Congruences for Convolutions of Hilbert Modular Forms

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Abstract

Let $\mathbf{f}$ be a primitive, cuspidal Hilbert modular form of parallel weight. We investigate the Rankin convolution L-values $L(\mathbf{f}, \mathbf{g}, s)$, where $\mathbf{g}$ is a theta-lift modular form corresponding to a finite-order character. We prove weak forms of Kato's 'false Tate curve' congruences for these values, of the form predicted by conjectures in non-commmutative Iwasawa theory.

1 Introduction

In recent years there has been much interest in the generalisation of Iwasawa theory to non-abelian field extensions. Let p be an odd prime, E an elliptic curve defined over $\mathbb{Q}$, and $F_\infty/\mathbb{Q}$ a p-adic Lie extension. In the paper [4], Coates et al conjecture the existence of a non-abelian p-adic L-function in $K_1(\mathbb{Z}_p[[G]]_{\mathcal{S}^*})$ which interpolates the twisted L-functions $L(E, \rho, s)$ at $s = 1$ (modified by certain simple factors). Here, ρ ranges over the set of Artin representations of $G = \mathrm{Gal}(F_\infty/\mathbb{Q})$, and $\mathbb{Z}_p[[G]]_{\mathcal{S}^*}$ is the localisation of $\mathbb{Z}_p[[G]]$ at a certain Ore set $\mathcal{S}^*$. More general conjectures of this nature were made by Fukaya and Kato in [11].

Attacking this appears very difficult in general. However, in the case of the 'false Tate curve' extension of $\mathbb{Q}$, Kato proves in [14] that the existence of the non-abelian p-adic L-function is equivalent to a set of strong congruences between certain abelian p-adic L-functions. Investigating these conjectural congruences is the motivation for our paper, and we will explain them further below.

The false Tate curve extension is defined by

$$\mathbb{Q}_{FT} := \bigcup_{n \geq 1} \mathbb{Q}\left(\mu_{p^n},\ \sqrt[p^n]{\Delta}\right)$$

where μ_{p^n} denotes the group of p^n-th roots of unity, and Δ is a p-power free integer. The Galois group $G_{FT} := \mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q})$ is a semi-direct product of two p-adic Lie groups of dimension one:

$$G_{FT} \;\cong\; \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \;\lhd\; \mathrm{GL}_2(\mathbb{Z}_p).$$

This group has a unique self-dual representation of dimension $p^n - p^{n-1}$ (as is shown in [3] for example) which we denote by $\rho_{n,\mathbb{Q}}$. Putting $K_n = \mathbb{Q}(\mu_{p^n})$, this may be written

$$\rho_{n,\mathbb{Q}} \;=\; \mathrm{Ind}_{K_n}^{\mathbb{Q}}\, \chi_n$$

for a one-dimensional character χ_n of $\mathrm{Gal}(K_n(\sqrt[p^n]{\Delta})/K_n)$. In fact, all irreducible representations of G_{FT} have the form $\rho_{n,\mathbb{Q}} \otimes \psi$ for some $n \geq 0$ and some finite-order character ψ of $U^{(n)}$, where $U^{(n)} \cong \ker(\mathbb{Z}_p^\times \twoheadrightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times)$.

The structure of these Artin representations allows us to use the theory of Hilbert modular forms to make further progress. Let us write F_n for the maximal real subfield of K_n. We have a Hilbert modular form over F_n obtained as a theta-lift of χ_n (as defined in [16, §5]) which we denote by $\mathbf{g}_{\rho_n}$, identifying it with the two-dimensional induced representation $\rho_n := \mathrm{Ind}_{K_n}^{F_n} \chi_n$. Additionally, we know by the work of Wiles et al that $E/\mathbb{Q}$ is modular, and has an associated cusp form f_E. If we write $\mathbf{f}_E$ for the automorphic base-change of f_E to the field F_n, then the non-abelian twist $L(E, \rho_{n,\mathbb{Q}}, s)$ is essentially equal to the Rankin convolution $L(\mathbf{f}_E, \mathbf{g}_{\rho_n}, s)$.

This approach was used by Bouganis and V. Dokchitser in [3] to prove algebraicity properties for these L-values. It was then used in [5] by Delbourgo and the author to construct an abelian p-adic L-function $\mathbf{L}_p(E, \rho_n) \in \mathbb{Z}_p[[U^{(n)}]]$ interpolating the values $L(E, \rho_{n,\mathbb{Q}} \otimes \psi, 1)$ for characters $\psi : U^{(n)} \to \overline{\mathbb{Q}}^\times$.

In this case, Kato proved in [14] that the existence of the non-abelian p-adic L-function is equivalent to a family of congruences between the elements $\mathbf{L}_p(E, \rho_n)$. To be precise, there exists a map

$$\Theta_{G,\mathcal{S}^*} : K_1(\mathbb{Z}_p[[G]]_{\mathcal{S}^*}) \longrightarrow \prod_{n \geq 0} \mathrm{Quot}(\mathbb{Z}_p[[U^{(n)}]])^\times$$

whose image contains a sequence $(a_n)_{n \geq 0}$ if and only if

$$\prod_{1 \leq j \leq n} N_{j,n} \left(\frac{a_j}{N_{0,j}(a_0)} . \frac{\varphi \circ N_{0,j-1}(a_0)}{\varphi(a_{j-1})} \right)^{p^j} \equiv 1 \mod p^{2n} \qquad \text{for all } n \in \mathbb{N}.$$

Here we must explain that $N_{i,j} : \mathbb{Z}_p[[U^{(i)}]]^\times \to \mathbb{Z}_p[[U^{(j)}]]^\times$ denotes the norm map, and $\varphi : \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the ring homomorphism induced by the p-power map on $\mathbb{Z}_p^\times$.

In the case $p = 3$, the first level of these congruences for $a_n = \mathbf{L}_p(E, \rho_n)$ has been verified by Bouganis in [2]. In particular, he uses deep results of Wiles to prove a relationship between the motivic period $\Omega_E^+ \Omega_E^-$ appearing in the conjectures, and an automorphic period associated to the modular form f_E.

Delbourgo and the author also proved a congruence of this form (using Hilbert modular forms as described above) but modulo a smaller power of p. We did this for semistable elliptic curves in [5], and extended our results to the case of CM curves in [6].

In this paper, we will show that one may take a primitive Hilbert modular form $\mathbf{f}$ of arbitrary parallel weight, and achieve similar results for the convolution $L(\mathbf{f}, \mathbf{g}_{\rho_n}, s)$ at all critical values. Let $\mathbf{f}$ be a primitive, cuspidal Hilbert modular form over F_n, with parallel weight $k \geq 2$, conductor $\mathfrak{c}(\mathbf{f})$ and Hecke character η. We write $\mathfrak{p}$ for the unique prime ideal of $\mathcal{O}_{F_n}$ above p. To state our main results, we must impose the following hypotheses.

Hypothesis (Ord): $\mathfrak{p}$ does not divide $\mathfrak{c}(\mathbf{f})$ or $\Delta\mathcal{O}_F$, and $\mathfrak{c}(\mathbf{f}) + \Delta\mathcal{O}_F = \mathcal{O}_F$. Further, the Fourier coefficient $C(\mathfrak{c}(\mathbf{f}), \mathbf{f})$ is non-zero and $C(\mathfrak{p}, \mathbf{f})$ is a p-adic unit.

Hypothesis (Cong): there exists no congruence modulo $\mathfrak{M}_{\mathbb{C}_p}$ between $\mathbf{f}$ and another Hilbert modular form which lies outside the $\mathbf{f}$-isotypic component of $\mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$. Here, $\mathfrak{M}_{\mathbb{C}_p}$ denotes the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$.

Theorem 1.1 *Suppose that $p > k-2$, that $\mathbf{f}$ has rational Fourier coefficients and that Hypotheses (Ord) and (Cong) are satisfied. For each critical value $1 \le r \le k-1$ there exists a unique element $\mathbf{L}_p(\mathbf{f}, \rho_n, r) \in \mathbb{Z}_p[[U^{(n)}]]$ with the property*

$$\psi\Big(\mathbf{L}_p(\mathbf{f}, \rho_n, r)\Big) \;=\; \frac{\epsilon_{F_n}(\rho_n \otimes \psi, 1-r)_{\mathfrak{p}}}{\alpha(\mathfrak{p})^{f(\rho_n \otimes \psi, \mathfrak{p})}} \;\times\; \frac{P_{\mathfrak{p}}(\rho_n \otimes \psi^{-1},\, \alpha(\mathfrak{p})^{-1} p^{r-1})}{P_{\mathfrak{p}}(\rho_n \otimes \psi,\, \alpha(\mathfrak{p})\, p^{-r})}$$

$$\times\; \frac{\Psi_S(\mathbf{f}, \mathbf{g}^{\iota}_{\rho_n \otimes \psi}, r)}{D_F^{k-2}\; \langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}}$$

for each character $\psi : U^{(n)} \to \mathbb{C}^{\times}$.

Here we write $\Psi_S(\mathbf{f}, \mathbf{g}^{\iota}_{\rho_n \otimes \psi}, r)$ for the completed Rankin convolution of $\mathbf{f}$ and $\mathbf{g}^{\iota}_{\rho_n \otimes \psi}$ with the Euler factors at primes dividing $p.\Delta\mathcal{O}_F$ removed, and $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$ for the Petersson self-product of $\mathbf{f}$. The other terms in the formula are defined in §2 and §6.

Theorem 1.2 *Suppose that $\mathbf{f}$ satisfies the same hypotheses as in Theorem 1.1. Put $a_j = \mathbf{L}_p(\mathbf{f}, \rho_j, r)$ for some $1 \le r \le k-1$. Then we have the congruence*

$$\prod_{1 \le j \le n} N_{j,n} \left(\frac{a_j}{N_{0,j}(a_0)} . \frac{\varphi \circ N_{0,j-1}(a_0)}{\varphi(a_{j-1})} \right)^{p^j} \equiv 1 \mod p^{n+1}$$

for each $n \ge 0$.

