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The Early History of the Moment Problem

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In the present study it is discussed how the moment problem naturally arose within Stieltjes' creation of the analytical theory of continued fractions. Further it is shown how the moment problem in the work of Hamburger came to be regarded as an important problem in its own right. From then on it moved away from its origin into other fields of mathematics—complex function theory and functional analysis—in the work of Nevanlinna and M. Riesz respectively. In the end it was made completely independent from continued fractions. © 1993 Academic Press, Inc.

In dieser Arbeit wird dargelegt, wie das Momentproblem in natürlicher Weise während Stieltjes' Entwicklung der analytischen Theorie der Kettenbrüche entstand. Es wird weiterhin gezeigt, wie sich dieses Problem in den Arbeiten von Hamburger als eigenständiges Problem herauskristallisierte. Von da an löste es sich von seinem Ausgangspunkt und knüpfte an andere Teilgebiete der Mathematik an, so an die komplexe Funktionentheorie und an die Funktionalanalysis in den Arbeiten von Nevanlinna bzw. M. Riesz. Schliesslich wurde es vollkommen unabhängig von der Theorie der Kettenbrüche. © 1993 Academic Press, Inc.

Dans l'article l'auteur explique comment Stieltjes dans sa création de la théorie analytique des fractions continues est arrivé à concevoir le problème des moments. Elle démontre par la suite comment ce même problème des moments est devenu important en soi dans les travaux de Hamburger. Depuis l'intérêt pour ce problème s'est déplacé vers d'autres champs des mathématiques tels que la théorie des fonctions complexes et l'analyse fonctionnelle dans les travaux de Nevanlinna and M. Riesz respectivement. Le problème des moments est devenu finalement indépendent de la théorie des fractions continues. © 1993 Academic Press, Inc.

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INTRODUCTION

The purpose of this paper is twofold: First to analyze how the moment problem arose in 1894–1895 in the hands of Thomas Jan Stieltjes (1856–1894) as a means of studying the analytic behavior of continued fractions, in which connection he invented the important Stieltjes integral. Second, to show how the moment problem later on became entirely independent of the theory of continued fractions.

I shall analyze how Grommer in 1914, after a failed attempt by Van Vleck in 1903, extended Stieltjes' work to a more general class of continued fractions, and discuss how Hamburger then in 1920 was able to extend the "Stieltjes moment problem," which is only defined on the positive real axis, to the "Hamburger

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moment problem," which is defined on the whole real axis. With Hamburger's work the moment problem came to be regarded as an important problem in its own right, and from here on it moved further and further away from its origin. When Hausdorff in 1920 in connection with convergence-preserving matrices hit on the moment problem on a finite interval (the Hausdorff moment problem) there was no connection with continued fractions. In 1922 the general "Hamburger moment problem" moved from continued fractions into the field of complex function theory with the work of Nevanlinna and almost simultaneously M. Riesz observed the connection between the moment problem and the space of bounded linear functionals on $C([a, b])$.

Let me begin by explaining the moment problem in modern terms [1]:

Suppose μ is a positive Radon measure. If the number

$$c_n = \int x^n \, d\mu(x)$$

is well defined, it is called the nth moment of μ. If all the nth moments exists for $n = 0, 1, 2, 3, \ldots$ the sequence $(c_n)_{n \geq 0}$ is called the moment sequence of μ.

The moment problem then consists of the two following main problems:

(1) Given a sequence $(c_n)_{n \geq 0}$ of real numbers, determine if there exists a measure μ having it as its moment sequence (and find μ).

(2) Is μ uniquely determined by this sequence?

If the answer to the second question is yes the moment problem is called determinate; otherwise the problem is said to be indeterminate.

CHEBYSHEV AND THE METHOD OF MOMENTS

Even though it was Stieltjes who introduced the concept of the moment problem, he was not the first one who dealt with moments. Already in 1874 Chebyshev (1821–1894) [Chebyshev 1874], inspired by a work of the French statistician Irénée Jules Bienaymé (1796–1878) [Bienaymé 1853] [2], considered the problem of how an upper (resp. lower) bound for the value of the integral

$$\int_a^b f(x) \, dx \tag{1}$$

over an interval $[a, b]$ can be determined given the values of the integrals

$$\int_A^B f(x) \, dx, \int_A^B x f(x) \, dx, \ldots, \int_A^B x^m f(x) \, dx$$

over a larger interval $[A, B]$. Here $f(x)$ is an unknown function, which remains positive between the endpoints of integration. Chebyshev associated the problem with the expansion of the integral

$$\int_A^B \frac{f(x)}{z - x}\, dx$$

in a continued fraction

$$\cfrac{1}{\alpha z + \beta + \cfrac{1}{\alpha_1 z + \beta_1 + \cfrac{1}{\alpha_2 z + \beta_2 + \cdots}}}, \tag{2}$$

which can be done when the moments exist, and without proof he gave the inequalities which now bear his name:

$$\frac{\varphi(z_{l+1})}{\psi'(z_{l+1})} + \cdots + \frac{\varphi(z_{n-1})}{\psi'(z_{n-1})} \leq \int_{z_l}^{z_n} f(x)\, dx \leq \frac{\varphi(z_l)}{\psi'(z_l)} + \cdots + \frac{\varphi(z_n)}{\psi'(z_n)}.$$

Here $\varphi(z)/\psi(z)$ is one of the convergents of the continued fraction (2), and

$$z_1 < z_2 < \cdots < z_l < z_{l+1} < \cdots < z_{n-1} < z_n < \cdots < z_m$$

are the roots of the equation $\psi(z) = 0$.

Thus if the first $m + 1$ moments of a mass distribution are known over an arbitrary interval $[A, B]$, then the inequalities tell us something about the amount of mass distributed over a smaller interval $[z_1, z_2]$. This method of finding an upper and lower bound for the integral (1) is related to the moment problem and can be viewed as a precursor for it, but Chebyshev only worked with a finite number of moments and he primarily saw the method as a tool to prove some important limit theorems in probability theory [Chebyshev 1887].

HOW STIELTJES BECAME INTERESTED IN CONTINUED FRACTIONS

Stieltjes' main paper about continued fractions [Stieltjes 1894–1895], the first part of which came out just before he died in December 1894, is an incredibly beautiful piece of mathematics. It was the first general investigation of the analytic theory of continued fractions as part of complex function theory. Besides that, it introduced several new ideas and concepts such as the Stieltjes integral and the moment problem, and it became famous for its rigorous style.

Stieltjes' approach to continued fractions probably has its root in his interest in divergent series of the form

$$\frac{c_0}{x} + \frac{c_1}{x^2} + \frac{c_2}{x^3} + \cdots. \tag{3}$$

After Cauchy (1789–1857) in 1821 [Cauchy 1821] stressed that one cannot talk of a sum of a divergent series, mathematicians became more careful in their work with such series. However, in many branches of physics and in particular in celestial mechanics, divergent series were widely used. Stieltjes' career as a scientist began in the observatory of Leiden in Holland where he worked as an astronomer from 1877 to 1883, so he must have been familiar with the use of divergent series. But first of all Stieltjes was a mathematician. Indeed in the correspondence with Hermite (1822–1901), Stieltjes often complained that it was only in his rare leisure time that he could really study mathematics [Hermite 1905, Letter 2, Nov. 1882]. So from a mathematical point of view it must have been a challenge for Stieltjes to figure out why the use of divergent series in so many cases gave the right results. In his doctoral thesis from 1886 "Recherches sur quelques séries semiconvergentes" [Stieltjes 1886] he gave one of the first rigorous approaches to divergent series. Simultaneously Poincaré (1854–1912) published a more general treatment on asymptotic series [Poincaré 1886] and in the last two decades of the 19th century there was a great interest among mathematicians in the theory of summability.

