Image Analysis by Tchebichef Moments

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***Abstract*—This paper introduces a new set of orthogonal moment functions based on the discrete Tchebichef polynomials. The Tchebichef moments can be effectively used as pattern features in the analysis of two-dimensional images. The implementation of moments proposed in this paper does not involve any numerical approximation, since the basis set is orthogonal in the discrete domain of the image coordinate space. This property makes Tchebichef moments superior to the conventional orthogonal moments such as Legendre moments and Zernike moments, in terms of preserving the analytical properties needed to ensure information redundancy in a moment set. The paper also details the various computational aspects of Tchebichef moments and demonstrates their feature representation capability using the method of image reconstruction.**

***Index Terms*—Discrete orthogonal systems, image feature representation, orthogonal moments, Tchebichef polynomials.**

I. Introduction

MOMENT functions have been used as shape descriptors in a variety of applications in image analysis, like visual pattern recognition [1], [4], object classification [7], template matching [6], edge detection [5], pose estimation [13], robot vision [12], and data compression [9]. In all these applications, geometric moments and their extensions in the form of radial and complex moments have played important roles in characterizing the image shape, and in extracting features that are invariant with respect to image plane transformations. Teague [18] introduced moments with orthogonal basis functions, with the additional property of minimal information redundancy in a moment set. In this class, Legendre and Zernike moments have been extensively researched in the recent past, and several new techniques have emerged involving orthogonal moment based feature detectors [10], [14], [20].

In the following, we consider some of the major problems that are commonly encountered while implementing moment functions.

A. Numerical Approximation of Continuous Integrals

The general two-dimensional (2-D) moment definition using a moment weighting kernel (also known as the basis function) $\psi_{pq}(x,y)$, and an image intensity function $f(x,y)$ is given as

$$\Psi_{pq} = \int_x \int_y \psi_{pq}(x,y) f(x,y)\, dx\, dy, \quad p,q = 0,1,2,\ldots. \tag{1}$$

The integrals in the above equation are usually approximated by discrete summations, and this process not only leads to numerical errors in the computed moments, but also severely affects the analytical properties which they were intended to satisfy, such as invariance, orthogonality, etc.

B. Large Variation in the Dynamic Range of Values

The kernel ψ_{pq} often involves powers of p and q. For example, geometric moments of order $(p+q)$ on an image of size $N \times N$ pixels, are defined using the following kernel:

$$\psi_{pq}(x,y) = x^p y^q, \quad 0 \le x, \quad y \le N-1. \tag{2}$$

Similarly, a radial moment of order p and repetition q has the kernel

$$\psi_{pq}(r,\theta) = r^p e^{jq\theta}. \tag{3}$$

Moments computed with the above schemes will therefore have large variation in the dynamic range of values for different orders. Applications involving such moment functions will have to additionally include scale normalization to maintain equal weight for all the components in a set of feature vectors. Further, it may also be necessary to develop methods for avoiding numerical instabilities when the image size is large.

C. Coordinate Space Transformation

Orthogonal basis functions do not have the aforesaid problem of large dynamic range variation, but they generally have a domain which is completely different from the image coordinate space. For example, the Legendre polynomials are valid only in the range $[-1, 1]$, while the Zernike radial polynomials are defined inside the unit circle [2], [10], [11], [18]. The application of such orthogonal polynomials as basis functions in (1) will require an appropriate transformation of the image coordinate space [15]. The elemental area $dx \cdot dy$ in (1) also gets scaled by the corresponding factor, thus increasing the computational complexity.

The above problems motivate us to consider using discrete orthogonal polynomials as the basis set, and to define the corresponding moments directly on the image coordinate space. Since the implementation of discrete orthogonal moments does not involve any numerical approximations, the basis functions will exactly satisfy the orthogonality property, and thus yield a superior image reconstruction. Consider a discrete orthogonal system $\{f_n(i)\}$, where $a \le i \le b$. The orthogonality property in the above domain can then be written as

$$\sum_{i=a}^{i=b} w(i) f_m(i) f_n(i) = \rho(n,a,b)\delta_{mn} \tag{4}$$

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TABLE I
SOME IMPORTANT DISCRETE ORTHOGONAL POLYNOMIALS AND THEIR WEIGHT FUNCTIONS ($p, \alpha, c, \xi, \psi, \zeta$ ARE PARAMETERS ATTACHED TO THE RESPECTIVE POLYNOMIALS. n DENOTES THE DEGREE)