In Theorems 1.1 and 1.2, we make the assumption $p > k-2$. This is forced on us for technical reasons, and we expect it is not neccessary for the results to hold. In §7 we will give two numerical examples for $p = 3$, in which the first level congruences from Theorem 1.2 hold even without this condition.

However, we will also present an example in which Hypothesis (Cong) fails and the congruences fail. This is not surprising, given a well-known connection between our complex period $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$ and the congruence primes of $\mathbf{f}$ (we discuss this at the end of §6).

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2 Convolutions of Hilbert Modular Forms

In §2 and §3 we review some results on convolutions of Hilbert modular forms, and the Rankin-Selberg method. In §4 we will go on to construct p-adic measures which interpolate the critical values of these L-functions. Our principal reference for this theory is Panchishkin's book [15], and our methods are heavily based on his.

Let p be an odd prime, and let F be a totally real number field. We fix a prime ideal $\mathfrak{p}$ of F which lies above p. Consider two Hilbert modular forms $\mathbf{f}$

and $\mathbf{g}$ defined over F, which are primitive and cuspidal. Suppose that $\mathbf{f}$ has parallel weight $k \geq 2$, conductor $\mathfrak{c}(\mathbf{f})$ and Hecke character η; also suppose that $\mathbf{g}$ has parallel weight one, conductor $\mathfrak{c}(\mathbf{g})$ and character ω. Weights of Hilbert modular forms will always be assumed parallel in what follows.

We will adopt the following hypotheses throughout:

$$\mathfrak{p} \nmid \mathfrak{c}(\mathbf{f}), \quad \mathfrak{c}(\mathbf{f}) + \mathfrak{c}(\mathbf{g}) = \mathcal{O}_F, \quad C\big(\mathfrak{c}(\mathbf{f}), \mathbf{f}\big) \neq 0$$

where we have written $C(\mathfrak{m}, \mathbf{f})$ for the Fourier coefficient of $\mathbf{f}$ at the ideal $\mathfrak{m}$. Further, we adopt the hypothesis that $\mathbf{f}$ is $\mathfrak{p}$-ordinary: by this we mean that $C(\mathfrak{p}, \mathbf{f})$ is a p-adic unit.

The L-series associated to $\mathbf{f}$ is defined by

$$L(\mathbf{f}, s) \ := \ \sum_{\mathfrak{m}} C(\mathfrak{m}, \mathbf{f})\, N(\mathfrak{m})^{-s}.$$

As we assume $\mathbf{f}$ is primitive, we also have the Euler product expression:

$$L(\mathbf{f}, s) \ = \ \prod_{\mathfrak{q}} \big(1 \ - \ C(\mathfrak{q}, \mathbf{f})\, N(\mathfrak{q})^{-s} \ + \ \eta(\mathfrak{q})\, N(\mathfrak{q})^{k-1-2s}\big)^{-1}.$$

We write $\Psi(\mathbf{f}, \mathbf{g}, s)$ for the completed Rankin convolution of $\mathbf{f}$ and $\mathbf{g}$, which is given by

$$\Psi(\mathbf{f}, \mathbf{g}_\rho, s) \ := \ \left(\frac{\Gamma(s)}{(2\pi)^s}\right)^{2[F:\mathbb{Q}]} L_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})}(2s-k-1, \eta\omega) \sum_{\mathfrak{a}} C(\mathfrak{a}, \mathbf{f})\, C(\mathfrak{a}, \mathbf{g})\, N(\mathfrak{a})^{-s}.$$

These L-series only converge for $\mathrm{Re}(s)$ sufficiently large, but both may be continued to holomorphic functions on the whole complex plane, and satisfy functional equations of the usual form (see [16] for example).

For an integral ideal $\mathfrak{a}$ of $\mathcal{O}_F$ we have two linear operators $|\mathfrak{a}$ and $|U(\mathfrak{a})$ on the space $\mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$, which may be defined by their effect on the Fourier coefficients of any Hilbert modular form $\mathbf{h}$:

$$C(\mathfrak{m}, \mathbf{h}\big|\mathfrak{a}) \ = \ C(\mathfrak{m}\mathfrak{a}^{-1}, \mathbf{h}) \qquad \text{and} \qquad C(\mathfrak{m}, \mathbf{h}|U(\mathfrak{a})) = C(\mathfrak{m}\mathfrak{a}, \mathbf{h}),$$

where we put $C(\mathfrak{m}, \mathbf{h}) = 0$ whenever the ideal $\mathfrak{m}$ is not integral. Here we also have an involution $J_\mathfrak{a}$ on this space, defined in [15, Chapter 4].

Theorem 2.1 *Put $d = [F : \mathbb{Q}]$, and let $\mathfrak{d}$ be the different of $F/\mathbb{Q}$. Then, for any $\mathbf{F} \in \mathcal{S}_k(\mathfrak{c}(\mathbf{F}), \eta)$ and $\mathbf{G} \in \mathcal{S}_1(\mathfrak{c}(\mathbf{G}), \omega)$, we have the following integral representation for the Rankin convolution:*

$$\begin{aligned}\Psi(\mathbf{F}, \mathbf{G}, s) \ &= \ (-1)^{d(s-k+1)}\, 2^{dk}\, i^{d(1-k)}\, N\big(\mathfrak{c}(\mathbf{F})\mathfrak{c}(\mathbf{G})\mathfrak{d}^2\big)^{(k-1)/2-s}\, N\big(\mathfrak{c}(\mathbf{G})\big)^{1-k/2} \\ &\times \ \Big\langle \mathbf{F}^\iota, \Big(\mathbf{G}|J_{\mathfrak{c}(\mathbf{F})\mathfrak{c}(\mathbf{G})} \cdot E_{k-1}(s-k+1, \eta\omega^{-1})\Big)\Big|U\big(\mathfrak{c}(\mathbf{G})\big) \circ J_{\mathfrak{c}(\mathbf{F})}\Big\rangle_{\mathfrak{c}(\mathbf{F})},\end{aligned}$$

where $E_{k-1}(s, \psi)$ is the Eisenstein series specified in [15, chapter 4].

Proof. We take the integral representation from [16, 4.32], then apply the trace operator $\mathrm{Tr}_{\mathfrak{c}(\mathbf{F})}^{\mathfrak{c}(\mathbf{F})\mathfrak{c}(\mathbf{G})}$. Using the identity

$$V\big|\,\mathrm{Tr}_{\mathfrak{a}}^{\mathfrak{a}\mathfrak{b}} \ = \ N(\mathfrak{b})^{1-k/2}\ V\big|J_{\mathfrak{a}\mathfrak{b}} \circ U(\mathfrak{b}) \circ J_{\mathfrak{a}};$$

we obtain the desired result, following the same calculation as in [15, page 136, §4.4].

We return to our primitive Hilbert modular forms $\mathbf{f}$ and $\mathbf{g}$. For a prime ideal $\mathfrak{q}$ of F we will always write

$$1 - C(\mathfrak{q}, \mathbf{f})X + \eta(\mathfrak{q})N(\mathfrak{q})^{k-1}X^2 \;=\; \big(1 - \alpha(\mathfrak{q})X\big)\big(1 - \alpha'(\mathfrak{q})X\big),$$

for the factorisation of the local polynomial of $\mathbf{f}$ at v. For the prime $\mathfrak{p}$ we choose $\alpha(\mathfrak{p})$ to be the root which is a p-adic unit, with $\alpha'(\mathfrak{p})$ the non-unit root (which we can do as we assumed $\mathbf{f}$ to be ordinary at $\mathfrak{p}$). Similarly, for $\mathbf{g}$ we will write

$$\begin{aligned} 1 - C(\mathfrak{q}, \mathbf{g})X + \omega(\mathfrak{q})X^2 &= \big(1 - \beta(\mathfrak{q})X\big)\big(1 - \beta'(\mathfrak{q})X\big), \\ 1 - \overline{C(\mathfrak{q}, \mathbf{g})}X + \omega^{-1}(\mathfrak{q})X^2 &= \big(1 - \hat{\beta}(\mathfrak{q})X\big)\big(1 - \hat{\beta}'(\mathfrak{q})X\big). \end{aligned}$$

The convolution of $\mathbf{f}$ and $\mathbf{g}$ may be written as the following Euler Product:

$$L_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})}(2s-k-1, \eta\omega) \sum_{\mathfrak{a}} C(\mathfrak{a}, \mathbf{f})\, C(\mathfrak{a}, \mathbf{g})\, N(\mathfrak{a})^{-s} \;=\; \prod_{\mathfrak{q}} (1 - \alpha(\mathfrak{q})\beta(\mathfrak{q})N(\mathfrak{q})^{-s})^{-1}$$

$$\times\; (1 - \alpha(\mathfrak{q})\beta'(\mathfrak{q})N(\mathfrak{q})^{-s})^{-1}\, (1 - \alpha'(\mathfrak{q})\beta(\mathfrak{q})N(\mathfrak{q})^{-s})^{-1}\, (1 - \alpha'(\mathfrak{q})\beta'(\mathfrak{q})N(\mathfrak{q})^{-s})^{-1}.$$

Let us now fix a squarefree ideal $\mathfrak{m}_0$ which is divisible by $\mathfrak{p}$ and all the primes dividing $\mathfrak{c}(\mathbf{g})$. We define the $\mathfrak{m}_0$-stabilisation of $\mathbf{f}$ to be

$$\mathbf{f}_0 \;:=\; \sum_{\mathfrak{a}|\mathfrak{m}_0} \mu(\mathfrak{a})\, \alpha'(\mathfrak{a})\,.\,\mathbf{f}\big|\mathfrak{a}$$

where μ is the Möbius function on ideals. This definition is equivalent to the identity

$$L(\mathbf{f}_0, s) \;=\; L(\mathbf{f}, s) \;\times\; \prod_{\mathfrak{q}|\mathfrak{m}_0} (1 - \alpha'(\mathfrak{q})N(\mathfrak{q})^{-s}).$$