In his doctoral thesis Stieltjes wanted to examine those divergent series of the form (3) which arise naturally from the integrals

$$li(a) = \int_0^a \frac{1}{\log u}\,du, \int_0^\infty \frac{\sin au}{1+u^2}\,du, \int_0^\infty \frac{u \cos au}{1+u^2}\,du.$$

From Euler (1707–1783) [Euler 1748] it was well known that one could transform an infinite series into a continued fraction and vice versa. So the jump from divergent series of the form (3) to the study of continued fractions was not so big.

Another aspect of this relationship between definite integrals, infinite series, and continued fractions is that it sometimes provides a method to determine the value of the integral. Laguerre (1834–1886) was the first who explicitly pointed out this fact [Laguerre 1879]. He considered the integral

$$I = \int_x^\infty \frac{e^{-x}}{x}\,dx,$$

and by using partial integration he obtained the identity

$$\int_x^\infty \frac{e^{-x}}{x}\,dx = e^{-x}\cdot F(x) \mp 1\cdot 2\cdot 3\cdots n \int_x^\infty \frac{e^{-x}}{x^{n+1}}\,dx,$$

where

$$F(x) = \frac{1}{x} - \frac{1}{x^2} + \frac{1\cdot 2}{x^3} - \frac{1\cdot 2\cdot 3}{x^4} + \cdots \pm \frac{1\cdot 2\cdot 3\cdots(n-1)}{x^n}.$$

If $F(x)$ is continued to an infinite series it will diverge for all values of x, but Laguerre pointed out that it is possible to transform $F(x)$ into a continued fraction which is convergent,

$$\int_x^\infty \frac{e^{-x}}{x}\,dx = e^{-x}\cfrac{1}{x+1-\cfrac{1}{x+3-\cfrac{1}{\cfrac{x+5}{4}-\cfrac{1/4}{\cfrac{x+7}{9}-\cfrac{1/9}{\cfrac{x+9}{16}-\cfrac{1/16}{x+\cdots}}}}}}$$

and he showed that in the limit the continued fraction is equal to the integral.

It is very likely that Stieltjes' work with the relationship between definite integrals and divergent series of the form (3) led him toward the theory of continued fractions. Indeed he was fascinated by the fact that one can transform two such different analytic forms—a continued fraction and a definite integral—into one another and in his famous main work he wanted to generalize this observation. [Stieltjes 1894–1895, 497].

STIELTJES' THEORY ON CONTINUED FRACTIONS

Stieltjes' correspondence with Hermite reveals that he began his work on continued fractions in 1883. In 1889 he was so deeply involved in the subject that he wrote to Hermite "Je pense toujours aux fractions continues" [Hermite 1905, Letter 172, March 1889] and a month later "Je suis toujours entièrement abîmé dans mes fractions continues." [Hermite 1905, Letter 191, April 1889]. By this time it was his plan to divide the paper into two parts "l'une plutôt algébrique, tandis que, dans la seconde je m'occupe de la question de convergence surtout." [Hermite 1905, Letter 215, June 1889]. But Stieltjes became more and more interested in the analytic part and when the paper finally appeared in 1894–1895 it was without an algebraic part.

At the end of the 1880's Stieltjes expanded various definite integrals into continued fractions [Hermite 1905, Letter 178, March 1889]. Most of the continued fractions were of the form (or could be transformed to the form)

$$S(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cdots}}} \qquad a_i \in \mathbb{R}_+,\ z \in \mathbb{C} \tag{4}$$

which is now called a Stieltjes continued fraction. Stieltjes devoted his paper to a complete investigation of the question of convergence of these continued fractions. I will now show how I think this analysis inspired Stieltjes to introduce the moment problem.

One of his main theorems states that if $z \in \mathbb{C} \backslash \mathbb{R}_-$ then the even convergents

$$\frac{P_{2n}(z)}{Q_{2n}(z)}$$

as well as the odd convergents

$$\frac{P_{2n+1}(z)}{Q_{2n+1}(z)}$$

of $S(z)$ will converge toward analytic functions $F(z)$, $F_1(z)$ respectively:

$$\lim_{n\to\infty} \frac{P_{2n}(z)}{Q_{2n}(z)} = F(z), \qquad \lim_{n\to\infty} \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = F_1(z) \qquad z \in \mathbb{C}\backslash\mathbb{R}_-. \tag{5}$$

Already in 1892 Stieltjes communicated this general theorem to Hermite [Hermite 1905, letter 325, Jan. 1892] but did not give the full proof of it. He mentioned that for $z \in \mathbb{R}_+$ the theorem was an easy consequence of a result due to Stern (1807–1894) [Stern 1847], and he continued with a proof showing that the theorem is true also for $z = a + ib$, $a > 0$. But the extension to the whole plane except $\mathbb{R}_-$ is a rather complicated task, and even though Stieltjes claimed that he had known the proof for a long time it seems unlikely that he could have proved this result before 1894. As a matter of fact two years later (in 1894) he discovered what is now called the "Stieltjes–Vitali" theorem, which says that if $f_1(z), f_2(z), \ldots$ is a sequence of holomorphic functions such that $|f_1(z) + \cdots + f_n(z)|$, $n = 1, 2, 3, \ldots$, is uniformly bounded in a domain $S \subseteq \mathbb{C}$, then the uniform convergence of the series

$$\sum_1^\infty f_k(z) \tag{6}$$

in some small disc in S is enough to ensure that the series (6) will converge uniformly in the whole domain S. This result is exactly what he needed in order to prove the above statement (5). The "Stieltjes–Vitali" theorem is very strong and Stieltjes presented it to Hermite with the following remark

> Grâce à mon théorème, j'en conclus très facilement que la fraction continue [$S(z)$] est convergente dans tout le plan, excepté la coupure formée par la partie négative de l'axe réel. C'est là un perfectionnement notable de ma théorie que j'ai cherché depuis bien longtemps. [Hermite 1905, Letter 399, Feb. 1894]

Thus he actually did not find the proof of his main theorem until the beginning of 1894, and even then he was not quite sure of himself and asked Hermite to show the proof to Picard (1856–1941), Hermite's son-in-law, to get his opinion [Hermite 1905, Letter 399, Feb. 1894].

How and when did Stieltjes encounter moments in this connection? Well, in the very same letter from 1892, where he announced the main theorem, he also told Hermite that he had found the analytic form of the limit functions $F(z)$, $F_1(z)$. Indeed he claimed that one can always find two positive functions $f(u)$, $f_1(u)$,

$u \in \mathbb{R}$ such that

$$F(z) = \int_0^\infty \frac{f(u)}{z+u}\,du \qquad \text{and} \qquad F_1(z) = \int_0^\infty \frac{f_1(u)}{z+u}\,du. \tag{7}$$

This, as we shall see later, is not always true. To the question of how one actually finds these two positive functions $f(u), f_1(u)$ Stieltjes just said at that time that he had to think more about it but they appear as the limits of certain discontinuous functions. So even though he didn't quite know at this time how to prove it, there was no doubt in his mind that f and f_1 could be constructed. Stieltjes then noticed that in the case where the series $\Sigma\, a_n$ of the coefficients of $S(z)$ converges, $f_1 \not\equiv f$ and—here comes the remarkable thing—even though they are different the following identity holds:

$$\int_0^\infty u^k(f(u) - f_1(u))\,du = 0 \qquad \text{for all } k = 0, 1, \ldots. \tag{8}$$

In relation to the moment problem this is a very important consequence, because it shows that $f(u)$ and $f_1(u)$ are not determined by their moments. I think that this discovery made by Stieltjes was the main reason for the emergence of the moment problem.