Name	Notation	a	b	w(x)
Tchebichef	$t_n(x)$	0	$N-1$	1
Krawtchouk	$k_n^{(p)}(x)$	0	N	$p^x(1-p)^{N-x}\binom{N}{x}$
Charlier	$c_n^{(a)}(x)$	0	∞	$\frac{e^{-\alpha}\alpha^x}{x!}$
Meixner	$m_n^{(c,\xi)}(x)$	0	∞	$\frac{c^x(\xi)_x}{x!}$
Hahn	$h_n^{(\xi,\psi,\zeta)}(x)$	0	$N-1$	$\frac{(\xi)_x(\psi)_x}{x!(\zeta)_x}$

where $w(i)$ is the weighting function (also called the jump function), and $\rho(\cdot)$ is the squared norm. The weighting functions of some important discrete orthogonal systems [3], [16] are given in Table I. The simplest among these systems is the Tchebichef polynomial which has a unit weight, and has a domain of definition that is ideally suited for square images of size $N \times N$ pixels.

In this paper, we propose a new set of orthogonal moment features based on discrete Tchebichef polynomials. The organization of the paper is as follows. A brief outline of the properties of continuous orthogonal moments is given in Section II. The equations related to Legendre and Zernike moments are presented to highlight their implementation aspects and to use them later as a reference to compare the performance of Tchebichef moments. A review of discrete orthogonal systems, and the prerequisites for defining a set of moments on a discrete coordinate space are given in Section III. This section also provides the definition of Tchebichef polynomials. The scaled Tchebichef polynomials and the Tchebichef moments are introduced in Section IV. A few important properties of Tchebichef polynomials, which can be effectively used in moment computation, are given in Section V. Experimental results validating the theoretical derivations, and a comparative analysis of performance of Tchebichef moments with Legendre and Zernike moments are included in Section VI.

II. CONTINUOUS ORTHOGONAL MOMENTS

The two most important orthogonal moments that have found several applications in the field of image shape representation are the Legendre moments and the Zernike moments. The Legendre moments of order $(p+q)$ are defined as

$$\lambda_{pq} = \frac{(2p+1)(2q+1)}{4} \times \int_{-1}^{1}\int_{-1}^{1} P_p(x)P_q(y)f(x,y)\,dx\,dy, \quad x,y \in [-1,1], \quad p,q = 0,1,2\ldots \tag{5}$$

where $P_m(x)$ is the Legendre polynomial of degree m.

On an image coordinate space $(i,j) \in [0, N-1]$, the above moment integral has the following discrete approximation:

$$\lambda_{pq} = \frac{(2p+1)(2q+1)}{N^2} \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} P_p\left(\frac{2i-N+1}{N-1}\right) \times P_q\left(\frac{2j-N+1}{N-1}\right) f(i,j). \tag{6}$$

The inverse moment transform which follows from the orthogonality of Legendre polynomials in the continuous domain, can be similarly expressed as

$$f(i,j) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} P_m\left(\frac{2i-N+1}{N-1}\right) \times P_n\left(\frac{2j-N+1}{N-1}\right), \quad i,j = 0,1,2,\ldots N-1. \tag{7}$$

The reconstruction of an image from a set of moments from order 0 to order $n_{\max}$ uses the truncated form of the series in (7), to get a polynomial approximation of $f(i,j)$.

The Zernike moments Z_{nl} of order n, and repetition l are defined using polar coordinates (r,θ) inside a unit circle

$$Z_{nl} = \frac{(n+1)}{\pi}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} R_{nl}(r)e^{-\hat{j}\,l\theta} f(r,\theta) r\,dr\,d\theta, \quad \hat{j} = \sqrt{-1}, \quad |l| \le n, \quad n-|l| \text{ is even.} \tag{8}$$

In this equation, $R_{nl}(r)$ denotes the Zernike radial polynomials of degree n. A discrete approximation of (8) is

$$Z_{nl} = \frac{2(n+1)}{\pi(N-1)^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} R_{nl}(r_{ij}) \times e^{-\hat{j}\,l\theta_{ij}} f(i,j), \quad 0 \le r_{ij} \le 1 \tag{9}$$

where the image coordinate transformation to the interior of the unit circle is given by

$$r_{ij} = \sqrt{(c_1 i + c_2)^2 + (c_1 j + c_2)^2}, \quad c_1 = \frac{\sqrt{2}}{N-1}, \quad c2 = -\frac{1}{\sqrt{2}}, \quad \theta_{ij} = \tan^{-1}\left(\frac{c_1 j + c_2}{c_1 i + c_2}\right), \quad i,j = 0,1,2,\ldots N-1. \tag{10}$$