We also define $\mathbf{g}_{\mathfrak{m}_0} \in \mathcal{S}(\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2, \omega)$ by

$$\mathbf{g}_{\mathfrak{m}_0} \;=\; \sum_{\mathfrak{a}|\mathfrak{m}_0} \mu(\mathfrak{a}) \cdot \mathbf{g}\big|U(\mathfrak{a}) \circ \mathfrak{a}.$$

Equivalently, $\mathbf{g}_{\mathfrak{m}_0}$ is the non-primitive Hilbert modular form whose Fourier coefficients are given by

$$C(\mathfrak{n}, \mathbf{g}_{\mathfrak{m}_0}) = \begin{cases} C(\mathfrak{n}, \mathbf{g}) & \text{if } \mathfrak{n} \text{ and } \mathfrak{m}_0 \text{ are coprime;} \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of this section, we will write $\mathfrak{m}'$ for an auxiliary ideal supported on the primes dividing $\mathfrak{m}_0$, such that $\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2|\mathfrak{m}'$. We substitute

$$\mathbf{F} \;=\; \mathbf{f}_0 \;\in\; \mathcal{S}_k\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}_0, \eta\big), \qquad \text{and} \qquad \mathbf{G} \;=\; \mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'} \;\in\; \mathcal{S}_1\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}', \omega^{-1}\big)$$

into Theorem 2.1 to obtain the formula

$$\Psi\big(\mathbf{f}_0, \mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'}, s\big) \;=\; (-1)^{dk}\, 2^{dk}\, i^{d(1-k)}\, N(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-k/2}\, N\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{d}^2\big)^{(k-1)/2-s}$$

$$\times\quad \Big\langle \mathbf{f}_0^{\iota},\; \mathbf{g}_{\mathfrak{m}_0}\,.\,E_{k-1}(s-k+1, \eta\omega^{-1})\Big|U(\mathfrak{m}'\mathfrak{m}_0^{-1}) \circ J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}_0} \Big\rangle_{\mathfrak{c}(\mathbf{f})\mathfrak{m}_0}.$$

We define a linear functional

$$\begin{array}{rcl} \mathcal{L}_F \ : \ \mathcal{M}_k(\mathfrak{c}\mathfrak{m}_0,\eta) & \longrightarrow & \mathbb{C} \\ \Phi & \longmapsto & \dfrac{\left\langle \mathbf{f}_0^\iota, \Phi|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}_0}\right\rangle_{\mathfrak{c}(\mathbf{f})\mathfrak{m}_0}}{\langle \mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}. \end{array}$$

We also write $\mathcal{H}ol$ for the holomorphic projection operator constructed in [15, page 138, §4.6]. This operator maps the space $\tilde{\mathcal{M}}_k(\mathfrak{c}\mathfrak{m}_0,\eta)$ of C^∞-Hilbert modular forms to $\mathcal{M}_k(\mathfrak{c}\mathfrak{m}_0,\eta)$, and is related to the Petersson inner product by the formula

$$\langle \mathbf{h},\Phi\rangle_{\mathfrak{c}\mathfrak{m}_0} \ = \ \langle \mathbf{h},\mathcal{H}ol(\Phi)\rangle_{\mathfrak{c}\mathfrak{m}_0}$$

for any $\mathbf{h}$ in $\mathcal{S}_k(\mathfrak{c}\mathfrak{m}_0,\eta)$. Holomorphic projection was not required in [5], where the only weight considered was $k=2$; however the Eisenstein series $E_{k-1}(s-k+1)$ will be non-holomorphic at some critical values if $k>2$. For convenience we will put

$$\Phi(\mathbf{g},s) \ := \ \mathcal{H}ol\left(\mathbf{g}_{\mathfrak{m}_0}E_{k-1}(s,\eta\omega^{-1})\right).$$

Then, applying holomorphic projection to the above formula for $\Psi(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'},s)$ and writing it in terms of the linear functional $\mathcal{L}_F$, we obtain

$$\begin{aligned} \frac{\Psi(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}\mathfrak{m}'},s)}{\langle \mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} \ &= \ (-1)^{dk}\,2^{dk}i^{d(1-k)}\,N(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-k/2}N(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{d}^2)^{(k-1)/2-s} \\ &\times \quad \mathcal{L}_F\left(\Phi(\mathbf{g},s-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\right). \end{aligned}$$

Next we will rewrite the convolution $\Psi\big(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'},s\big)$ in terms of $\Psi(\mathbf{f},\mathbf{g}^\iota,s)$. We define the contragredient Euler factor by

$$\begin{aligned} \mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^\iota,s) \quad := \quad & \prod_{v|\mathfrak{m}_0}(1-\alpha'(v)\hat{\beta}(v)N(v)^{-s})(1-\alpha'(v)\hat{\beta}'(v)N(v)^{-s}) \\ & \times (1-\alpha^{-1}(v)\beta(v)N(v)^{s-1})(1-\alpha^{-1}(v)\beta'(v)N(v)^{s-1}). \end{aligned}$$

Lemma 2.2 *We have the formula*

$$\begin{aligned} \Psi(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}\mathfrak{m}'},s) \quad &= \quad N\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'}{\mathfrak{c}(\mathbf{g})}\right)^{1/2-s}\frac{\alpha(\mathfrak{m}')}{\alpha\big(\mathfrak{c}(\mathbf{g})\big)} \\ &\times \quad \Lambda(\mathbf{g})\ \ C\big(\mathfrak{c}(\mathbf{f}),\mathbf{f}\big)\ \ \mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^\iota,s)\ \ \Psi(\mathbf{f},\mathbf{g}^\iota,s). \end{aligned}$$

Proof. Let us put $\mathbf{F}=\mathbf{f}_0$ and $\mathbf{G}=\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}\mathfrak{m}'}$. Quoting [15, page 125, 1.22] we have the identity

$$\mathbf{h}\big|J_{\mathfrak{m}\mathfrak{c}(\mathbf{h})} \ = \ N(\mathfrak{m})^{\mathrm{wt}(\mathbf{h})/2}\left(\mathbf{h}\big|J_{\mathfrak{c}(\mathbf{h})}\right)\big|\mathfrak{m}$$

which holds for all Hilbert modular forms $\mathbf{h}$ and ideals $\mathfrak{m}$.

By assumption, $\mathfrak{m}'=\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2\mathfrak{r}$ for some integral ideal $\mathfrak{r}$ which is supported on the primes dividing $\mathfrak{m}_0$. The level of $\mathbf{g}_{\mathfrak{m}_0}$ is $\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2$, so we apply the above identity to obtain

$$\mathbf{G} \ = \ \mathbf{g}_{\mathfrak{m}_0}\big|J_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2\mathfrak{r}} \ = \ N\big(\mathfrak{c}(\mathbf{f})\mathfrak{r}\big)^{1/2}\left(\mathbf{g}_{\mathfrak{m}_0}\big|J_{\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2}\right)\Big|\mathfrak{c}(\mathbf{f})\mathfrak{r}$$

Let us write $\tilde{\mathbf{g}} = \mathbf{g}_{\mathfrak{m}_0}\big|J_{\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2}$ so that $\mathbf{G} = N(\mathfrak{c}(\mathbf{f})\mathfrak{r})^{1/2}\,\tilde{\mathbf{g}}\,|\mathfrak{c}(\mathbf{f})\mathfrak{r}$. Then we have

$$\begin{aligned}
\Psi(\mathbf{F},\mathbf{G},s) &= N\big(\mathfrak{c}(\mathbf{f})\mathfrak{r}\big)^{1/2}\Psi\big(\mathbf{f}_0,\tilde{\mathbf{g}}|\mathfrak{c}(\mathbf{f})\mathfrak{r},s\big)\\
&= N\big(\mathfrak{c}(\mathbf{f})\mathfrak{r}\big)^{1/2-s}\Psi\big(\mathbf{f}_0|U(\mathfrak{c}(\mathbf{f})\mathfrak{r}),\tilde{\mathbf{g}},s\big)\\
&= N\big(\mathfrak{c}(\mathbf{f})\mathfrak{r}\big)^{1/2-s}\alpha(\mathfrak{r})C\big(\mathfrak{c}(\mathbf{f}),\mathbf{f}\big)\Psi\big(\mathbf{f}_0,\tilde{\mathbf{g}},s\big).
\end{aligned}$$

The second equality here follows from the identity

$$\Psi\big(\mathbf{f},\mathbf{g}|\mathfrak{a},s\big) \;=\; N(\mathfrak{a})^{-s}\,\Psi\big(\mathbf{f}|U(\mathfrak{a}),\mathbf{g},s\big)$$

which is clear when we recall that $C(\mathfrak{m},\mathbf{g}|\mathfrak{a}) \;=\; C(\mathfrak{m}\mathfrak{a}^{-1},\mathbf{g})$ and $C(\mathfrak{m},\mathbf{f}|U(\mathfrak{a})) = C(\mathfrak{m}\mathfrak{a},\mathbf{f})$. The third equality is deduced from the fact that $\mathbf{f}_0|U(\mathfrak{q}) = \alpha(\mathfrak{q})\,\mathbf{f}_0$ when $\mathfrak{q}$ divides $\mathfrak{m}_0$ or $\mathfrak{c}(\mathbf{f})$, as well as the observation $\alpha(\mathfrak{c}(\mathbf{f})) = C(\mathfrak{c}(\mathbf{f}),\mathbf{f})$. We also have

$$\begin{aligned}
\Psi(\mathbf{f}_0,\tilde{\mathbf{g}},s) &= \Psi(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{g})\mathfrak{m}_0^2},s)\\
&= N(\mathfrak{m}_0)^{1-2s}\;\alpha(\mathfrak{m}_0)^2\;\Lambda(\mathbf{g})\;\mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^{\iota},s)\;\Psi(\mathbf{f},\mathbf{g}^{\iota},s),
\end{aligned}$$

as quoted from [15, page 130, proposition 3.5]. Combining these two equations, we get the desired result.