To see why (8) is true despite the fact that $f_1(u) \not\equiv f(u)$ we have to know a little more about continued fractions:

The nth convergent

$$\frac{P_n(z)}{Q_n(z)}$$

of $S(z)$ can be developed into a power series of the form

$$\frac{P_n(z)}{Q_n(z)} = \frac{c_0^{(n)}}{z} - \frac{c_1^{(n)}}{z^2} + \frac{c_2^{(n)}}{z^3} - \cdots + (-1)^{(n-1)}\frac{c_{n-1}^{(n)}}{z^n} + (-1)^n \frac{c_n^{(n)}}{z^{n+1}} + \cdots \tag{9}$$

which converge for $|z|$ big enough and where $c_0^{(n)}, \ldots, c_{n-1}^{(n)}$ are positive. If

$$\frac{P_{n+m}(z)}{Q_{n+m}(z)}$$

is a convergent of $S(z)$ different from

$$\frac{P_n(z)}{Q_n(z)},$$

then for all $m > 0$ we have that

$$c_\nu^{(n)} = c_\nu^{(n+m)} \qquad \text{for } \nu = 0, 1, 2, \ldots, n-1.$$

This determines a sequence of positive coefficients

$$c_0, c_1, c_2, c_3, \ldots$$

(here we have set $c_\nu = c_\nu^{(n)}$ for $\nu = 0, 1, \ldots, n-1$), and the power series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots \tag{10}$$

is the power series that corresponds to $S(z)$.

Now because of (5), $S(z)$, $F(z)$, and $F_1(z)$ will have the same formal power series expansion. But this means that

$$\frac{1}{z}\int_0^\infty f(u)\,du - \frac{1}{z^2}\int_0^\infty uf(u)\,du + \frac{1}{z^3}\int_0^\infty u^2 f(u)\,du - \cdots$$

is equal to (10), and therefore we necessarily have

$$c_n = \int_0^\infty u^n f(u)\,du.$$

Similarly

$$c_n = \int_0^\infty u^n f_1(u)\,du,$$

and therefore, as Stieltjes observed,

$$\int_0^\infty u^k(f(u) - f_1(u))\,du = 0 \qquad \text{for all } k \geq 0.$$

As mentioned above Stieltjes had here encountered an example of an indeterminate moment problem. Stieltjes did not write about moments in the letter, but remarked:

> L'existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que
>
> $$\int_0^\infty u^k \varphi(u)\,du = 0 \qquad (k = 0, 1, 2, 3, \ldots)$$
>
> me paraît très remarquable. [Hermite 1905, Letter 325, Jan. 1892].

I find it very likely that this observation was the point of departure for his study of the moment problem.

Actually in this letter from 1892 we can see that Stieltjes had a good sense of what later became the essential part of his paper—the convergence of

$$\frac{P_{2n}(z)}{Q_{2n}(z)} \quad \text{and} \quad \frac{P_{2n+1}(z)}{Q_{2n+1}(z)}$$

in the whole plane except the semiaxis $\mathbb{R}_-$ and the analytic form of the limits of the convergents of $S(z)$. But there was still much for Stieltjes to do in order to make the proofs rigorous. The Stieltjes–Vitali theorem, as we have already discussed, was one part of this; another part was that in order to find the analytic forms of $F(z)$ and $F_1(z)$ it was—as we shall see later—necessary for Stieltjes to extend the notion of a definite integral.

The Case Where $\sum a_n < \infty$

Half a year later, still in 1892 [Hermite 1905, Letter 349, Oct. 1892], Stieltjes discovered that when the coefficients a_n of the Stieltjes continued fraction $S(z)$ form a convergent series the limits of the convergents of the continued fraction $S(z)$ turn out to be analytic functions representable by a sum of simple fractions. Indeed if

$$\sum_1^\infty a_n < \infty,$$

the following four limits exist,

$$\begin{aligned} \lim P_{2n}(z) &= p(z) \\ \lim Q_{2n}(z) &= q(z) \\ \lim P_{2n+1}(z) &= p_1(z) \\ \lim Q_{2n+1}(z) &= q_1(z), \end{aligned}$$

and the limit functions p, q, p_1, and q_1 are holomorphic in $\mathbb{C}$. He was also able to show that one can write

$$p(z) = c\left(1 + \frac{z}{l_1}\right)\left(1 + \frac{z}{l_2}\right)\left(1 + \frac{z}{l_3}\right)\cdots$$

$$q(z) = c'\left(1 + \frac{z}{m_1}\right)\left(1 + \frac{z}{m_2}\right)\left(1 + \frac{z}{m_3}\right)\cdots$$

and that the zeros of p and q are real and separate from each other in the following way:

$$0 < m_1 < l_1 < m_2 < l_2 < m_3 < \cdots \quad -m_i \text{ root of } q(z), \ -l_i \text{ root of } p(z)$$

(i.e., the zeros of $q(z)$ stay apart from each other; they do not accumulate at any point). Therefore it is possible to develop the fraction

$$\frac{p(z)}{q(z)}$$

in simple fractions

$$\frac{p(z)}{q(z)} = \frac{M_1}{z + m_1} + \frac{M_2}{z + m_2} + \frac{M_3}{z + m_3} + \cdots \quad m_k \in \mathbb{R}_+. \tag{11}$$

A similar result holds for

$$\frac{p_1(z)}{q_1(z)}.$$

This is a very beautiful piece of mathematics, and Hermite who was thrilled by this discovery, wrote back to Stieltjes two days later:

> Vous êtes un merveilleux géomètre, les recherches nouvelles sur les fractions continues algébriques que vous me communiquez sont un modèle d'invention et d'élégance; ni Gauss, ni Jacobi ne m'ont jamais causé plus de plaisir. [Hermite 1905, letter 350, Oct. 1892]

As we saw above Stieltjes had earlier in the year claimed that the limits of the convergents of $S(z)$ could be written in the form (7), but as we can see now this is obviously not true in the case where $\Sigma\, a_n < \infty$ because the sum

$$\sum \frac{M_i}{z + m_i}$$

in (11) cannot be written as a Riemann integral.

The Analytic Form of the Limit Functions $F(z)$, $F_1(z)$

But what about the case where $\Sigma\, a_n$ diverges? It could not be treated in the same neat way as when $\Sigma\, a_n < \infty$, because one cannot be sure of the behavior of the singularities of the convergents. In fact expansions of the form

$$\sum \frac{M_i}{z + x_i}$$

are generally impossible because the poles of the convergents may have finite limit points; they may even accumulate at each point of $\mathbb{R}_-$. So in order to find the analytic form of $F(z)$ and $F_1(z)$ it seemed natural to try to find it as an integral expression, both because of the behavior of the poles and because it was well known that the integral

$$\int_0^\infty \frac{f(u)}{z + u}\, du,$$

where $f(u) \geq 0$ for $u \in (0, \infty)$ and where

$$c_k \int_0^\infty u^k f(u)\, du$$

existed for all k, could be transformed into a continued fraction of the form $S(z)$. In October 1892 Stieltjes had the solution and wrote to Hermite:

> Dans le second cas, où la serie $\Sigma\, a_n$ est divergente, le résultat est aussi simple, mais pour l'énoncer dans tout sa simplicité, il faut d'abord quelques préliminaires, il est nécessaire d'élargir un peu la notion de l'intégrale définie. . . . la fraction continue est convergente (il n'y a pas lieu de distinguer les réduites d'ordre pair et impair) dans tout le plan, excepté la partie négative de l'axe réel. C'est là, en général, une ligne singulière, et il est impossible de continuer la fonction analytique en franchissant cette ligne. Mais ce qui est surtout remarquable c'est la forme analytique sous forme d'intégrale définie qu'on peut donner à cette fonction . . . . [Hermite 1905, Letter 351, Oct. 1892]