The Zernike moment Z_{nl} of order n has $(n+1)$ components including both positive and negative values of l, satisfying the conditions in (8). The components are $Z_{n,-l}, Z_{n,-l+2}, \ldots Z_{n,l-2}, Z_{nl}$. If we write

$$Z_{nl} = Z_{nl}^{(c)} - \hat{j}\,Z_{nl}^{(s)}, \quad (\hat{j} = \sqrt{-1}) \tag{11}$$

then using the definition (8) and the properties $Z^{(c)}_{n,-l} = Z^{(c)}_{nl}; Z^{(s)}_{n,-l} = -Z^{(s)}_{nl}$, the inverse moment transform can be conveniently expressed as follows:

$$f(r,\theta) = \sum_n Z^{(c)}_{n0} R_{n0}(r) + \left\{ 2 \sum_{l>0} \left(Z^{(c)}_{nl} \cos(l\theta) + Z^{(s)}_{nl} \sin(l\theta) \right) R_{nl}(r) \right\} \quad \left(Z^{(c)}_{n0} \text{ exists only if } n \text{ is even} \right). \tag{12}$$

The above formula can be used to reconstruct an image from its Zernike moments computed up to a maximum order.

III. Discrete Orthogonal Moments

The following well-known theorem on orthogonal functions provides the mathematical basis for arriving at a definition for discrete orthogonal moments of an image intensity distribution $f(x,y)$: If $\{t_n(x)\}$ is a set of discrete orthogonal polynomials with unit weight, satisfying the condition

$$\sum_{x=0}^{N-1} t_m(x) t_n(x) = \rho(n,N)\delta_{mn}, \quad 0 \le m,n \le N-1 \tag{13}$$

then any bounded function $f(x,y)$, $0 \le \{x,y\} \le N-1$, has the following polynomial representation in terms of the functions $t_n(x)$

$$f(x,y) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} T_{mn} t_m(x) t_n(y) \tag{14}$$

where the coefficients T_{pq} are given by

$$T_{pq} = \frac{1}{\rho(p,N)\rho(q,N)} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} t_p(x) t_q(y) f(x,y) \quad p,q = 0,1,2,\ldots N-1. \tag{15}$$

The above theorem can be generalized for orthogonal polynomials with weight $w(x)$, by replacing each orthogonal function $t_n(x)$ by the function $t_n(x)\sqrt{w(x)}$ in (13)–(15).

Equation (15) is easily obtained by substituting for $f(x,y)$ using (14) in the expression $\sum_{x=0}^{N-1} \sum_{y=0}^{N-1} t_p(x) t_q(y) f(x,y)$, and noting that

$$\rho(p,N) = \sum_{x=0}^{N-1} \{t_p(x)\}^2. \tag{16}$$

Conversely, (14) follows from (15). In the context of image moments, it means that if we define a discrete orthogonal moment function as in (15) with $\{t_p(x)\}$ as the basis set, then the image may be reconstructed from the moments, using (14) as the inverse moment transform. The moment definition as given in (15) completely eliminates the need for any approximation of continuous integrals, and does not require coordinate space transformations. We propose a modified version of Tchebichef polynomials as a convenient set of discrete orthogonal basis functions with unit weight, for defining moments of the above type.

The discrete Tchebichef polynomials [3], [8] are defined as

$$t_n(x) = (1-N)_n \; {}_3F_2(-n,-x,1+n;1,1-N;1), \quad n,x,y = 0,1,2,\ldots N-1 \tag{17}$$

where $(a)_k$ is the Pochhammer symbol given by

$$(a)_k = a(a+1)(a+2)\ldots(a+k-1) \tag{18}$$

and ${}_3F_2(\cdot)$ is the generalized hypergeometric function

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{z^k}{k!}. \tag{19}$$

With the above definitions, (17) can also be written as

$$t_n(x) = n! \sum_{k=0}^{n} (-1)^{n-k} \binom{N-1-k}{n-k} \binom{n+k}{n} \binom{x}{k}. \tag{20}$$

The Tchebichef polynomials satisfy the property of orthogonality (13), with

$$\rho(n,N) = \frac{N(N^2-1)(N^2-2^2)\ldots(N^2-n^2)}{2n+1} = (2n)! \binom{N+n}{2n+1}, \quad n = 0,1,\ldots,N-1 \tag{21}$$

and have the following recurrence relation:

$$(n+1)t_{n+1}(x) - (2n+1)(2x-N+1)t_n(x) + n(N^2-n^2)t_{n-1}(x) = 0, \quad n = 1,2\ldots,N-1. \tag{22}$$

This set of polynomials is, however, not suitable for defining moments, as it can be easily verified that the value of $t_n(x)$ grows as N^n, and the value of the moment T_{pq} given in (15) grows as $N^{-(p+q)}$. We therefore introduce the *scaled* Tchebichef polynomials, and analyze the properties of the corresponding moment functions.