Combining Lemma 2.2 with our integral representation, we obtain

$$\begin{aligned}
\frac{\Psi\big(\mathbf{f}_0,\mathbf{g}_{\mathfrak{m}_0}|J_{\mathfrak{c}(\mathbf{f})\mathfrak{m}'},s\big)}{\langle\mathbf{f},\mathbf{f}\rangle} &= (-1)^{dk}\,2^{dk}\imath^{d(1-k)}\,N(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-k/2}\,N(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{d}^2)^{(k-1)/2-s}\\
&\times\quad \mathcal{L}_F\left(\Phi(\mathbf{g},s-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\right)\\
= \quad N\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'}{\mathfrak{c}(\mathbf{g})}\right)^{1/2-s} &\Lambda(\mathbf{g})\;\frac{\alpha(\mathfrak{m}')}{\alpha\big(\mathfrak{c}(\mathbf{g})\big)}\;C\big(\mathfrak{c}(\mathbf{f}),\mathbf{f}\big)\;\mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^{\iota},s)\;\frac{\Psi(\mathbf{f},\mathbf{g}^{\iota},s)}{\langle\mathbf{f},\mathbf{f}\rangle}
\end{aligned}$$

This implies

$$\begin{aligned}
\alpha\big(\mathfrak{c}(\mathbf{g})\big)^{-1}\;\Lambda(\mathbf{g})\;\mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^{\iota},s)\;\frac{\Psi(\mathbf{f},\mathbf{g}^{\iota},s)}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} &= \frac{(-1)^{dk}2^{dk}\imath^{d(1-k)}}{\alpha(\mathfrak{m}')\,C\big(\mathfrak{c}(\mathbf{f}),\mathbf{f}\big)}N(\mathfrak{m}'\mathfrak{m}_0^{-1})^{1-k/2}\\
\times\;N\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}'\mathfrak{d}^2\big)^{(k-1)/2-s}\;N\left(\frac{\mathfrak{c}(\mathbf{f})\mathfrak{m}'}{\mathfrak{c}(\mathbf{g})}\right)^{s-1/2}\quad &\mathcal{L}_F\Big(\Phi(\mathbf{g},s-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\Big)
\end{aligned}$$

and we get the following corollary.

Corollary 2.3 *We have the formula*

$$N\big(\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{s-1/2}\;\;\Lambda(\mathbf{g})\;\;\alpha\big(\mathfrak{c}(\mathbf{g})\big)^{-1}\;\;\mathrm{Eul}_{\mathfrak{m}_0}(\mathbf{g}^{\iota},s)\;\;\frac{\Psi(\mathbf{f},\mathbf{g}^{\iota},s)}{\langle\mathbf{f},\mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}$$

$$=\;\;\frac{(-1)^{dk}\,2^{dk}\,\imath^{d(1-k)}}{\alpha(\mathfrak{m}')\,C\big(\mathfrak{c}(\mathbf{f}),\mathbf{f}\big)}\;\;N\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}_0\mathfrak{d}^2\big)^{k/2-1}\;\;\mathcal{L}_F\Big(\Phi(\mathbf{g},s-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\Big).$$

This is a more general version of [5, corollary 11]. We observe that the left-hand side of this equation does not depend on the choice of $\mathfrak{m}'$, so neither does the right-hand side; this is the 'distribution property' which we will require in §4.

3 Fourier Expansions

We will need an explicit formula for the Fourier coefficients of the Hilbert modular form

$$\Phi(\mathbf{g}, s) \ := \ \mathcal{H}ol\left(\mathbf{g}_{\mathfrak{m}_0} E_{k-1}(s, \eta\omega^{-1})\right).$$

We now very briefly recall some facts about Fourier expansions of Hilbert modular forms (full details can be found in [15, chapter 4] or [16]). Let $h = |\tilde{Cl}_F|$ be the narrow ideal class number of the field F, and choose finite ideles $t_1, ..., t_h$ such that the set of ideals $\{\tilde{t}_\lambda : 1 \leq \lambda \leq h\}$ form a complete set of representatives for $\tilde{Cl}_F$, and are all coprime to $\mathfrak{m}_0$. Here we write $\tilde{t}_\lambda$ for the ideal of $\mathcal{O}_F$ generated by t_λ. A Hilbert modular form $\mathbf{f}$ may be naturally identified with an h-tuple $(f_1, ..., f_h)$ of modular forms on $\mathbb{H}^n$, where $\mathbb{H}$ denotes the complex upper half-plane. For $1 \leq \lambda \leq h$ we will refer to f_λ as the 'λ-component' of $\mathbf{f}$. This function has a Fourier expansion of the form

$$f_\lambda(z) \ = \sum_\xi a_\lambda(\xi)\, e_F(\xi z)$$

where the sum ranges over all $\xi \in \tilde{t}_\lambda$ which are totally positive (denoted '$0 \ll \xi$') and $\xi = 0$. Further, if the ideal $\mathfrak{m} = \xi\tilde{t}_\lambda^{-1}$ is integral, then $C(\mathfrak{m}, \mathbf{f}) = a_\lambda(\xi) N(\tilde{t}_\lambda)^{-k/2}$.

Let us write the λ-component of $\Phi(\mathbf{g}, s)$ as

$$\Phi(\mathbf{g}, s)_\lambda(z) \ = \sum_{0 \ll \xi \in \tilde{t}_\lambda} \phi_\lambda(\xi, \mathbf{g}, s) e_F(\xi z)$$

and the λ-component of $\mathbf{g}$ as

$$\mathbf{g}_\lambda(z) \ = \sum_{0 \ll \xi \in \tilde{t}_\lambda} b_\lambda(\xi) e_F(\xi z).$$

We quote [15, page 143, 5.8], specialising to our case of $\mathrm{wt}(\mathbf{g}) = 1$ as before: for an integer $1 \leq r \leq k-1$, we have

$$\begin{aligned}\phi_\lambda(\xi, \mathbf{g}, r-k+1) \ &= \ N(\tilde{t}_\lambda)^{\frac{k+1}{2}-1-r} \sum_{\xi=\xi_1+\xi_2} b_\lambda(\xi_1) \sum_{\tilde{\xi_2} = \tilde{b}\,\tilde{c}} \mathrm{sign}(N(\tilde{b}))^{k-2}\, N(\tilde{b})^{2r-k} \\ &\times \ (\eta\omega^{-1})(\tilde{c}) \prod_\nu P_{r-k+1}(\xi_{2,\nu}, \xi_\nu)\end{aligned}$$

where the second sum ranges over all $b \in \tilde{t}_\lambda^{-1}$ and $c \in \mathcal{O}_F$ such that $\tilde{\xi_2} = \tilde{b}\,\tilde{c}$. Here, $\xi_{2,\nu}$ denotes the image of ξ_2 under the real embedding $\nu : F \hookrightarrow \mathbb{R}$, and $P_s(\xi_{2,\nu}, \xi_\nu)$ denotes the polynomial

$$P_s(\xi_{2,\nu}, \xi_\nu) \ = \ \sum_{i=0}^{-s} (-1)^i \binom{-s}{i} \frac{\Gamma(k-1+s)}{\Gamma(k-1+s-i)} \frac{\Gamma(k-1-i)}{\Gamma(k-1)} \xi_{2,\nu}^{-s-i}\ \xi_\nu^{\,i}$$

where $s \leq 0$. For $s \in \mathbb{Z}$, this polynomial has rational coefficients. Now, recall that

$$C(\mathfrak{m}, \Phi(\mathbf{g}, s-k+1)) = \begin{cases} N(\tilde{t}_\lambda)^{-k/2}\ \phi_\lambda(\xi, \mathbf{g}, s-k+1) & \text{if } \mathfrak{m} = \xi\tilde{t}_\lambda^{-1} \text{ is integral;} \\ 0 & \text{if } \mathfrak{m} \text{ is not integral.} \end{cases}$$

Therefore,

$$\begin{aligned} C(\mathfrak{m}, \Phi(\mathbf{g}, s-k+1)) \;&=\; N(\tilde{t}_\lambda)^{-\frac{1}{2}-s} \sum_{\xi=\xi_1+\xi_2} b_\lambda(\xi_1) \sum_{\tilde{\xi_2}=\tilde{b}\tilde{c}} \operatorname{sign}(N(\tilde{b}))^k \, N(\tilde{b})^{2s-k} \\ &\times\;\; (\eta\omega^{-1})(\tilde{c}) \prod_\nu P_{s-k+1}(\xi_{2,\nu}, \xi_\nu) \end{aligned}$$

where $\mathfrak{m} = \xi\tilde{t}_\lambda^{-1}$ for $\xi \gg 0$ as above. If we substitute $\xi_2 = \xi - \xi_1$, we can write

$$\prod_\nu P_s(\xi_{2,\nu}, \xi_\nu) \;=\; (-1)^{ds} \, N(\xi_1)^{-s} \;+\; N(\xi) \;\times\; (\text{other terms})$$

and therefore

$$C\big(\mathfrak{m}, \Phi(\mathbf{g}, s-k+1)\big) \;=\; N(\tilde{t}_\lambda)^{-\frac{1}{2}-s} \Big(N(\xi)\, u_\lambda(\xi)$$

$$+\,(-1)^{d(s-k+1)} \sum_{\xi=\xi_1+\xi_2} b_\lambda(\xi_1)\; N(\xi_1)^{s-k+1} \sum_{\tilde{\xi_2}=\tilde{b}\tilde{c}} \operatorname{sign}(N(\tilde{b}))^k \; N(\tilde{b})^{2s-k}\; (\eta\omega^{-1})(\tilde{c}) \Big)$$

where $u_\lambda(\xi)$ is a linear combination of the coefficients $b_\lambda(\xi)$ and the values of the Hecke character $\eta\omega^{-1}$ (just as in [15, page 143, 5.9]).