Instead of then telling Hermite how to deal with the case where the series $\Sigma\, a_n$

diverges Stieltjes returned to the previously discussed case where $\Sigma\, a_n < \infty$ and showed that also in this case the limits of the convergents can be expressed as an integral in his generalized sense. He defined two decreasing functions φ and φ_1 by

$$\begin{aligned}
\varphi(u) &= M_1 + M_2 + M_3 + \cdots + M_n + \cdots && (0 < u < m_1) \\
\varphi(u) &= \quad\;\; M_2 + M_3 + \cdots + M_n + \cdots && (m_1 < u < m_2) \\
\varphi(u) &= \qquad\quad\; M_3 + \cdots + M_n + \cdots && (m_2 < u < m_3) \\
& \quad \cdots
\end{aligned}$$

and

$$\begin{aligned}
\varphi_1(u) &= N_0 + N_1 + N_2 + \cdots + N_n + \cdots && (0 < u < n_1) \\
\varphi_1(u) &= \quad\;\; N_1 + N_2 + \cdots + N_n + \cdots && (n_1 < u < n_2) \\
\varphi_1(u) &= \qquad\quad\; N_2 + \cdots + N_n + \cdots && (n_2 < u < n_3) \\
& \quad \cdots,
\end{aligned}$$

where $-m_i$ are the roots of $q(z)$ and $-n_i$ the roots of $q_1(z)$, and M_i and N_i are the numerators in the simple fractional expansion

$$\frac{p(z)}{q(z)} = \sum \frac{M_i}{z + m_i}; \qquad \frac{p_1(z)}{q_1(z)} = \sum \frac{N_i}{z + n_i}.$$

Stieltjes then defined

$$\begin{aligned}
\int_a^b f(x)\, d\varphi(x) = \lim_{n=\infty} \{ & f(\xi_1)[\varphi(x_1) - \varphi(a)] \\
& + f(\xi_2)[\varphi(x_2) - \varphi(x_1)] \\
& + \ldots \\
& + \ldots \\
& + f(\xi_n)[\varphi(b) - \varphi(x_{n-1})]\},
\end{aligned}$$

where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a division of $[a, b]$ and $\xi_k \in [x_{k-1}, x_k]$. With this extension of the Riemann integral Stieltjes could indeed write

$$\frac{p(z)}{q(z)} = -\int_0^\infty \frac{d\varphi(u)}{z + u}; \qquad \frac{p_1(z)}{q_1(z)} = -\int_0^\infty \frac{d\varphi_1(u)}{z + u}.$$

To this Stieltjes remarked:

> ici intervient la généralisation de l'intégrale définie à laquelle j'ai fait allusion plus haut; $\varphi(x)$ étant une fonction qui varie toujours dans le même sens (mais qui peut ne pas admettre de dérivée et même avoir de discontinuité dans tout intervalle) [Hermite 1905, Letter 351, Oct. 1892]

So Stieltjes was now able to integrate a function with respect to a discontinuous mass distribution.

As mentioned above

$$\frac{p(z)}{q(z)} \quad \text{and} \quad \frac{p_1(z)}{q_1(z)}$$

have the same asymptotic expansion (10), which means that we necessarily have

$$c_n = \int_0^\infty u^n \, d\varphi(u) = \int_0^\infty u^n \, d\varphi_1(u), \tag{12}$$

so the mass distributions φ and φ_1 will solve the moment problem. But Stieltjes still did not talk about that, he just stated the relationship (12).

In the second case where $\Sigma \, a_n = \infty$ Stieltjes claimed that the continued fraction $S(z)$ is convergent and expressible in the form

$$-\int_0^\infty \frac{d\varphi(u)}{z+u}$$

and therefore is a holomorphic function in $\mathbb{C}\backslash\mathbb{R}_-$. About the nature of $\varphi(u)$ he said:

> $\varphi(u)$ est une fonction décroissante, mais qui, en général n'admet pas de dérivée et qui aura des discontinuités dans tout intervalle [Hermite 1905, Letter 351, Oct. 1892]

For a modern reader it feels quite natural to extend the integral in the way Stieltjes did, but in 1892 it may not have been such a natural way of thinking. In any case Hermite was confused; he did not quite understand what Stieltjes was doing and asked him if one should not write

$$\frac{p(z)}{q(z)} = -\int_0^\infty d\frac{\varphi(u)}{z+u}$$

instead of

$$\frac{p(z)}{q(z)} = -\int_0^\infty \frac{d\varphi(u)}{z+u}$$

and

$$\int_a^b f(x)\varphi(x)\,dx$$

instead of

$$\int_a^b f(x)\,d\varphi(x)$$

(see [Hermite 1905, Letter 352, Oct. 1892]).

It is hard to say what Hermite had in mind, but it is clear that he did not understand what Stieltjes wanted to do, and one can hardly blame him for that.

Stieltjes did not go into detail and he did not say a word about how to find the function $\varphi(u)$ in the case where $\Sigma\, a_n$ diverge. It is obvious that it cannot be constructed in the same way as in the case where $\Sigma\, a_n$ converge. This case is much more complicated and to see how Stieltjes did construct $\varphi(u)$ we have to look in his main paper [Stieltjes 1894–1895, 469 ff.]. The method he used here to construct φ (and φ_1) was independent of the behavior of the series $\Sigma\, a_n$, so this was a general method which could be used in both cases. He introduced the correspondence between mass distributions on $\mathbb{R}_+$ and increasing functions of x. Instead of using decreasing step functions as he did in the letter to Hermite he used the simple fraction expansion

$$\frac{P_{2n}(z)}{Q_{2n}(z)} = \sum_{i=1}^{n} \frac{M_i^{(2n)}}{z + x_i^{(2n)}} \tag{13}$$

of the convergents of $S(z)$ to construct a sequence of *increasing* step functions $\varphi_n(u)$,

$$\begin{aligned}
\varphi_n(u) &= 0 && 0 \leq u < x_1^{(2n)} \\
\varphi_n(u) &= M_1^{(2n)} && x_1^{(2n)} \leq u < x_2^{(2n)} \\
\varphi_n(u) &= M_1^{(2n)} + M_2^{(2n)} && x_2^{(2n)} \leq u < x_3^{(2n)} \\
&\;\;\cdot\,\cdot \\
&\;\;\cdot\,\cdot \\
&\;\;\cdot\,\cdot \\
\varphi_n(u) &= M_1^{(2n)} + \cdots + M_n^{(2n)} && x_n^{(2n)} \leq u < \infty,
\end{aligned}$$

and then

$$\frac{P_{2n}(z)}{Q_{2n}(z)} = \int_0^\infty \frac{1}{z+u}\, d\varphi_n(u).$$

In order to construct the function φ which "works" in the limit, Stieltjes introduced the concept of limit superior and limit inferior, an idea he borrowed from du Bois-Reymond (1831–1889) [Stieltjes 1894–1895, 479]. He then defined

$$\varphi(u) = \frac{\limsup \varphi_n(u) + \liminf \varphi_n(u)}{2}$$

and showed that

$$\lim_{n=\infty} \frac{P_{2n}(z)}{Q_{2n}(z)} = \int_0^\infty \frac{1}{z+u}\, d\varphi(u)$$

and similarly

$$\lim_{n=\infty} \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = \int_0^\infty \frac{1}{z+u}\, d\varphi_1(u),$$

where $\varphi_1(u)$ is defined in the same way as $\varphi(u)$, using the simple fraction expansion of the odd convergents, and for the same reason as before, the identity (12) is valid for φ and φ_1. When the series $\Sigma\ a_n$ diverges the continued fraction $S(z)$ converges, and it is not necessary to distinguish between odd and even convergents.

