilde{\lambda}_{12} = f\, d\mu$ and $d\tilde{\lambda}_{21} = \bar{f}\, d\mu$. Then, $\tilde{\Lambda} = (\tilde{\lambda}_{ij}) \in V_\Lambda$ and $0 = \lambda_{12} \neq \tilde{\lambda}_{12}$. This shows that λ_{12} is not k-determinate.

In the preceding example λ_{11} and λ_{22} were not separately k-determinate. In the next example λ_{11} and λ_{22} are separately k-determinate, but not k-determinate.

EXAMPLE 3.4. Let $\mu \in \mathscr{M}^*(\mathbb{R})$ be an indeterminate $\mathscr{N}$-extremal measure. Then μ is of the form $\sum_{n=1}^\infty a_n \delta_{x_n}$, where $a_n > 0$ for $n \in \mathbb{N}$. Put $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = \mu$ and $\Lambda = (\lambda_{ij})_{i,j=1,2}$. If $\mu_1 \in V_\mu$, then the $(2, 2)$ matrix for which all entries equal μ_1 is in V_μ. That is, λ_{11}, λ_{12}, λ_{21}, and λ_{22} are not k-determinate. We show that λ_{11}, λ_{12}, λ_{21}, and λ_{22} are separately k-determinate.

We first prove this for λ_{11}. The proof for λ_{22} is the same. Suppose that $\tilde{\Lambda} = (\tilde{\lambda}_{ij}) \in V_\Lambda$ and $\lambda_{ij} = \tilde{\lambda}_{ij}$ for $(i, j) \neq (1, 1)$. Since $\tilde{\lambda}_{12} = \tilde{\lambda}_{21} = \tilde{\lambda}_{22} = \mu$ has positive mass at x_n for $n \in \mathbb{N}$, $\tilde{\lambda}_{11}$ must have positive mass, say b_n, at x_n. Because $\tilde{\Lambda}$ is a nonnegative matrix, $b_n a_n - a_n^2 \geqslant 0$, that is, $b_n \geqslant a_n$ for $n \in \mathbb{N}$. Since $\mu = \sum a_n \delta_{x_n}$ is $\mathscr{N}$-extremal, the mass $a_n = \rho(x_n)$ is larger than the mass concentrated at x_n by any other $\mu_1 \in V_\mu$ [10, Theorem 2.13, p. 60]. Since $\tilde{\lambda}_{11} \in V_\mu$, it follows that $b_n = a_n$ for $n \in \mathbb{N}$ and $\lambda_{11} = \tilde{\lambda}_{11}$.

We now show that λ_{12} is separately k-determinate. Suppose that $\tilde{\Lambda} = (\tilde{\lambda}_{ij}) \in V_\Lambda$, $\lambda_{11} = \tilde{\lambda}_{11}$ and $\lambda_{22} = \tilde{\lambda}_{22}$. From $\operatorname{supp} \tilde{\lambda}_{12} \subseteq \operatorname{supp} \tilde{\lambda}_{11} \cap \operatorname{supp} \tilde{\lambda}_{22}$ (see Lemma 1.1) it follows that $\tilde{\lambda}_{12} = \sum_{n=1}^\infty b_n \delta_{x_n}$ with some $b_n \in \mathbb{C}$ for $n \in \mathbb{N}$. $\tilde{\Lambda} \in \mathscr{M}_2^*(\mathbb{R})$ gives $|b_n| \leqslant a_n$ for all $n \in \mathbb{N}$. Combined with $\tilde{\Lambda} \in V_\Lambda$, the latter implies that $\mu - \frac{1}{2}(\tilde{\lambda}_{12} + \tilde{\lambda}_{21}) = \sum_n (a_n - \operatorname{Re} b_n)\, \delta_{x_n}$ is in $\mathscr{M}^*(\mathbb{R})$ and has zero moments. This clearly yields $a_n = \operatorname{Re} b_n$ for $n \in \mathbb{N}$. From $|b_n| \leqslant a_n$ we get $b_n = a_n$ for $n \in \mathbb{N}$, i.e., $\lambda_{12} = \tilde{\lambda}_{12}$ and $\lambda_{21} = \tilde{\lambda}_{21}$.

(3.4)

Arguing as in the proof of Proposition 3.2, we obtain

PROPOSITION 3.3. *Let $\Lambda = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ and $r \in \{1, ..., k\}$. If $\mathscr{P}_n$ is dense in $L^2(\lambda_{rr})$, then λ_{mr} and λ_{rm} are separately k-determinate with respect to Λ for each $m \in \{1, ..., k\}$, $m \neq r$.*

In this opposite direction we have

PROPOSITION 3.4. *Let $\Lambda = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$ and $r \in \{1, ..., k\}$. Suppose that there is a $\delta > 0$ such that*

$$\sum_{i,j=1}^{k} \lambda_{ij}(m)\, t_i \bar{t}_j \geq \delta \sum_{j=1}^{k} |t_j|^2 \lambda_\Lambda(\mathfrak{M})$$

for all $t_1, ..., t_k \in \mathbb{C}$ *and all Borel sets* $\mathfrak{M}$ *of* $\mathbb{R}^n$.

If λ_{rr} *is not V-extremal, then* λ_{mr} *(and* λ_{rm}*) is not separately k-determinate for every* $m \in \{1, ..., k\}$, $m \neq r$.

Proof. Since λ_{rr} is not V-extremal, there is a nonzero $f \in L^\infty(\lambda_{rr})$ which is orthogonal to $\mathscr{P}_n$ in $L^2(\lambda_{rr})$. Fix $m \in \{1, ..., k\}$, $m \neq r$. Let $\varepsilon > 0$. We define a (k, k) matrix $\tilde{\Lambda} = (\tilde{\lambda}_{ij})$ of measures by $d\tilde{\lambda}_{mr} := d\lambda_{mr} + \varepsilon f\, d\lambda_{rr}$, $\tilde{\lambda}_{rm} := \tilde{\lambda}_{mr}$ and $\tilde{\lambda}_{ij} = \lambda_{ij}$ otherwise. By the above definiteness assumption, the matrix $\tilde{\Lambda} = (\tilde{\lambda}_{ij})$ becomes nonnegative for sufficiently small $\varepsilon > 0$. Then $\tilde{\Lambda} \in \mathscr{M}_k^*(\mathbb{R}^n)$. It is clear that $\tilde{\Lambda} \in V_\Lambda$. Since $f \neq 0$ in $L^\infty(\lambda_{rr})$, λ_{mr} is not separately k-determinate.

Concluding Remarks. A further study of the determinacy seems to be desirable. Let us mention two questions in this direction.

(1) Is Proposition 3.4 true without the definiteness assumption?

(2) Suppose that $\lambda_{11}, ..., \lambda_{kk}$ are indeterminate for $\Lambda = (\lambda_{ij}) \in \mathscr{M}_k^*(\mathbb{R}^n)$. Does it follow that Λ is not k-determinate? (The answer is obviously "yes" if Λ is a diagonal matrix.)

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