4 Bounded p-adic Measures

Let us now fix an imaginary quadratic extension K/F, where F is our totally real field as before. We will construct a bounded measure on the group

$$\mathcal{G}(\mathfrak{M}_0) \;:=\; \operatorname{Gal}\big(K(\mathfrak{M}_0 p^\infty)/K\big)$$

where $\mathfrak{M}_0 = \mathfrak{m}_0\mathcal{O}_K$ and $K(\mathfrak{M}_0 p^\infty)$ denotes the maximal ray class field modulo $\mathfrak{M}_0 p^\infty$ over K.

For every finite-order character $\chi : \mathcal{G}(\mathfrak{M}_0) \to \mathbb{C}^\times$, we have a theta-lift Hilbert modular form $\mathbf{g} \in \mathcal{M}_1(\mathfrak{c}(\mathbf{g}), \omega)$ defined over F (see [16, §5]). We denote this by $\mathbf{g}_\rho$, identifying it with the induced Artin representation

$$\rho = \operatorname{Ind}_{\operatorname{Gal}(M/K)}^{\operatorname{Gal}(M/F)} \chi$$

where M/K is a finite extension through which the character χ factors. This identification is natural since $L(\mathbf{g}_\rho, s)$ is the same as the Artin L-function $L(\rho/F, s)$.

In what follows we will write $\epsilon_F(\rho, s)$ for the global epsilon factor that appears in the functional equation of $L(\rho/F, s)$; to be precise we have

$$\hat{L}(\rho, s) \;=\; \epsilon_F(\rho, s)\; \hat{L}(\rho^\vee, 1-s)$$

where $\hat{L}(\rho, s) = L_\infty(\rho, s)\, L(\rho, s)$ is the L-function with the Euler factors at infinite places included, and $\rho^\vee$ is the contragredient representation. We will also write $\mathfrak{c}(\rho)$ for the Artin conductor of ρ, and $\operatorname{Eul}_{\mathfrak{m}_0}(\rho^\vee, s)$ for the Euler factor $\operatorname{Eul}_{\mathfrak{m}_0}(\mathbf{g}_\rho^\iota, s)$ we defined in §2.

Proposition 4.1 *Given a primitive cusp form* $\mathbf{f} \in \mathcal{S}_k\big(\mathfrak{c}(\mathbf{f}), \eta\big)$ *there exists an algebraic-valued, bounded measure* $\mu_{\mathfrak{m}_0}(\mathbf{f}, r)$ *(for each* $1 \le r \le k-1$*) on* $\mathcal{G}(\mathfrak{M}_0)$ *taking the value*

$$\int_{\mathcal{G}(\mathfrak{M}_0)} \chi \, d\mu_{\mathfrak{m}_0}(\mathbf{f}, r) \;=\; \epsilon_F(\rho, 1-r)\ \mathrm{Eul}_{\mathfrak{m}_0}(\rho^\vee, r)\ \alpha\big(\mathfrak{c}(\rho)\big)^{-1}\ \frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}$$

at every finite-order character $\chi : \mathcal{G}(\mathfrak{M}_0) \to \mathbb{C}^\times$*, where* $\rho = \mathrm{Ind}_K^F \chi$*.*

Proof. Firstly, we know that these values are algebraic, by results from the key paper [16] of Shimura. It is a simple consequence of [16, 2.48] that

$$\Lambda(\mathbf{g}_\rho)\ N(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{d}^2)^{r-1/2}\ i^{[F:\mathbb{Q}]} \;=\; \epsilon_F(\rho, 1-r).$$

Further, by definition of the cusp form $\mathbf{g}_\rho$, the conductor $\mathfrak{c}(\mathbf{g}_\rho)$ is equal to $\mathfrak{c}(\rho)$. Therefore, by Corollary 2.3 we may write

$$\epsilon_F(\rho, 1-r)\ \frac{\mathrm{Eul}_{\mathfrak{m}_0}(\rho^\vee, r)}{\alpha\big(\mathfrak{c}(\rho)\big)}\ \frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}} \;=\; \gamma(\mathfrak{m}')\ \mathcal{L}_F\Big(\Phi(\mathbf{g}_\rho, r-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\Big)$$

for any $\mathfrak{m}'$ such that $\mathfrak{c}(\rho)\mathfrak{m}_0^2|\mathfrak{m}'$. Here we have written $\gamma(\mathfrak{m}')$ for the constant

$$\gamma(\mathfrak{m}') \;=\; \frac{(-1)^d\ 2^{dk}\ i^{dk}}{\alpha(\mathfrak{m}')\ C\big(\mathfrak{c}(\mathbf{f}), \mathbf{f}\big)} N\big(\mathfrak{c}(\mathbf{f})\mathfrak{m}_0\mathfrak{d}^2\big)^{k/2-1}.$$

Quoting [15, page 144, 5.11] we know that that the linear functional $\mathcal{L}_F$ may be written as a linear combination of Fourier coefficients:

$$\mathcal{L}_F(\Theta) \;=\; \sum_{\mathfrak{a}} \kappa_{\mathfrak{a}}\, C(\mathfrak{a}, \Theta),$$

where the $\kappa_{\mathfrak{a}}$ are algebraic numbers, fixed independently of Θ, and all but finitely many $\kappa_{\mathfrak{a}}$ are zero. As Panchishkin states in [15], this follows from a version of Atkin-Lehner theory for Hilbert modular forms. Therefore it suffices to prove an appropriate set of abstract 'Kummer congruences' for the Fourier coefficients

$$C\big(\mathfrak{a}, \Phi(\mathbf{g}_{\mathrm{Ind}\,\chi}, r-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\big)$$

as the character χ varies.

Applying the formula from the end of §3, we have

$$C\big(\mathfrak{a}, \Phi(\mathbf{g}_\rho, r-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\big) \;\equiv\; N(\tilde{t}_\lambda)^{-\frac{1}{2}-r}(-1)^{d(r-k+1)} \;\times$$

$$\sum_{\xi=\xi_1+\xi_2} b_\lambda(\xi_1)\ N(\xi_1)^{r-k+1} \sum_{\tilde{\xi_2}=\tilde{b}\tilde{c}} \mathrm{sign}(N(\tilde{b}))^k\ N(\tilde{b})^{2r-k}\ (\eta\omega^{-1})(\tilde{c}) \quad \mathrm{mod}\ N(\mathfrak{m}'\mathfrak{m}_0^{-1})$$

where $\mathfrak{a} = \xi\tilde{t}_\lambda^{-1}$. All the terms in this sum are p-integral, and the only ones which depend on χ are the Fourier coefficient $b_\lambda(\xi_1)$ and the value of the Hecke character ω. By definition of the theta-lift modular form $\mathbf{g}_{\mathrm{Ind}\,\chi}$, we may write

$$b_\lambda(\xi_1)N(\tilde{t}_\lambda)^{-1/2} \;=\; \sum_{\mathfrak{A}} \chi(\mathfrak{A}).$$

where the sum ranges over all ideals $\mathfrak{A}$ of $\mathcal{O}_K$ such that $N_{K/F}(\mathfrak{A}) = \xi_1 N(\tilde{t}_\lambda)^{-1}$. Further, the Hecke character ω is given by

$$\omega(\mathfrak{a}) \;=\; \theta_{K/F}(\mathfrak{a})\; \chi(\mathfrak{a}\mathcal{O}_K)$$

where $\theta_{K/F}$ is the quadratic Hecke character defined by

$$\theta_{K/F}(\mathfrak{q}) = \begin{cases} 1 & \text{if } \mathfrak{q} \text{ splits in } K/F \\ -1 & \text{if } \mathfrak{q} \text{ is inert in } K/F \\ 0 & \text{if } \mathfrak{q} \text{ ramifies in } K/F \end{cases}$$

for any prime ideal $\mathfrak{q}$ (these properties are easily verified from the discussion of theta-lifts in [16, §5]). Therefore we have

$$C\big(\mathfrak{a}, \Phi(\mathbf{g}_\rho, r{-}k{+}1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\big) \;\equiv\; \sum_{\xi_1,b,c,\mathfrak{A}} w(\xi_1,b,c)\; \chi(\mathfrak{A})\; \chi^{-1}(\tilde{c}\,\mathcal{O}_K) \bmod N(\mathfrak{m}'\mathfrak{m}_0^{-1})$$

for some algebraic numbers $w(\xi_1,b,c)$ which are all p-integral.

Suppose that we have coefficients $B_\chi \in \mathbb{C}_p$ (almost all zero) satisfying

$$\sum_\chi B_\chi\, \chi(\sigma) \;\in p^m\, \mathcal{O}_{\mathbb{C}_p}$$

for all $\sigma \in \mathcal{G}(\mathfrak{M}_0)$. For the finite set of characters χ such that $B_\chi \neq 0$, we choose $\mathfrak{m}'$ large enough that $p^m \mathfrak{c}(\mathbf{g}_{\mathrm{Ind}\,\chi})\mathfrak{m}_0^2 | \mathfrak{m}'$ for each χ. We also fix ideals $\mathfrak{C}_c$ such that $\chi(\mathfrak{C}_c) = \chi^{-1}(\tilde{c}\,\mathcal{O}_K)$ for each χ; it is clear this can be done, since the characters are all defined modulo $\mathfrak{M}_0\, p^R$ for R sufficiently large. Then we may write

$$\sum_\chi B_\chi\; C\big(\mathfrak{a}, \Phi(\mathrm{Ind}\,\chi, r-k+1)\big|U(\mathfrak{m}'\mathfrak{m}_0^{-1})\big)$$
$$\equiv\; \sum_\chi \sum_{\xi_1,b,c,\mathfrak{A}} B_\chi\; w(\xi_1,b,c)\; \chi(\mathfrak{A}\,\mathfrak{C}_c) \quad \bmod N(\mathfrak{m}'\mathfrak{m}_0^{-1}).$$

By assumption, $\sum_\chi B_\chi\, \chi(\mathfrak{A}\,\mathfrak{C}_c) \equiv 0 \mod p^m$; and we chose $\mathfrak{m}'$ divisible by p^m, so we conclude that the whole sum lies in $p^m\, \mathcal{O}_{\mathbb{C}_p}$. Thus, we have checked the Kummer congruences for these special values, and we conclude that they define a bounded p-adic measure on $\mathcal{G}(\mathfrak{M}_0)$.