It is often believed that Stieltjes introduced his integral in order to solve the moment problem, but until this point he had not mentioned the moment problem at all; the letter to Hermite in which he wrote about his integral for the first time clearly shows that the purpose of the integral was to find the analytic form of $S(z)$. Actually Stieltjes' definition of the integral remained within the limits of the theory of continued fractions and was hardly noticed in the main stream of integral theory until 1909 (see [Hawkins 1970])—14 years after its publication. It was F. Riesz (1880–1956) who brought the Stieltjes integral into the main stream and turned Lebesgue's (1875–1941) interest toward it, when he in 1909 [Riesz 1909] discovered one of his famous theorems—the representation theorem—stating that bounded linear functionals A on $C([a, b])$ can be represented by a Stieltjes integral

$$A(f) = \int_a^b f(x)\, d\alpha(x),$$

where the definition of the Stieltjes integral is extended to functions α of bounded variation on $[a, b]$. In a foot note to this theorem Riesz [Riesz 1909] remarked that Julius König (1849–1914) had already used the Stieltjes integral in a course two years before Stieltjes. But he did not publish anything about it until 1897—two years after Stieltjes—and it was then only a little note in Hungarian "Mathematikai ès Tennészettudomànyi Ertesitö" (see [Riesz 1909]). It had no real influence on the development of mathematics, and I still think it is right to attribute the Stieltjes integral to Stieltjes.

Introduction of the Moment Problem

Stieltjes never talked about the moment problem in the correspondence with Hermite, but in September 1893 [Hermite 1905, Letter 385, Sept. 1893] he returned to that astonishing discovery he had made the year before, that the continued fractions give rise to functions $f(x) \not\equiv 0$ which satisfy the equations

$$\int_0^\infty x^n f(x)\, dx = 0 \qquad \text{for all } n \geq 0.$$

Stieltjes had noticed that if the interval of integration is replaced by a finite interval, say $[0, a]$, one cannot have

$$\int_0^a x^n f(x)\, dx = 0 \qquad \text{for all } n \geq 0$$

without having $f(x) \equiv 0$. He wrote:

Je réfléchis de temps en temps sur ces fonctions paradoxales (c'est ainsi que je les désigne provisoirement) $f(x)$ telle que

$$\int_0^\infty x^n f(x)\,dx = 0 \qquad (n = 0, 1, 2, \ldots).$$

[Hermite 1905, Letter 385, Sept. 1893]

By this time he had given those functions a special name. This indicates that he had become quite interested in this phenomenon and his examination of it led him to the moment problem.

He introduced the moment problem in his main paper in the part where he studied the convergence/divergence of $S(z)$ in the case where $\Sigma\, a_n$ converges [Stieltjes 1894–1895, 437–451]. He showed that in this case the continued fraction $S(z)$ will diverge. A comparison of the simple fraction expansion (13) of the convergents P_{2n}/Q_{2n}, P_{2n+1}/Q_{2n+1} of $S(z)$ and their expansion in power series (9) gives that

$$\begin{aligned} c_k &= \sum_{i=1}^{n} M_i^{(2n)}(x_i^{(2n)})^k \qquad && k = 0, 1, 2, \ldots, 2n-1 \\ c_k &= \sum_{i=0}^{n} N_i^{(2n+1)}(x_i^{(2n+1)})^k \qquad && k = 0, 1, 2, \ldots, 2n;\ x_0^{(2n+1)} = 0. \end{aligned} \tag{14}$$

Going to the limit we have, as we saw above,

$$\lim \frac{P_{2n}(z)}{Q_{2n}(z)} = \frac{p(z)}{q(z)} = \sum_{i=1}^{\infty} \frac{M_i}{z + m_i}$$

$$\lim \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = \frac{p_1(z)}{q_1(z)} = \frac{N_0}{z} + \sum_{i=1}^{\infty} \frac{N_i}{z + n_i},$$

where $M_i = \lim M_i^{(2n)}$, $m_i = \lim x_i^{(2n)}$, $N_i = \lim N_i^{(2n+1)}$, $n_i = \lim x_i^{(2n+1)}$.

The identity (14) also holds in the limit:

$$\begin{aligned} c_i &= \sum_{k=1}^{\infty} M_k m_k^i \qquad && (i = 0, 1, 2, \ldots) \\ c_0 &= \sum_{k=0}^{\infty} N_k && \\ c_i &= \sum_{k=1}^{\infty} N_k n_k^i \qquad && (i = 1, 2, 3, \ldots). \end{aligned} \tag{15}$$

This relationship gave Stieltjes the idea of considering a system (m_i, ξ_i) as a mass distribution along the positive real axis, where the mass m_i is located at the point whose distance to the origin is ξ_i. He defined the kth moment in relation to the origin as the sum

$$\sum m_i \xi_i^k$$

and then introduced the moment problem as follows:

> nous appellerons problème des moments le problème suivant: Trouver une distribution de masse positive sur une droite $(0, \infty)$ les moments d'ordre k ($k = 0, 1, 2, 3, \ldots$) étant donnés [Stieltjes 1894–1895, 449]

So (15) shows that the mass distribution (M_i, m_i) and (N_i, n_i) solve the moment problem corresponding to $S(z)$ in the case where $\Sigma\, a_n < \infty$. From these two solutions infinitely many solutions can be constructed by taking convex linear combinations.

The Stieltjes integral provides a tool to give a general solution of the moment problem no matter whether the series $\Sigma\, a_n$ converges or not because of the relationships

$$\lim \frac{P_{2n}(z)}{Q_{2n}(z)} = F(z) = \int_0^\infty \frac{1}{z+u}\, d\varphi(u)$$

$$\lim \frac{P_{2n+1}(z)}{Q_{2n+1}(z)} = F_1(z) = \int_0^\infty \frac{1}{z+u}\, d\varphi_1(u)$$

and

$$c_n = \int_0^\infty x^n\, d\varphi(u)$$
$$c_n = \int_0^\infty x^n\, d\varphi_1(u) \tag{16}$$

which show that both φ and φ_1 solve the moment problem corresponding to $S(z)$. If $\Sigma\, a_n$ diverges, φ and φ_1 will be equal and there is no need to distinguish between odd and even convergents. If $\Sigma\, a_n < \infty$ then $S(z)$ will diverge, $\varphi \not\equiv \varphi_1$, and the mass distributions characterized by φ and φ_1 are the ones given by the systems (M_i, m_i) and (N_i, n_i). Stieltjes showed that in the first case, φ is the only solution to the moment problem, and he called the moment problem determinate if $\Sigma\, a_n = \infty$ and indeterminate if $\Sigma\, a_n < \infty$. Stieltjes showed that if the sequence c_0, c_1, $c_2, \ldots$ arises as described above from a continued fraction $S(z)$ then the determinants

$$A_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{vmatrix} \quad \text{and} \quad B_n = \begin{vmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{vmatrix} \tag{17}$$

are positive for all $n \in \mathbb{N}$. In this case we know that the moment problem is solvable. By converting these arguments Stieltjes could show that

> The moment problem corresponding to a given sequence $c_0, c_1, c_2, \ldots$ is solvable if and only if $A_n > 0$ and $B_n > 0$ for all natural numbers n.

and this is the complete solution of the moment problem on the positive real axis—now called the Stieltjes moment problem.

He also managed to show that his definition of the determinate moment problem is equivalent to ours, namely that

> The moment problem is determinate if and only if it has exactly one solution.

Conclusion on Stieltjes

This work by Stieltjes is a very remarkable piece of mathematics. It contains several new ideas, which came to him in a strangely awkward way. As a matter of fact he had an intuitive understanding of the main results very early and then during his work with the proofs of those main theorems he hit on exciting things such as the "Stieltjes–Vitali" theorem, the Stieltjes integral, and in a certain way also the moment problem.