IV. Tchebichef Moments

We define the scaled Tchebichef polynomials as

$$\tilde{t}_n(x) = \frac{t_n(x)}{\beta(n,N)} \tag{23}$$

where $t_n(x)$ is the discrete Tchebichef polynomial of degree n, given by (17), and $\beta(n,N)$ is a suitable constant which is independent of x. Under the above transformation, the squared-norm

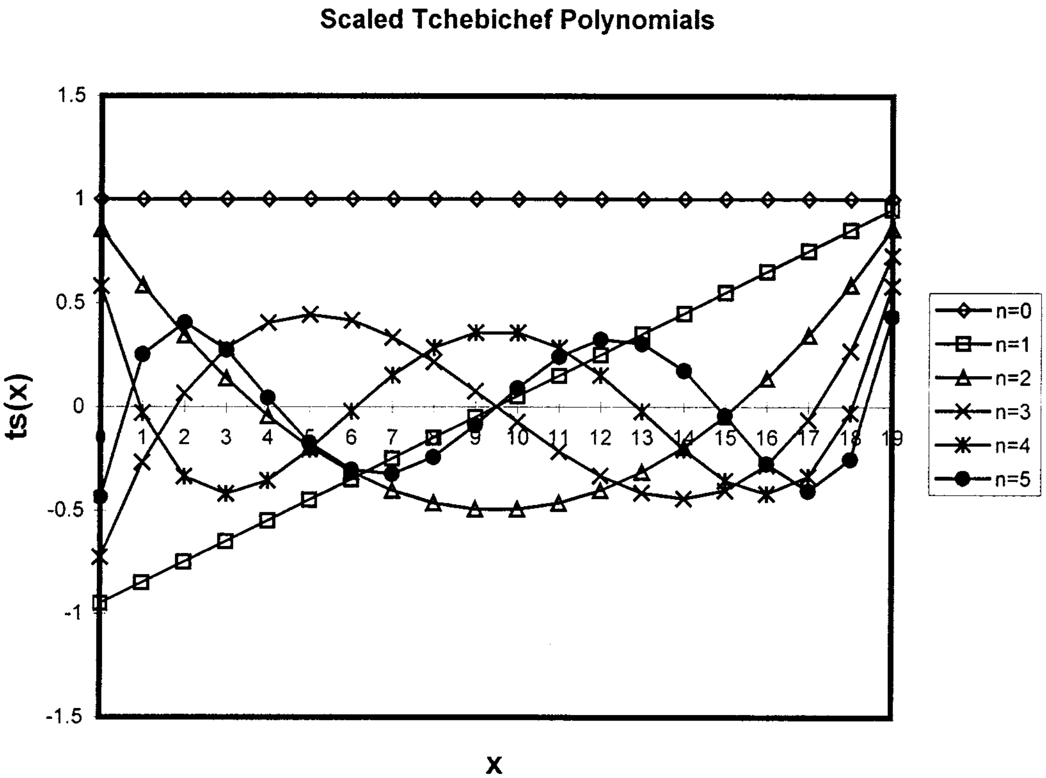

Fig. 1. Plot of scaled Tchebichef polynomials for $N = 20$.

of the scaled polynomials gets modified according to the formula

$$\tilde{\rho}(n, N) = \frac{\rho(n, N)}{\beta(n, N)^2}. \quad (24)$$

We now define the Tchebichef moments as

$$T_{pq} = \frac{1}{\tilde{\rho}(p, N)\tilde{\rho}(q, N)} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \tilde{t}_p(x)\tilde{t}_q(y) f(x, y), \quad p, q = 0, 1, 2, \ldots N-1. \quad (25)$$

This equation also leads to the following inverse moment transform:

$$f(x, y) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} T_{mn}\tilde{t}_m(x)\tilde{t}_n(y), \quad x, y = 0, 1, \ldots N-1. \quad (26)$$

The scale factor $\beta(n, N)$ is typically a function of N which grows as N^n. The above framework for computing the scaled Tchebichef polynomials and the associated moments, then guarantees that there will not be large variations in the dynamic range of values of moments, nor any numerical instabilities for large values of N. The simplest choice for $\beta(n, N)$ is

$$\beta(n, N) = N^n \quad (27)$$

in which case, we have the following recurrence formula for $\tilde{t}_n(x)$ as shown at the bottom of the page, where $\tilde{t}_0(x) = 1$ and $\tilde{t}_1(x) = (2x + 1 - N)/N$ and

$$\tilde{\rho}(n, N) = \frac{N\left(1 - \frac{1}{N^2}\right)\left(1 - \frac{2^2}{N^2}\right)\cdots\left(1 - \frac{n^2}{N^2}\right)}{2n+1}, \quad n = 0, 1, \ldots, N-1. \quad (29)$$

A plot of the polynomial values for $N = 20$, obtained from (28) is given in Fig. 1. Other possible choices of $\beta(n, N)$ are as follows.