5 Integrality of Special Values

In order to establish our set of congruences, we need to know that our p-adic measures take integral values. In fact, it appears that they become increasingly p-integral with the discriminant of the totally real field F. We now impose our hypothesis on the congruence properties of $\mathbf{f}$:

Hypothesis (Cong): there exists no congruence modulo $\mathfrak{M}_{\mathbb{C}_p}$ between $\mathbf{f}$ and another Hilbert modular form which lies outside the $\mathbf{f}$-isotypic component of $\mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$.

Here, when we refer to a congruence of Hilbert modular forms, we mean a congruence of their Fourier expansions. We will discuss why this hypothesis is necessary at the end of §6.

Proposition 5.1 *If $p > k-2$ and $\mathbf{f}$ satisfies Hypothesis (Cong), then we have*

$$\nu_p \left(\epsilon_F(\rho, 1-r) \, \frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f} \rangle} \right) \ \geq \ \nu_p \left(D_F^{k-2} \right)$$

for each integer $1 \leq r \leq k-1$, where ν_p is the p-adic valuation and D_F is the discriminant of F.

Proof. We have the integral representation

$$\begin{aligned} \Psi(\mathbf{f}, \mathbf{g}^\iota, s) \ &= \ (-1)^{d(s-k+1)} \, 2^{dk} \, i^{d(1-k)} \, N\big(\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})\mathfrak{d}^2\big)^{(k-1)/2-s} \, N\big(\mathfrak{c}(\mathbf{g})\big)^{1-k/2} \\ & \qquad \times \quad \Lambda(\mathbf{f}^\iota) \, \Big\langle \mathbf{f}, \Theta(s-k+1) \Big\rangle_{\mathfrak{c}(\mathbf{f})} \end{aligned}$$

where

$$\Theta(s) \ = \ \mathcal{H}ol\Big(\big(\mathbf{g}^\iota \big| J_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})}\big) \, . \, E_{k-1}(s, \eta\omega^{-1}) \Big) \Big| U\big(\mathfrak{c}(\mathbf{g})\big).$$

This follows after substituting $\mathbf{F} = \mathbf{f}$ and $\mathbf{G} = \mathbf{g}^\iota$ into Theorem 2.1 and applying the formula

$$\begin{aligned} \big\langle \mathbf{f}^\iota, \Theta | J_{\mathfrak{c}(\mathbf{f})} \big\rangle \ &= \ \big\langle \mathbf{f}^\iota | J_{\mathfrak{c}(\mathbf{f})}, \Theta \big\rangle \\ &= \ \Lambda(\mathbf{f}^\iota) \big\langle \mathbf{f}, \Theta \big\rangle. \end{aligned}$$

The identity above holds because $J_{\mathfrak{c}}$ is self-adjoint with respect to the Petersson inner product, and $\mathbf{f}^\iota | J_{\mathfrak{c}(\mathbf{f})} = \Lambda(\mathbf{f}^\iota)\, \mathbf{f}$ for any primitive $\mathbf{f}$.

We know $\Lambda(\mathbf{f}^\iota)$ is a root of unity, and we always assume that $\mathfrak{p} \nmid 2\mathfrak{c}(\mathbf{f})$; therefore we can write

$$\frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f} \rangle} \ = \ (\,p\text{–adic unit}) \times N\big(\mathfrak{d}^2\big)^{(k-1)/2-r} N\big(\mathfrak{c}(\mathbf{g})\big)^{1/2-r} \frac{\big\langle \mathbf{f}, \Theta(r-k+1) \big\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle}$$

for $1 \leq r \leq k-1$. Recall that

$$\epsilon_F(\rho, 1-r) \ = \ i^d \, \Lambda(\mathbf{g}_\rho) \, N\big(\mathfrak{c}(\mathbf{g}_\rho)\mathfrak{d}^2\big)^{r-1/2}.$$

Also, the norm of the different $\mathfrak{d}$ is equal to the absolute discriminant $|D_F|$, so

$$\epsilon_F(\rho, 1-r) \, \frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f} \rangle} \ = \ (\,p-\text{adic unit}) \ \times \ D_F^{k-2} \, \frac{\big\langle \mathbf{f}, \Theta(r-k+1) \big\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}}.$$

By choosing a basis for the finite-dimensional vector space $\mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$ which includes $\mathbf{f}$, we may write

$$\Theta(r-k+1) \ = \ c\,\mathbf{f} \ + \ \sum_{\mathbf{f}_i \neq \mathbf{f}} c_i \, \mathbf{f}_i \big| \mathfrak{b}_i$$

for scalars c and $c_i \in \overline{\mathbb{Q}}$ (almost all zero), and primitive forms $\mathbf{f}_i \in \mathcal{M}_k(\mathfrak{a}_i, \eta)$ such that $\mathfrak{a}_i\mathfrak{b}_i$ divides $\mathfrak{c}(\mathbf{f})$. We deduce that

$$\frac{\big\langle \mathbf{f}, \Theta(r-k+1) \big\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}} \ = \ c + \sum_{\mathbf{f}_i \neq \mathbf{f}} c_i \, \frac{\big\langle \mathbf{f}, \mathbf{f}_i \big| \mathfrak{b}_i \big\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}}.$$

For each i we have $\langle \mathbf{f}, \mathbf{f}_i \rangle_{\mathfrak{c}(\mathbf{f})} = 0$ as $\mathbf{f}$ and $\mathbf{f}_i$ are distinct primitive forms. This implies that $\big\langle \mathbf{f}, \mathbf{f}_i \big| \mathfrak{b}_i \big\rangle_{\mathfrak{c}(\mathbf{f})} = 0$ (one can see this from [16, proposition 4.13]) so we have

$$\frac{\big\langle \mathbf{f}, \Theta(r-k+1) \big\rangle_{\mathfrak{c}(\mathbf{f})}}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}} \;=\; c.$$

Therefore we obtain the equation

$$\epsilon_F(\rho, 1-r)\, \frac{\Psi(\mathbf{f}, \mathbf{g}_\rho^\iota, r)}{\langle \mathbf{f}, \mathbf{f} \rangle} \;=\; (\,p\,-\text{adic unit}) \;\times\; D_F^{k-2}\; c$$

and to prove the proposition it suffices to show that c is p-integral.

Suppose not; then $c^{-1} \equiv 0 \mod \mathfrak{M}_{\mathbb{C}_p}$. We have an explicit formula for the Fourier coefficients of

$$\Theta(r-k+1) \;=\; \mathcal{H}ol\Big(\big(\mathbf{g}^\iota \big| J_{\mathfrak{c}(\mathbf{f})\mathfrak{c}(\mathbf{g})}\big) \cdot E_{k-1}(r-k+1, \eta\omega^{-1}) \Big) \Big| U\big(\mathfrak{c}(\mathbf{g})\big)$$

almost identical to that for $C(\mathfrak{m}, \Phi(\mathbf{g}, r-k+1))$ in Section 3; they can be written as p-integral linear combinations of the polynomials $P_{r-k+1}(\xi_{2,\nu}, \xi_\nu)$. Since we assume $p > k-2$, these polynomials have p-integral coefficients, so it is clear that $\Theta(r-k+1)$ has a p-integral Fourier expansion. Therefore

$$c^{-1}\, \Theta(r-k+1) \;\equiv\; 0 \mod \mathfrak{M}_{\mathbb{C}_p}.$$

This means that

$$\begin{aligned} \mathbf{f} \;&=\; c^{-1}\, \Theta(r-k+1) \;-\; \sum_{\mathbf{f}_i \neq \mathbf{f}} c^{-1} c_i\; \mathbf{f}_i \big| \mathfrak{b}_i \\ &\equiv\; -\sum_{\mathbf{f}_i \neq \mathbf{f}} c^{-1} c_i\; \mathbf{f}_i \big| \mathfrak{b}_i \qquad \mod \mathfrak{M}_{\mathbb{C}_p}, \end{aligned}$$

and we see that $\mathbf{f}$ is congruent modulo $\mathfrak{M}_{\mathbb{C}_p}$ to an element of $\mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$ which does not lie in the $\mathbf{f}$-isotypic component. This contradicts Hypothesis (Cong) and we have proved the proposition.

6 False Tate Curve Congruences

We now restrict our attention to the setting of the introduction. We consider the p-power cyclotomic field $K_n = \mathbb{Q}(\mu_{p^n})$ and its maximal real subfield $F_n = K_n^+$, for an integer $n \geq 1$. The field F_n is totally real, and from now on we assume our primitive cusp form $\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}(\mathbf{f}), \eta)$ is defined over F_n. The prime p is totally ramified in the extension $F_n/\mathbb{Q}$, and we set $\mathfrak{p}$ to be the unique prime ideal of $\mathcal{O}_{F_n}$ above p.

We fix a false Tate curve extension of $\mathbb{Q}$, defined by

$$\mathbb{Q}_{FT} \;:=\; \bigcup_{n \geq 1} \mathbb{Q}\big(\mu_{p^n},\; \sqrt[p^n]{\Delta}\big)$$

where Δ is a p-power free integer. Recall from §1 that $G_{FT} := \mathrm{Gal}(\mathbb{Q}_{FT}/\mathbb{Q})$ has a unique irreducible self-dual representation $\rho_{n,\mathbb{Q}} = \mathrm{Ind}_{K_n}^{\mathbb{Q}}\, \chi_n$ of dimension $p^n - p^{n-1}$, for each $n \geq 1$, and all irreducible representations of G_{FT} may be written in the form $\rho_{n,\mathbb{Q}} \otimes \psi$ for some $n \geq 0$ and some finite-order character ψ of $U^{(n)}$.

As before, we have a theta-lift Hilbert modular form over F_n, corresponding to the Hecke character χ_n. We denote this by $\mathbf{g}_{\rho_n}$, identifying it with the two-dimensional Artin representation $\rho_n = \mathrm{Ind}_{K_n}^{F_n} \chi_n$. We point out that this notation is canonical, as $L(\mathbf{g}_{\rho_n}, s)$ is equivalent to the Artin L-series $L(\rho_n/F_n, s)$ and we have $L(\rho_n/F_n, s) = L(\rho_{n,\mathbb{Q}}, s)$ by the Artin formalism.