Seen in relation to the main purpose of this article the story told so far has shown how a new concept in mathematics, "the moment problem," came out of Stieltjes' development of the theory of analytic continued fractions. For Stieltjes the moment problem was interesting as a tool for determining the convergence/divergence of the continued fraction $S(z)$ which is equivalent to the determinateness/indeterminateness of the corresponding moment problem. Also his definition of determinateness/indeterminateness shows that it was not the moment problem as an independent theory that occupied him but the analysis of the continued fraction $S(z)$. Only in the last sections of his main paper did Stieltjes show an interest in the moment problem as an isolated phenomenon. He characterized some special solutions of the indeterminate moment problem which belongs to those called N-extremal solutions due to Rolf Nevanlinna (1895–1980). This can be regarded as the first little step toward a theory of its own. Still, as we shall see in the following, it took more than 20 years before the moment problem was fully emancipated.

AFTER STIELTJES

After Stieltjes there were attempts to extend his theory to other types of continued fractions. At this early stage the moment problem was closely connected to continued fractions, and the successors' work on Stieltjes' theory also tells something about the extension of the Stieltjes moment problem to the whole real axis and the difficulties mathematicians had to deal with in order to solve that problem.

In fact, it was not just a triviality to extend Stieltjes' theory. In 1903 the American mathematician Edward Burr Van Vleck (1833–1912) tried it without success [Van Vleck 1903]. Van Vleck wanted to find the necessary and sufficient conditions for the convergence of continued fractions of the same form as Stieltjes', i.e.,

$$V(z) = \cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cfrac{1}{a_4 + \cdots}}}}$$

His restriction on the coefficients a_n was that only the odd or the even coefficients had to be positive; the others could be positive as well as negative.

Van Vleck ran into the problem that he could not be sure that the odd and even convergents of $V(z)$ will converge. This fact made the question of convergence and divergence of $V(z)$ much more complicated and Van Vleck only succeeded in extending the theory to $V(z)$ in specific cases. This wrongly made him think that an extension was impossible:

> No necessary and sufficient test for convergence [of $V(z)$] has been found, and it seems quite probable that no such test is possible. [Van Vleck 1903, 299]

Van Vleck's work was almost a parallel to Stieltjes', and gave no new ways of treating the problems, so from a historical point of view Van Vleck's article is not interesting because of its mathematics, but more because of its failure to extend Stieltjes' theory. It showed that new methods were needed in order to overcome the difficulties.

THE METHOD OF CHOICE

The problem was solved 10 years later by J. Grommer in a paper [Grommer 1914] the main point of which was to characterize entire transcendental functions with only real zeroes. To prove his main result Grommer used Stieltjes' theory of continued fractions, but he could not just use it in Stieltjes' form but needed an extension to a larger class of continued fractions:

$$K(z) = \cfrac{k_1}{z + l_1 + \cfrac{k_2}{z + l_2 + \cfrac{k_3}{z + l_3 + \cdots}}} \tag{18}$$

where $z \in \mathbb{C}$, $k_n \in \mathbb{R}\backslash\{0\}$, and $l_n \in \mathbb{R}$. To a series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots \qquad c_0 \neq 0,\ c_n \in \mathbb{R} \tag{19}$$

corresponds a continued fraction of the form $K(z)$ if and only if the coefficients satisfy the condition that the determinants $A_n \neq 0$ (17) for all natural numbers n.

If $A_n > 0$ for all $n \in \mathbb{N}$ then it is possible to expand the convergents $U_n(z)/V_n(z)$ of $K(z)$ in simple fractions

$$\frac{U_n(z)}{V_n(z)} = \sum_{\nu=1}^{n} \frac{N_\nu^{(n)}}{z + \lambda_\nu^{(n)}}, \qquad -\lambda_\nu^{(n)} \text{ are the roots of } V_n(z),\ N_\nu^{(n)} > 0\ \forall \nu.$$

To each convergent of $K(z)$ Grommer could then like Stieltjes form an increasing step function $\varphi_n(u)$ and write

$$\frac{U_n(z)}{V_n(z)} = \int_{-\infty}^{\infty} \frac{1}{z+u}\, d\varphi_n(u) \qquad z \in \mathbb{C}\backslash\mathbb{R}.$$

Grommer now had the same problem as Van Vleck: there was no guarantee that the convergents $U_n(z)/V_n(z)$ of $K(z)$ would converge!

Grommer showed that the way out of this problem was to use the "method of choice." As a matter of fact he found that it was always possible to choose a subsequence

$$\frac{U_{n_h}(z)}{V_{n_h}(z)}$$

that will converge uniformly on compact subsets and moreover its limit can be represented analytically by

$$\lim_{h=\infty} \frac{U_{n_h}(z)}{V_{n_h}(z)} = \int_{-\infty}^{\infty} \frac{d\psi(u)}{z+u} = \int_{-\infty}^{\infty} \frac{d\chi(u)}{z+u} = \int_{-\infty}^{\infty} \frac{d\phi(u)}{z+u},$$

where

$$\psi(u) = \limsup_{h=\infty} \varphi_{n_h}(u), \qquad \chi(u) = \liminf_{h=\infty} \varphi_{n_h}(u),$$

$$\phi(u) = \frac{p\psi(u) + q\chi(u)}{p+q}, \qquad p \geq 0, q \geq 0, p + q > 0.$$

Grommers "method of choice" is very important in the development of the moment problem, because it ensures that if only the determinants A_n are positive for all $n \in \mathbb{N}$, then one can, as we shall see, always find a solution to the moment problem extended to the whole real axis. This "method of choice" was the tool by which Hamburger in 1920 could develop a general extension of the Stieltjes moment problem. The very idea of choice is due to Hilbert:

> Der Gedanke der Auswahl tritt erst in der Theorie der unendlich vielen Veränderlichen von Hilbert auf. [Grommer 1914, 137].

THE MOMENT PROBLEM ESTABLISHED AS A THEORY

The German mathematician Hans Ludwig Hamburger (1889–1956) introduced the extension of the moment problem to the whole real axis in 1919. Hamburger's main purpose was to determine the convergence/divergence of continued fractions $K(z)$ (18), and $S(z)$ (4) when it exists, corresponding to a given power series of the form (19) by looking directly at the sequence $(c_n)_{n\geq 0}$ instead of the coefficients of the continued fractions [Hamburger 1919]. Hamburger was inspired by his countryman Oskar Perron (1880–1975) who in 1913 gave a criterion by which one could decide whether the moment problem corresponding to a given continued fraction $S(z)$ was determinate or not by operating directly with the c_n's [Perron 1913].

In this paper Hamburger showed a stronger version of Perron's result and at the end of the paper he extended the result to sequences $(c_n)_{n\geq 0}$ that satisfy only the condition $A_n > 0\ \forall n \geq 0$, where A_n is defined in (17). Hamburger could apply Grommer's method of choice to the corresponding continued fraction $K(z)$. This gave him a subsequence of convergents

$$K_{n_\nu}(z)$$

satisfying

$$K_{n_\nu}(z) \rightarrow \int_{-\infty}^{\infty} \frac{d\varphi(u)}{z+u}.$$

He then claimed that the integrals

$$\int_{-\infty}^{\infty} u^n \, d\varphi(u) \tag{20}$$

existed for all n and equal c_n. He did not prove it in this paper but wrote in a footnote:

> Der Beweis dieser Behauptung, die unseres Wissens bisher noch nicht ausgesprochen worden ist, bietet keine prinzipielle Schwierigkeiten, und soll and andrer Stelle veröffentlicht werden. [Hamburger 1919, 213]

On the basis of (20) he then introduced what is now called the Hamburger moment problem:

> Die Frage nach einer nirgends abnehmenden Funktion $\varphi(u)$ mit der Eigenschaft, dass die Integrale
>
> $$\int_{-\infty}^{\infty} u^n \, d\varphi(u) = c_n$$
>
> sind, wollen wir das Momentproblem im weiteren Sinne nennen. [Hamburger 1919, 214].