1) $\beta(n, N) = (N-1)(N-2)\ldots(N-n)$. With this scale factor, the scaled polynomials satisfy the condition $\tilde{t}_n(0) = 1$ if n is even, and $\tilde{t}_n(0) = -1$ if n is odd.
2) $\beta(n, N) = \sqrt{(N^2 - 1^2)(N^2 - 2^2)\cdots(N^2 - n^2)}$. This choice of β leads to the equation $\tilde{\rho}(n, N) = (N/(2n + 1))$, which is equal to the corresponding squared-norm of the Legendre polynomials in the discrete domain.
3) $\beta(n, N) = \sqrt{\rho(n, N)}$. Obviously, this selection is helpful in making the scaled polynomial set orthonormal, with the property $\tilde{\rho}(n, N) = 1$.

The recurrence formulae for the above cases can be obtained in a compact form. It is also interesting to note that the reconstructed image intensity function obtained from the inverse mo-

$$\tilde{t}_n(x) = \frac{(2n-1)\tilde{t}_1(x)\tilde{t}_{n-1}(x) - (n-1)\left(1 - \frac{(n-1)^2}{N^2}\right)\tilde{t}_{n-2}(x)}{n}, \quad n = 2, 3, \ldots N-1 \quad (28)$$

ment transform, is independent of the choice of $\beta(n, N)$. This can be easily verified using (23)–(26).

V. Properties of Tchebichef Moments

In this section, we analyze some of the computational aspects of Tchebichef moments, using well known properties of Tchebichef polynomials.

A. Symmetry

The symmetry property can be used to considerably reduce the time required for computing the Tchebichef moments. The scaled Tchebichef polynomials have the same symmetry property which the classical Tchebichef polynomials satisfy

$$\tilde{t}_n(N-1-x) = (-1)^n \tilde{t}_n(x). \tag{30}$$

This relation suggests the subdivision of the domain of an $N \times N$ image (where N is even) into four equal parts (Fig. 2), and performing the computation of the polynomials only in the first quadrant where $0 \le x, y \le (N/2-1)$. The expression for Tchebichef moments in (25) can be modified with the help of (30), as follows:

$$T_{pq} = \frac{1}{\tilde{\rho}(p,N)\tilde{\rho}(q,N)} \times \sum_{x=0}^{(N/2)-1} \sum_{y=0}^{(N/2)-1} \tilde{t}_p(x)\tilde{t}_q(y) \times \left\{ \begin{array}{l} f(x,y) + (-1)^p f(N-1-x, y) \\ \quad +(-1)^q f(x, N-1-y) \\ \quad +(-1)^{p+q} f(N-1-x, N-1-y) \end{array} \right\}. \tag{31}$$

In addition to reducing the computation time by a factor of four, the symmetry property is also useful in minimizing the storage required for the scaled Tchebichef polynomials. If an application needs the storage of polynomials up to a maximum degree M, then a two-dimensional (2-D) array of size $M \times (N/2)$ would suffice. However, in this case, the application has to incorporate the symmetry condition in its formulation, to determine the value of $\tilde{t}_n(x)$ when $x \ge (N/2)$. As an example, the reconstruction formula in (26) will have to be modified as

$$\begin{aligned} f(x,y) &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} T_{mn}\tilde{t}_m(x)\tilde{t}_n(y), \\ &\qquad\qquad \text{if } x, y < (N/2), \\ &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} (-1)^m T_{mn}\tilde{t}_m(N-1-x)\tilde{t}_n(y), \\ &\qquad\qquad \text{if } y < (N/2); \quad x \ge (N/2), \\ &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} (-1)^n T_{mn}\tilde{t}_m(x)\tilde{t}_n(N-1-y), \\ &\qquad\qquad \text{if } x < (N/2); \quad y \ge (N/2), \\ &= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} (-1)^{m+n} T_{mn}\tilde{t}_m(N-1-x) \\ &\qquad \times \tilde{t}_n(N-1-y), \quad \text{if } x, y \ge (N/2). \end{aligned} \tag{32}$$

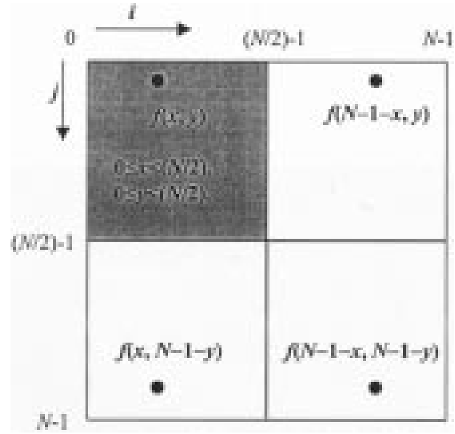

Fig. 2. Owing to symmetry, Tchebichef polynomials need be computed only on one quadrant image.