Additionally, for $0 \leq j \leq n$, we have the restricted character $\chi_{j,n} := \mathrm{Res}_{K_n}(\chi_j)$ and a corresponding induced representation $\rho_{j,n} := \mathrm{Ind}_{K_n}^{F_n} \chi_{j,n}$. In fact, the theta-lift modular form $\mathbf{g}_{\rho_{j,n}}$ coincides with the automorphic base-change of $\mathbf{g}_{\rho_j}$ from F_j to F_n.

Before we state the main result of this section, we must comment on our conventions for local epsilon factors: we follow those of Deligne from [7] (see also Tate [17]). Recall that we may write the global epsilon factor as a product:

$$\epsilon_F(\rho, s) \;=\; \prod_v \epsilon_F(\rho, s)_v$$

where v ranges over all places of F and $\epsilon_F(\rho, s)_v$ is a local epsilon factor at v. In fact, the local factor at v depends on a choice of additive character of F_v and a Haar measure on F_v. We assume we have fixed these as in [17, (3.5)] so that the above product formula holds. In particular, at the archimedean places we can use the standard characters and measures given by [17, (3.2.4) and (3.2.5)].

Proposition 6.1 *Suppose that $p > k-2$ and that $\mathbf{f}$ and satisfies Hypotheses (Cong) and (Ord). Suppose further that $\mathbf{f}$ has rational Fourier coefficients; then for each $0 \leq j \leq n$ there exists a unique element $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r) \in \mathbb{Z}_p[[U^{(n)}]]$ with the property*

$$\psi\Big(\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r)\Big) \;=\; \frac{\epsilon_{F_n}(\rho_{j,n} \otimes \psi, 1-r)_{\mathfrak{p}}}{\alpha(\mathfrak{p})^{f(\rho_{j,n}\otimes\psi, \mathfrak{p})}} \;\times\; \frac{P_{\mathfrak{p}}(\rho_{j,n} \otimes \psi^{-1},\, \alpha(\mathfrak{p})^{-1} p^{r-1})}{P_{\mathfrak{p}}(\rho_{j,n} \otimes \psi,\, \alpha(\mathfrak{p})\, p^{-r})}$$
$$\times\; \frac{\Psi_S(\mathbf{f}, \mathbf{g}^{\iota}_{\rho_{j,n}\otimes\psi}, r)}{D_F^{k-2}\;\langle \mathbf{f}, \mathbf{f}\rangle_{\mathfrak{c}(\mathbf{f})}}$$

for each character $\psi : U^{(n)} \to \mathbb{C}^{\times}$. Here we have written $\epsilon_F(\rho, 1-r)_{\mathfrak{p}}$ for the local epsilon factor at $\mathfrak{p}$ (normalised as above). Also, we write $P_{\mathfrak{p}}(\rho, T)$ for the local polynomial of ρ at $\mathfrak{p}$, $f(\rho, \mathfrak{p})$ for the power of $\mathfrak{p}$ dividing the conductor of ρ, and S for the set of all primes dividing $p\Delta\mathcal{O}_{F_n}$.

Further, we have a congruence

$$\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r) \;\equiv\; \mathbf{L}_p(\mathbf{f}, \rho_n, r) \mod p\,\mathbb{Z}_p[[U^{(n)}]]$$

for each $0 \leq j \leq n$.

Proof. We will prove this lemma using the measure $\mu_{\mathfrak{m}_0}(\mathbf{f}, r)$ which we constructed in Section 4. We observe that

$$\rho_{j,n} \otimes \psi \;=\; \mathrm{Ind}_{K_n}^{F_n}(\chi_{j,n} \cdot \psi).$$

The characters $\chi_{j,n}$ and ψ extend naturally to $\mathcal{G}(\mathfrak{M}_0)$, where $\mathfrak{M}_0$ is the product of $\mathfrak{P}$ and all primes dividing $\Delta\mathcal{O}_{K_n}$. Therefore we may define a bounded p-adic measure $\mu(\mathbf{f}, \rho_{j,n}, r)$ on $U^{(n)}$ by setting

$$\int_{U^{(n)}} \psi \; d\mu(\mathbf{f}, \rho_{j,n}, r) \;=\; \frac{1}{D_F^{k-2}} \int_{\mathcal{G}(\mathfrak{M}_0)} \chi_{j,n} \cdot \psi \; d\mu_{\mathfrak{m}_0}(\mathbf{f}, r)$$

for each finite-order character $\psi : U^{(n)} \to \mathbb{C}^\times$. Under our hypotheses we may apply Proposition 5.1 to see that these values are integral. This implies the existence of an element $\widetilde{\mathbf{L}_p}(\mathbf{f}, \rho_{j,n}, r) \in \mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$ which produces the value above when evaluated at the character ψ (making the usual identification of integral p-adic measures with elements of the Iwasawa algebra).

Comparing the values of the measure $\mu_{\mathfrak{m}_0}(\mathbf{f}, r)$ with those proposed for $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r)$, we see that our desired element is equal to

$$\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r) \;=\; \gamma_{j,n}(\mathbf{f}, r) \cdot \widetilde{\mathbf{L}_p}(\mathbf{f}, \rho_{j,n}, r)$$

where $\gamma_{j,n}(\mathbf{f}, r) \in \mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$ is defined by the interpolation property

$$\psi\big(\gamma_{j,n}(\mathbf{f}, r)\big) \;=\; \prod_{\mathfrak{q} | \Delta \mathcal{O}_{F_n}} \frac{\alpha(\mathfrak{q})^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{c}(\rho_{j,n}))}}{\epsilon_{F_n}(\rho_{j,n} \otimes \psi, 1-r)_{\mathfrak{q}}}.$$

Since p and Δ are coprime, each $\alpha(\mathfrak{q})$ is a p-adic unit. Further, since ψ is unramified at $\mathfrak{q}|\Delta.\mathcal{O}_{F_n}$, we may apply [17, 3.4.6] to write

$$\epsilon_{F_n}(\rho_{j,n} \otimes \psi, 1-r)_v \;=\; \psi(\mathfrak{q}^{A_{j,n}}) \cdot \epsilon_{F_n}(\rho_{j,n}, 1-r)_v$$

where the exponent $A_{j,n}$ does not depend on ψ. It is clear that such a $\gamma_{j,n}(\mathbf{f}, r)$ exists, which establishes the existence of $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r) \in \mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$.

To demonstrate that this element actually lies in $\mathbb{Z}_p[[U^{(n)}]]$, we apply [16, theorem 4.2] which shows that the value

$$\tau(\omega)^{-1} \, \frac{\Psi_S(\mathbf{f}, \mathbf{g}, r)}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}}$$

is Galois-equivariant with respect to $(\mathbf{f}, \mathbf{g}) \mapsto (\mathbf{f}^\sigma, \mathbf{g}^\sigma)$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$; here, $\tau(\omega)$ denotes the Gauss sum of the Hecke character ω associated to $\mathbf{g}$. In the case $\mathbf{g} = \mathbf{g}^\iota_{\rho_{j,n} \otimes \psi}$, it is clear that $\tau(\omega)^{-1}$ will be a rational multiple of the epsilon factor $\epsilon_{F_n}(\rho_{j,n} \otimes \psi, 1-r)$ and we have the same Galois-equivariance property for

$$\epsilon_{F_n}(\rho_{j,n} \otimes \psi, 1-r) \, \frac{\Psi_S(\mathbf{f}, \mathbf{g}^\iota_{\rho_{j,n} \otimes \psi}, r)}{\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}}.$$

Further, we can replace the global epsilon factor by its local counterpart at $\mathfrak{p}$ without affecting this property. We may do this because the conductors of the representations $\rho_{j,n}$ away from p are squares, and the representations are self-dual so their local root numbers are ± 1. Therefore, the local factors at primes other than $\mathfrak{p}$ lie in $\mathbb{Q}$ and they do not affect the rationality of the values.

Since we now assume that $\mathbf{f}$ has rational Fourier coefficients, and $\rho_{j,n}$ may be realised over $\mathbb{Q}$, this Galois-equivariance property shows that the values of $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r)$ lie in $\mathbb{Z}_p$.

It remains to prove that these elements are congruent modulo p as j varies. The character $\chi_{j,n}$ takes values in μ_{p^n} for all $0 \le j \le n$. Therefore we have $\chi_{j,n} \equiv \chi_n \mod \mathfrak{M}_{\mathbb{C}_p}$, which implies

$$\frac{1}{D_F^{k-2}} \int_{\mathcal{G}(\mathfrak{M}_0)} \psi \cdot \chi_{j,n} \, d\mu_{\mathfrak{m}_0}(\mathbf{f}, r) \;\equiv\; \frac{1}{D_F^{k-2}} \int_{\mathcal{G}(\mathfrak{M}_0)} \psi \cdot \chi_n \, d\mu(\mathbf{f}, \rho_n, r) \mod \mathfrak{M}_{\mathbb{C}_p}$$

for any ψ (as this measure is integral). So the values $\psi\big(\widetilde{\mathbf{L}}_p(\mathbf{f}, \rho_{j,n}, r)\big)$ and $\psi\big(\widetilde{\mathbf{L}}_p(\mathbf{f}, \rho_n, r)\big)$ are congruent for all ψ, implying the elements of $\mathcal{O}_{\mathbb{C}_p}[[U^{(n)}]]$ themselves are congruent. Similarly, we have a congruence

$$\gamma_{0,n}(\mathbf{f}, r) \;\equiv\; \gamma_{1,n}(\mathbf{f}, r) \;\equiv\; \ldots \;\equiv\; \gamma_{n,n}(\mathbf{f}, r) \mod \mathfrak{M}_{\mathbb{C}_p}[[U^{(n)}]];$$

this fact is analogous to Claim ($\star$) in [5], and follows easily from the proof of that assertion. This implies that $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r)$ and $\mathbf{L}_p(\mathbf{f}, \rho_n, r)$ are congruent modulo $\mathfrak{M}_{\mathbb{C}_p} \cap \mathbb{Z}_p = p\,\mathbb{Z}_p$.