He also introduced the modern definition of determinateness and indeterminateness, which does not depend on the corresponding continued fractions.

The next year, in 1920, Hamburger published the first part of the extensive work "Über eine Erweiterung des Stieltjesschen Momentproblems" [Hamburger 1920–1921]. This was the first profound and complete treatment of the moment problem. From being primarily a tool for the determination of convergence/divergence of continued fractions it now became established as a theory of its own. In order to solve the generalized moment problem and give criteria of determinateness/indeterminateness Hamburger used continued fractions, the Stieltjes integral theory, and Grommer's method of choice, so in a certain why he stayed on the trail pointed out by Stieltjes. One of his most important tools both in the construction of a solution to the Hamburger moment problem and to prove a criterion of determinateness was the so-called generalized convergent $K_n(z; t)$ of the continued fraction $K(z)$:

$$K_n(z; t) = \cfrac{k_1}{z + l_1 + \cfrac{k_2}{z + l_2 + \dots + \cfrac{k_{n-1}}{z + l_{n-1} + \cfrac{k_n}{z + l_n + t}}}}$$

With this tool he solved the moment problem completely:

> The moment problem corresponding to a power series (19) has a solution if and only if the determinants $A_n > 0$ (17) for all $n \in \mathbb{N}$ [Hamburger 1920–1921, 289].

In order to establish criteria for determinateness he introduced, inspired by Georg Hamel (1877–1954) [Hamel 1918], another concept of convergence of $K(z)$, which he called complete convergence. The continued fraction $K(z)$ converges completely towards the function $f(z)$ if for every $\varepsilon > 0$ and for every closed bounded set $B \subseteq \mathbb{C} \backslash \mathbb{R}$ one can find an $N \in \mathbb{N}$, which only depends upon ε and B, such that

$$|K_n(z; t) - f(z)| \leq \varepsilon$$

for all $n \geq N$, all $t \in \mathbb{R}$, and all $z \in B$. [Hamburger 1920–1921, 289]. He could then show that the moment problem corresponding to a power series with positive determinants A_n is determinate if and only if $K(z)$ converges completely.

Hamburger was the last one who investigated the moment problem entirely within the theory of continued fractions, but he did suggest other possibilities. One of them was based on a redefinition of the concept of "complete convergence" which led to a characterization of determinateness building upon certain circle considerations. Later this became very widespread. Indeed Nevanlinna used them two years later and today they play an important role in the theory of the moment problem. But Hamburger preferred to work directly with the continued fractions instead of the more geometrical interpretation of complete convergence which, he said,

> dienen nur dazu, dem Leser eine Anschauung von der Eigenart dieses Begriffes zu geben und werden im Folgenden nicht mehr benutzt werden. Der Leser kann daher ohne Schaden für das Verständnis des Folgenden gleich mit der Lektüre des Abschnittes 3 dieses Paragraphen auf Seite 292 fortfahren. [Hamburger 1920–1921, 290]

Hamburger also suggested another way of dealing with the moment problem, that, as he put it himself:

> Ermöglicht eine andere mehr funktionentheoretische Formulierung des Momentproblems, die das selbständige Interesse, das dieses Problem beansprucht, besser hervortreten lässt. [Hamburger 1920–1921, 275]

In fact, he showed that if the integral

$$f(z) = \int_{-\infty}^{\infty} \frac{1}{z + u}\, d\varphi(u) \tag{21}$$

converges uniformly in every bounded closed subset of $\mathbb{C}\backslash\mathbb{R}$ and if the function $f(z)$ has the asymptotic development

$$\beta(z) = \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{z^{n+1}} \qquad \text{for } z = iy, y \rightarrow \infty$$

then all the moments

$$\int_{-\infty}^{\infty} u^n\, d\varphi(u)$$

exist and are equal to c_n; conversely, if all the moments exist, then the function (21) is analytic in every bounded closed subset of $\mathbb{C}\backslash\mathbb{R}$ and has the asymptotic development $\beta(z)$ for $|z| \rightarrow \infty$ in the angles

$$\delta \leq \arg z \leq \pi - \delta, \qquad -\pi + \delta \leq \arg z \leq -\delta. \tag{22}$$

The next main contribution to the general theory of the moment problem given by Nevanlinna in 1922 used the function-theoretic interpretation of the moment problem, and from then on continued fractions were no longer the main way of dealing with the theory.

THE FIRST TREATMENT OF THE MOMENT PROBLEM WITHOUT CONTINUED FRACTIONS

In 1920 (published 1921) the German mathematician Felix Hausdorff (1868–1942) solved the moment problem on a bounded interval almost by accident. This is called the Hausdorff moment problem and consists of finding an increasing function $\chi(u)$ defined on [0, 1] such that

$$\mu_n = \int_0^1 u^n\, d\chi(u) \qquad n = 0, 1, 2, \ldots,$$

where $(\mu_n)_{n\geq 0}$ is a given sequence of real numbers.

Hausdorff almost stumbled over the solution of this problem in connection with a work on summability methods [Hausdorff 1921]. He wanted to examine matrix operations

$$A_p = \sum_m \lambda_{p,m} a_m$$

between sequences (a_n), (A_n), where $\lambda = (\lambda_{p,m})_{p,m=0,1,2,\dots}$ is a matrix with only a finite number of $\lambda_{p,m} \neq 0$ for every p. He called the matrix λ "konvergenzerhaltende" (C-matrix in the following) if it would transform convergent sequences (a_n) into convergent sequences (A_n), the limit of which need not be the same as the limit of (a_n). If a C-matrix could be written as

$$\lambda = \rho^{-1}\mu\rho,$$

where

$$\mu = \begin{pmatrix} \mu_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \qquad \rho = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 1 & -3 & 3 & -1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

μ is called a C-sequence. It turned out that for what is called Cesàro (resp. Hölder) summability the C-sequences exist and can be represented as

$$\mu_n = \alpha \int_0^1 u^n(1-u)^{\alpha-1}\,du \qquad (\alpha > 0)$$

$$\nu_n = \frac{1}{\Gamma(\alpha)} \int_0^1 u^n \left(\log\frac{1}{u}\right)^{\alpha-1} du \qquad (\alpha > 0)$$

which means that they are moment sequences corresponding to the density functions $\alpha(1-u)^{\alpha-1}$ and $1/\Gamma(\alpha)(\log 1/u)^{\alpha-1}$, respectively. Here Hausdorff realized the connection to the moment problem [Hausdorff 1921, 84] and using Stieltjes' theory for continued fractions he solved the Hausdorff moment problem. In the same breath he mentioned that the problem could be solved in a very simple manner, without extensive algebraic and function-theoretic preparations, and he did so, in an article from 1923, "Momentprobleme für ein endliches Intervall" [Hausdorff 1923]. This was the first treatment of moment problems without any appearance of continued fractions at all.

Hausdorff used the newly developed functional analysis to characterize the moment sequences $(\mu_n)_{n\geq 0}$ for increasing functions $\chi(u)$ on $[0, 1]$. To every polynomial $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ he assigned the number

$$Mf(x) = \alpha_0\mu_0 + \alpha_1\mu_1 + \cdots + \alpha_n\mu_n.$$

The Hausdorff moment problem then becomes a matter of finding an increasing function $\chi(u)$ such that

$$\int_0^1 f(x)\, d\chi(x) = Mf$$

for every polynomial $f(x)$. In modern terminology, Hausdorff had defined a linear functional M on the set of polynomials and asked if this could be represented as a Stieltjes integral. He was not the only one who used functional analysis in order to cope with the moment problem. As we shall see later, Marcel Riesz (1886–1969) used it also.