B. Polynomial Expansion

The scaled Tchebichef polynomial $\tilde{t}_n(x)$ can be expressed as a polynomial of x. Using (20) and (23), we can write

$$\tilde{t}_n(x) = \frac{1}{\beta(n,N)} \sum_{k=0}^{n} C_k(n,N) \sum_{i=0}^{k} s_k^{(i)} x^i \tag{33}$$

where

$$C_k(n,N) = (-1)^{n-k}\frac{n!}{k!}\binom{N-1-k}{n-k}\binom{n+k}{n} \tag{34}$$

and $s_k^{(i)}$ are the Stirling numbers of the first kind [19], which satisfies

$$\frac{x!}{(x-k)!} = \sum_{i=0}^{k} s_k^{(i)} x^i. \tag{35}$$

C. Representation Using Geometric Moments

The polynomial expansion given above is useful in writing the Tchebichef moments (25) in terms of the geometric moments. If the geometric moments of an image $f(x, y)$ are expressed using the discrete sum approximation as

$$m_{pq} = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} x^p y^q f(x,y) \tag{36}$$

then using (33), the Tchebichef moments of the same image may be expressed in terms of geometric moments as follows:

$$T_{pq} = A_p A_q \sum_{k=0}^{p} C_k(p,N) \sum_{l=0}^{q} C_l(q,N) \times \sum_{i=0}^{k}\sum_{j=0}^{l} s_k^{(i)} s_l^{(j)} m_{ij} \tag{37}$$

where

$$A_p = \frac{1}{\beta(p,N)\tilde{\rho}(p,N)}.$$

From (37), it is seen that the Tchebichef moments depend on the geometric moments up to the same order. The explicit expressions of the Tchebichef moments in terms of geometric moments up to the second order (for $\beta(n,N) = N^n$) are as follows:

$$
\begin{aligned}
T_{00} &= \frac{m_{00}}{N^2}\\
T_{10} &= \frac{6m_{10} + 3(1-N)m_{00}}{N(N^2-1)}\\
T_{01} &= \frac{6m_{01} + 3(1-N)m_{00}}{N(N^2-1)}\\
T_{20} &= \frac{30m_{20} + 30(1-N)m_{10} + 5(1-N)(2-N)m_{00}}{(N^2-1)(N^2-2^2)}\\
T_{02} &= \frac{30m_{02} + 30(1-N)m_{01} + 5(1-N)(2-N)m_{00}}{(N^2-1)(N^2-2^2)}\\
T_{11} &= \frac{36m_{11} + 18(1-N)(m_{10}+m_{01}) + 9(1-N)^2 m_{00}}{(N^2-1)^2}.
\end{aligned}
\tag{38}
$$

D. Recurrence Relation with Respect to x

The polynomials $\tilde{t}_n(x)$ have the following recurrence relation [3], with respect to the variable x

$$
\begin{aligned}
&x(N-x)\tilde{t}_n(x)\\
&\quad = (-n(n+1) - (2x-1)(x-N-1) - x)\tilde{t}_n(x-1)\\
&\qquad + ((x-1)(x-N-1))\tilde{t}_n(x-2).
\end{aligned}
\tag{39}
$$

The starting values for the above recursion can be obtained from the following equations:

$$
\begin{aligned}
\tilde{t}_n(0) &= \frac{(1-N)_n}{\beta(n,N)},\\
\tilde{t}_n(1) &= \tilde{t}_n(0)\left(1 + \frac{n(n+1)}{1-N}\right)
\end{aligned}
\tag{40}
$$

VI. Experimental Results

This section presents the test data and results used to validate the theoretical framework presented above, and also to establish the feature representation capability of Tchebichef moments through image reconstruction. A comparative analysis between Tchebichef moments, Legendre moments and Zernike moments is also given. A binary image of the letter "E" (see Fig. 3) on a 20 × 20 pixel grid ($N = 20$) was used to analyze the values of the moment functions. The Tchebichef, Legendre, and geometric moments of the first few orders of the test image, computed using (31), (6), and (36), respectively, are given in Table II. In addition to the relationship between the two moments T_{pq} and m_{pq} as given in (37), the table also shows the uniform range of Tchebichef moments for different orders. The remarkable similarity between the values of Tchebichef moments and Legendre moments can be attributed to the fact [3], [17]

$$
\lim_{N\to\infty} \frac{t_n(xN)}{N^n} = P_n(2x-1), \quad x \in [0,1], \quad n = 0,1,2\ldots. \tag{41}
$$