As an immediate consequence of this proposition, we have proved Theorem 1.1. Now, let φ and $N_{i,j}$ be the maps defined in §1, and put $a_j = \mathbf{L}_p(\mathbf{f}, \rho_j, r)$ for a critical value $1 \le r \le k-1$. A simple induction argument (given in detail in [5]) allows us to deduce the following result from Proposition 6.1 (which establishes Theorem 1.2).

Corollary 6.2 *Assume that $p > k-2$, $\mathbf{f}$ has rational Fourier coefficients, and f satisfies Hypotheses (Cong) and (Ord). Then we have the 'false Tate curve' congruence*

$$\prod_{1 \le j \le n} N_{j,n} \left(\frac{a_j}{N_{0,j}(a_0)} . \frac{\varphi \circ N_{0,j-1}(a_0)}{\varphi(a_{j-1})} \right)^{p^j} \;\equiv\; 1 \mod p^{n+1}$$

for each $n \ge 0$.

Remark: It is not difficult to find cases in which Hypothesis (Cong) fails and Theorem 1.2 does not hold. Let us give an explicit example: taking $p = 3$ we have $F_1 = \mathbb{Q}$, so if we work over this field we are reduced to the case of elliptic modular forms. This allows us to compute the values of $\mathbf{L}_p(f, \rho_{j,1}, r)$ using MAGMA [1] (we will discuss the methods used in §7).

We have a primitive cusp form f in $S_4^{\mathrm{new}}(\Gamma_0(19))$, with q-expansion $f(z) = q - 3q^2 - 5q^3 + q^4 + \ldots$. This cusp form is congruent modulo 3 to another modular form $\tilde{f}(z) = q + 9q^2 + 28q^3 + 73q^4 + \ldots$ at level 19. We compute the value of $\mathbf{L}_p(f, \rho_{0,1}, r)\big|_{r=1}$ and $\mathbf{L}_p(f, \rho_{1,1}, r)\big|_{r=1}$ evaluated at the trivial character:

$$\mathbf{1}\Big(\mathbf{L}_p(f, \rho_{0,1}, 1)\Big) \;=\; 3^2 + 2.3^5 + O(3^7), \quad \mathbf{1}\Big(\mathbf{L}_p(f, \rho_{1,1}, 1)\Big) \;=\; 1 + 2.3^1 + 2.3^2 + O(3^4).$$

If Theorem 1.2 held here, these values would be congruent modulo 3, which is not the case; the congruence at $r = 3$ also fails.

Let us say briefly why this is not surprising. The complex period in the interpolation formula for $\mathbf{L}_p(\mathbf{f}, \rho_{j,1}, r)$ is the Petersson inner product $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathfrak{c}(\mathbf{f})}$, which is equal to a twisted adjoint L-value of $\mathbf{f}$, up to certain simple factors (see [13, theorem 7.1]).

It is known that the adjoint L-series is closely related to the congruence module of the Hecke eigenform $\mathbf{f}$; this is discussed at length by Doi, Hida and

Ishii in [9]. In particular, p will be a congruence prime for $\mathbf{f}$ if and only if p divides the algebraic part of the adjoint L-value. This relationship is conjectural for Hilbert modular forms in full generality, but has been proved under certain assumptions: see [12] and [8] for example.

As a consequence, if $\mathfrak{p}$ is a congruence prime for $\mathbf{f}$ then $\mathfrak{p}$ should divide the algebraic part of the adjoint L-value at $s = 1$, making the values of $\mathbf{L}_p(\mathbf{f}, \rho_{j,n}, r)$ less p-integral. Therefore Hypothesis (Cong) may be a natural condition to impose, when we use this particular automorphic period.

7 Numerical Examples over $\mathbb{Q}$

We have established Proposition 5.1 and Corollary 6.2 subject to the assumption $p > k-2$. More precisely: to prove Proposition 5.1 we needed the coefficients of the polynomials $P_{r-k+1}(\xi_{2,\nu}, \xi_\nu)$ to be p-integral, and their denominators could only be divisors of $(k-2)!$. Despite the fact that we were unable to remove this hypothesis, we expect that our results should hold even when $p \leq k-2$ and we will conclude our paper by giving some numerical evidence for this.

Unfortunately it is difficult to compute convolution L-values of Hilbert modular forms in general, but if we restrict to the case $p = 3$ and $n = 1$, our base field is $F_1 = \mathbb{Q}$ and we can work with elliptic modular forms. Using the L-series functions in the computer package MAGMA [1], we calculate the value of $\psi\big(\mathbf{L}_p(f, \rho_{j,1}, r)\big)$ in the case $\psi = \mathbf{1}$ for several examples of f (where we write f instead of $\mathbf{f}$ to emphasise the fact that we are now working with a classical modular form). To compute the interpolation factors, we use the methods of T. and V. Dokchitser which are described in the paper [10].

To compute the Petersson inner product $\langle f, f\rangle_N$ for a cusp form $f \in S_k(\Gamma_0(N))$ we use a well-known formula which relates it to the adjoint L-series:

$$\langle f, f\rangle_N \;=\; 2^{-2k}\,(k-1)!\,N\,\phi(N)\,\frac{L\big(\mathrm{Ad}(f), 1\big)}{\pi^{k+1}}.$$

We can then compute the algebraic parts of the L-values for $\rho = \rho_{0,1}$ and $\rho_{1,1}$, which we write as follows:

$$\Psi_f^*(\rho, r) \;=\; \sqrt{N_\rho}\cdot\frac{\Psi_S(f, g_\rho, r)}{\langle f, f\rangle_N}.$$

Here N_ρ denotes the conductor of ρ. Finally we compute the values

$$\mathcal{L}_f(\rho, r) = \frac{\epsilon(\rho, 1-r)_p}{\alpha(p)^{f(\rho,p)}} \;\times\; \frac{P_p(\rho^\vee,\, \alpha(p)^{-1}p^{r-1})}{P_p(\rho,\, \alpha(p)\,p^{-r})} \;\times\; \frac{\Psi_S(f, g_\rho, r)}{\langle f, f\rangle_N}.$$

This is equal to the evaluation of $\mathbf{L}_p(f, \rho, r)$ at the trivial character. By Proposition 6.1 we expect the congruence

$$\mathcal{L}_f(\rho_{0,1}, r) \;\equiv\; \mathcal{L}_f(\rho_{1,1}, r) \mod 3.$$

We tested two primitive cusp forms of weight 6 for which Hypothesis (Cong) is satisfied. Table 1 shows our results for the newform $f \in S_6^{\mathrm{new}}(\Gamma_0(5))$ which has q-expansion $q + 2q^2 - 4q^3 - 28q^4 + ...$ and Table 2 shows the same data for $f \in S_6^{\mathrm{new}}(\Gamma_0(7))$ having q-expansion $q - 10q^2 - 14q^3 + 68q^4 + ...$ As $p = 3$ the assumption $p > k-2$ is not satisfied, but we still observe the congruence for each critical value of r.

Table 1: values for f of weight 6, level 5, with $\Delta = 2$

r	$\Psi_f^*(\rho_{0,1}, r)$	$\Psi_f^*(\rho_{1,1}, r)$	$\mathcal{L}_f(\rho_{0,1}, r)$	$\mathcal{L}_f(\rho_{1,1}, r)$
1	$\frac{2^5.5^5}{31^1}$	$2^{14}.5^5.661^1$	$1.3^0 + 1.3^1 + O(3^3)$	$1.3^0 + 2.3^1 + O(3^2)$
2	$\frac{2^2.5^2}{3^1}$	$\frac{2^5.5^3.1759^1}{3^3}$	$2.3^0 + 2.3^2 + O(3^3)$	$2.3^0 + 2.3^1 + O(3^2)$
3	$\frac{2^3}{3^2}$	$\frac{2^5.5^2}{3^6}$	$1.3^0 + 2.3^2 + O(3^4)$	$1.3^0 + 1.3^1 + O(3^2)$
4	$\frac{2^2}{3^3}$	$\frac{2^1.5^1.1759^1}{3^9}$	$2.3^0 + 1.3^1 + O(3^2)$	$2.3^0 + 2.3^4 + O(3^5)$
5	$\frac{2^5.5^1}{3^4.31^1}$	$\frac{2^6.5^1.661^1}{3^{12}}$	$1.3^0 + 2.3^1 + O(3^2)$	$1.3^0 + 2.3^2 + O(3^3)$

Table 2: values for f of weight 6, level 7, with $\Delta = 2$

r	$\Psi_f^*(\rho_{0,1}, r)$	$\Psi_f^*(\rho_{1,1}, r)$	$\mathcal{L}_f(\rho_{0,1}, r)$	$\mathcal{L}_f(\rho_{1,1}, r)$
1	$\frac{2^8.7^2.19^1}{3^1.43^1}$	$2^{10}.5^2.7^3.13^3$	$2.3^0 + 1.3^1 + O(3^2)$	$2.3^0 + 1.3^2 + O(3^3)$
2	$\frac{2^2.7^1}{3^1}$	$\frac{2^8.7^2.181^1}{3^3}$	$2.3^0 + 1.3^3 + O(3^4)$	$2.3^0 + 2.3^1 + O(3^2)$
3	0	0	0	0
4	$\frac{2^2}{3^3.7^1}$	$\frac{2^4.181^1}{3^9}$	$1.3^0 + 2.3^1 + O(3^3)$	$1.3^0 + 1.3^4 + O(3^5)$
5	$\frac{2^8.19^1}{3^5.7^2.43^1}$	$\frac{2^2.5^2.13^3}{3^{12}.7^1}$	$1.3^0 + 1.3^1 + O(3^2)$	$1.3^0 + 2.3^1 + O(3^2)$

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