Hausdorff showed that his moment problem could be solved if and only if

$$\mu_{m,n} = \mu_m - \binom{n}{1}\mu_{m+1} + \binom{n}{2}\mu_{m+2} - \cdots + (-1)^n \mu_{m+n} \geq 0 \qquad \text{for all } m \text{ and } n.$$

We have now seen how the moment problem came out of the theory of continued fractions and how 20 years later Hamburger's work established it as a theory of its own. Hausdorff's work was the first in which the moment problem was completely disconnected from its roots—the continued fractions—and simultaneously the same happened for the moment problem on the whole real axis.

THE GENERAL THEORY OF THE MOMENT PROBLEM FINALLY SEPARATES FROM CONTINUED FRACTIONS

Stieltjes' and Hamburger's moment problems were separated from continued fractions in Rolf Nevanlinna's paper "Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentproblem" [Nevanlinna 1922]. Its title indicates that its primary concern was bounded holomorphic functions. He examined what conditions a real sequence $(c_n)_{n\geq 0}$ must satisfy in order that there exist a function f which maps the upper half plane holomorphically into the lower half plane and allows the asymptotic expansion

$$f \approx \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}} \qquad |z| \to \infty$$

in the angles (22). It turns out that the sequences in question are precisely the Hamburger moment sequences. With Nevanlinna the moment problem entered into a new area of mathematics, namely complex function theory. He did use a few results from the theory of continued fractions, but they were not the tools by which he got his main results. Two new things came out of Nevanlinna's work: a formula by which one can determine all the solutions to a given indeterminate moment problem, and a characterization of what today is called the N-extremal solutions, which are very special extreme points in the convex set of solutions.

In 1923 the moment problem became connected with the theory of functional analysis, which was growing rapidly at that time. It was Marcel Riesz who, inspired by the work of his brother Frederic Riesz, especially his representation theorem, was the first one to solve the Hamburger moment problem with the use of functional analysis [Riesz 1923]. He showed that the moment problem corresponding to a sequence

$$c_0, c_1, c_2, \ldots$$

has a nondecreasing solution $\varphi(t)$ if and only if the linear functional

$$T\colon \{f(t) | f \in \mathbb{R}[t]\} \rightarrow \mathbb{R}$$

defined by

$$T(t^n) = c_n$$

is positive. In the proof of this theorem he used what we now call Hahn–Banach methods. Actually he used this "Hahn–Banach" method already in 1918 in a lecture given in Stockholm, but he did not write anything about it until this paper came out in 1923. Another new important theorem proved by Riesz was a characterization of the N-extremal solutions:

φ is an N-extremal solution if and only if the polynomials are dense in $L^2(\mathbb{R}, \varphi)$.

With the work of Nevanlinna and Riesz the moment problem freed itself from continued fractions and moved into other fields of mathematics—complex function theory and functional analysis—and today it no longer bears the marks of its origin.

NOTES

1. For further information on the theory of the moment problem see for example [Shohat & Tamarkin 1943; Akhiezer 1965].
2. For further information on the work of Chebyshev and Bienaymé see [Heyde & Seneta 1977].

REFERENCES

Akhiezer, N. 1965. *The classical moment problem*. Edinburg/London: Oliver & Boyd.

Bienaymé, I. 1853. Considérations à l'appui de la découverte de Laplace sur la loi de probabilité dans la méthode des moindres carré. *Comptes Rendus Académie des Sciences, Paris* **37,** 309–326; also in 1867, *Journal de Mathematiques pures et appliquées* (2) **12,** 158–176.

Cauchy, A. 1821. *Cours d'Analyse*. Paris: Debure. Also in 1882–1938. *Oeuvres complètes d'Augustin Cauchy*, II, vol. 3. Publiées sous la direction scientifique de l'Académie des Sciences et sous les auspices de M. le ministre de l'instruction publique. Paris: Gauthier-Villars.

Chebyshev, P. 1874. Sur les valeurs limites des intégrales. *Journal de Mathematiques pures et appliquées* (2) **19,** 157–160.

——— 1887. Sur deux théorèmes relatifs aux probabilités. All page references are to the reprint 1890–1891. *Acta Mathematica* **14,** 305–315.

Euler, L. 1748. *Introductio in analysin infinitorum*. Lausanne: Marcus-Michaelis Bousquet.

Grommer, J. 1914. Ganze transzendente Funktionen mit lauter reellen Nullstellen. *Journal für die reine und angewandte Matematik* **144,** 114–166.

Hamburger, H. 1919. Beiträge zur Konvergenztheorie der Stieltjesschen Kettenbrüche. *Mathematische Zeitschrift* **4,** 186–222.

——— 1920–1921. Über eine Erweiterung des Stieltjesschen Momentproblems. *Mathematische Annalen* **81,** 235–319; **82,** 120–164, 168–187.

Hamel, G. 1918. Über einen limitär-periodischen Kettenbruch. *Archiv der Mathematik und Physik* **27,** 37–43.

Hausdorff, F. 1921. Summationsmethoden und Momentfolgen. *Mathematische Zeitschrift* **9,** 74–109, 280–299.

——— 1923. Momentprobleme für ein endliches Intervall. *Mathematische Zeitschrift* **16,** 220–248.

Hawkins, T. 1970. *Lebesgue's theory of integration; Its origin and development.* New York: Chelsea.

Hermite, Ch. 1905. *Correspondance d'Hermite et de Stieltjes,* I and II. Paris: Gauthier-Villars.

Heyde, C., & Seneta, E. 1977. *I. J. Bienaymé; Statistical theory anticipated.* New York: Springer.

Laguerre, E. 1879. Sur L'intégrale $\int_x^\infty e^{-x}x^{-1}\,dx$. *Bulletin de la société mathématique de France* **7,** 72–81.

Nevanlinna, R. 1922. Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentproblem. *Annales Academiae Scientiarum Fennicae* **18,** 5, 1–53.

Perron, O. 1913. Erweiterung eines Markoffschen Satzes über die Konvergenz gewisser Kettenbrüche. *Mathematische Annalen* **74,** 545–554.

Poincaré, H. 1886. Sur les intégrales irrégulières des équations linéaires. *Acta Mathematica* **8,** 295–344.

Riesz, F. 1909. Sur les opérations fonctionelles linéaires. *Comptes Rendus Académie des sciences, Paris* **149,** 974–977.

Riesz, M. 1923. Sur le problème des moments. Troisième note. *Arkiv för matematik, Astronomi och Fysik* **17,** 1–52.

Shohat, J., & Tamarkin, J. 1943. *The problem of moments.* New York: Amer. Math. Soc.

Stern, M. 1847. Über die Kennzeichen der Convergenz eines Kettenbruchs. *Journal für die reine und angewandte Mathematik* **37,** 255–272.

Stieltjes, T. 1886. Recherches sur quelques séries semi-convergentes. *Annales Scientifiques de l'École Normale Supérieure* **3,** 3, 201–258.

——— 1894–1895. Recherches sur les fractions continues. *Annales de la Faculté des Sciences de l'Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques* (*1*) **8,** 1–122; **9,** 5–47. Also in 1914–1918. *Oeuvres complètes de Thomas Jan Stieltjes.* M. M. W. Kapteyn, J. C. Kluyver, E. F. Van de Sande Bakhuyzen. Eds., II, pp. 402–566. Groningen: Noordhoff. All page references are to *Oeuvres complètes.*

Van Vleck, E. 1903. On an extension of the 1894 memoir of Stieltjes. *Transactions of the American Mathematical Society* **4,** 297–332.