Equivalently

$$
\lim_{N\to\infty} \tilde{t}_n(i) = P_n(x'), \quad i = 0,1,\ldots N-1, \quad x' = (2i-N)/N. \tag{42}
$$

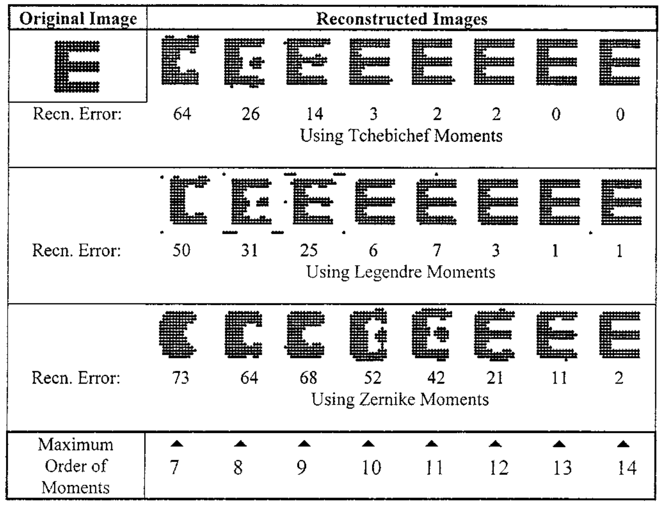

Fig. 3. Image reconstruction of letter "E" ($N = 20$).

TABLE II
TCHEBICHEF, LEGENDRE, AND GEOMETRIC MOMENTS OF THE TEST IMAGE

Order of Moment		Tchebichef	Legendre	Geometric
p	q	T_{pq}	λ_{pq}	m_{pq}
0	0	0.3525	0.3525	141.0
0	1	−0.0530075	−0.0556579	1269.0
1	0	−0.1342105	−0.1409210	1161.0
0	2	−0.2914483	−0.2261427	14539.0
1	1	0.0201820	0.0222507	10449.0
2	0	−0.4539758	−0.4039820	11631.0
0	3	−0.0221763	−0.0447540	186975.0
1	2	0.1877272	0.1785993	121059.0
2	1	0.0682670	0.0637866	104679.0
3	0	0.2141892	0.1910559	132111.0
0	4	−0.3144268	−0.3091446	2555671.0
1	3	−0.0190416	−0.0146005	1575855.0
2	2	0.3236577	0.1972807	1221269.0
3	1	−0.0322089	−0.0301667	1188999.0
4	0	−0.1482432	−0.1969217	1618203.0

The sequence of reconstructed images, as the maximum order of moments used in the reconstruction is varied from seven to 14, is shown in Fig. 3. We used the following formula to characterize the error between an input binary image $f(x,y)$, and the reconstructed image $\hat{f}(x,y)$

$$
\varepsilon = \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} |f(x,y) - \hat{f}(x,y)|. \tag{43}
$$

It may be noted that the total number of moment terms from order zero up to order n, in the case of both Tchebichef and Legendre moments is

$$
\Omega = \frac{(n+1)(n+2)}{2}. \tag{44}
$$

Zernike moments also satisfy the above condition, since there are $(n+1)$ components for an nth order moment (see Section II). Fig. 3 thus provides a comparison of the relative performances of Tchebichef, Legendre, and Zernike moments on the same scale.

Results of image reconstruction with a 60 × 60 image ($N = 60$) of a Chinese character are given in Fig. 4, with the maximum order of moments varied from 14 to 28. A more detailed

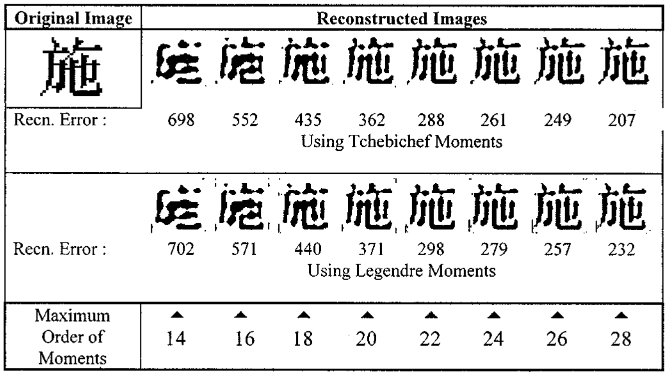

Fig. 4. Image reconstruction of a Chinese character without noise ($N = 60$).

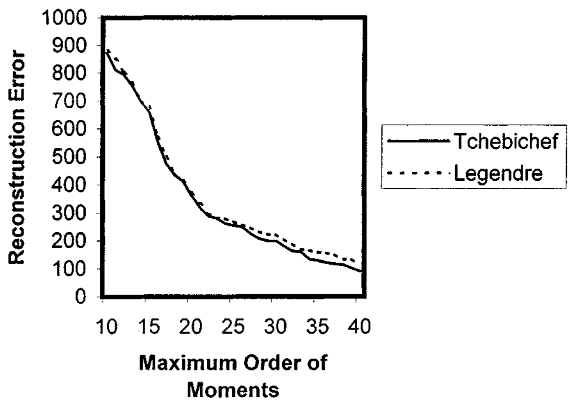

Fig. 5. Comparative analysis of reconstruction error without image noise.

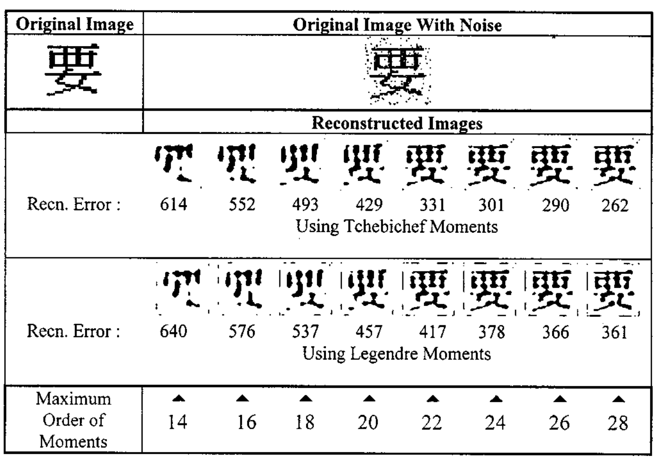

Fig. 6. Reconstruction from noisy image of a Chinese character ($N = 60$).

comparison plot of the variation of reconstruction error with the maximum order of moments is in Fig. 5. The analysis is repeated by adding 5% salt-and-pepper noise to the input image, and the corresponding results are given in Figs. 6 and 7. All of these results demonstrate the superior feature representation capability of Tchebichef moments over both Legendre and Zernike moments.

VII. CONCLUSION

A new set of discrete orthogonal moment features based on Tchebichef polynomials has been proposed in this paper. The

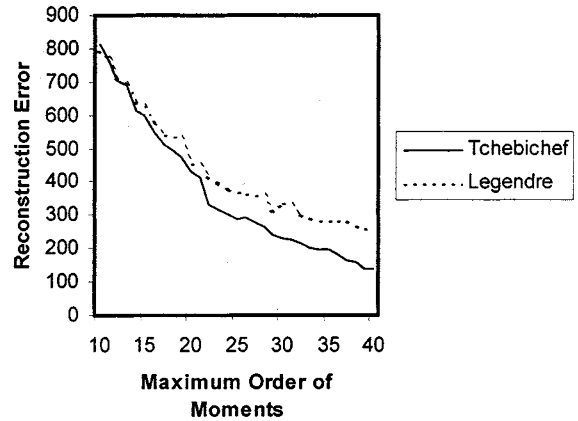

Fig. 7. Comparative analysis of reconstruction errors with image noise.

basis functions are orthogonal in the domain of the image coordinate space, and this feature completely eliminates the need for any discrete approximation in their numerical implementation. The coordinate space normalization required in Legendre and Zernike moment evaluation is also eliminated. Appropriate scale factors are introduced in the moment functions, so that the computed moments do not exhibit large variation in the dynamic range of values, nor any kind of numerical instabilities for large image sizes. Important analytical properties of Tchebichef polynomials and moments, together with their computational aspects have also been discussed.

Experimental results conclusively prove the effectiveness of Tchebichef moments as feature descriptors. Comparative analysis with Zernike and Legendre moments, shows the superior feature representation capability of Tchebichef moments.

Feature descriptors that are invariant with respect to rotations in the image plane, can be easily constructed using Zernike moments. Zernike moments are however computationally more complex than Tchebichef moments. Legendre and Tchebichef moments fall into the same class of orthogonal moments defined in the Cartesian coordinate space, where moment invariants (particularly rotation invariants) are not readily available. One method that is commonly adopted in such situations requiring invariant pattern recognition, is to first normalize the image to a standard image. Future work in the field of Tchebichef moments is directed toward the identification of invariants, and feasibility studies on the use of Tchebichef polynomials in two variables as basis functions.

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