NONVANISHING OF RANKIN-SELBERG L-FUNCTIONS FOR HILBERT MODULAR FORMS

SHENG-CHI LIU AND RIAD MASRI

Abstract. Let F be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and narrow class number one. Let $S_{2k}(\Gamma)$ be the vector space of cuspidal Hilbert modular forms of parallel weight $2k$ for $\Gamma = SL_2(\mathcal{O})$ and let B_{2k} be an orthogonal Hecke eigenbasis for this space. For any fixed Hecke eigenform $f \in S_{2k}(\Gamma)$ and any $\varepsilon > 0$ we prove that

$$\#\{g \in B_{2k} : L(f \times g, \tfrac{1}{2}) \neq 0\} \gg k^{n-\varepsilon}$$

where $L(f \times g, s)$ is the Rankin-Selberg L–function of f and g.

1. Introduction and statement of results

There is great interest in determining whether an automorphic L–function $L(\pi, s)$ is nonvanishing at the central point $s = 1/2$. To study this problem, one often uses analytic methods to study $L(\pi, s)$ as π varies in a family $\mathcal{F}$. A typical approach is to establish lower and upper bounds for the first and second moments, respectively, and combine these with Cauchy's inequality to deduce a lower bound for the number of π in $\mathcal{F}$ such that $L(\pi, 1/2) \neq 0$. In recent years, various novel methods based on period formulas have been used with spectacular success to obtain estimates of this type; see for example the excellent survey article of Michel and Venkatesh [MV]. In particular, the period formula approach can provide a very flexible and direct way to study the nonvanishing problem when more classical methods are difficult to apply.

In this note we continue this theme by studying the nonvanishing problem for the family of Rankin-Selberg L–functions $L(f \times g, s)$ where f is a fixed cuspidal Hilbert modular form of parallel weight $2k$ and g varies over an orthogonal Hecke eigenbasis for the vector space of such forms. We will evaluate the first moment using a Petersson trace formula for Hilbert modular forms due to Luo [Lu], which in particular allows us to handle difficulties arising from the presence of infinitely many units in totally real fields. To estimate the second moment using a classical approach via the approximate functional equation, trace formulae, Voronoi summation, etc. would be very difficult; see for example the discussion following [B, Corollary 1]. Here we will instead adapt a beautiful idea of Blomer [B] to establish an upper bound for the second moment using the period integral representation of the L–function $L(f \times g, s)$.

In order to state our main result we fix the following notation and assumptions. Let F be a totally real number field of degree n over $\mathbb{Q}$ with narrow class number one. Let $\mathcal{O}$ be the ring of integers, U be the unit group, $\mathcal{O}^*$ be the nonzero elements in $\mathcal{O}$, D be the discriminant, R_F be the regulator, and W_F be the number of roots of unity. For $\nu \in F$, let $\nu^{(i)} := \sigma_i(\nu)$ where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the embeddings of F. Let $\Gamma = SL_2(\mathcal{O})$ be the Hilbert modular group, which acts on the n-fold product $\mathbb{H}^n$ of the complex upper half-plane $\mathbb{H}$.

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For an integer $k \geq 2$, let $S_{2k}(\Gamma)$ be the space of cuspidal Hilbert modular forms of weight $(2k, \cdots, 2k)$ (see [Ga]). This is the space of holomorphic functions $f(z)$ on $\mathbb{H}^n$ which vanish in the cusps of Γ and satisfy

$$f(\gamma z) = N(cz+d)^{2k} f(z) \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where for $z = (z_1, \cdots, z_n) \in \mathbb{H}^n$ we have

$$N(cz+d) = \prod_{i=1}^{n} (\sigma_i(c) z_i + \sigma_i(d)).$$

It was shown by Shimizu [Sh] that

$$\dim(S_{2k}(\Gamma)) \sim \operatorname{vol}(\Gamma \backslash \mathbb{H}^n) \frac{(2k-1)^n}{(4\pi)^n}$$

as $k \to \infty$.

The Petersson inner-product of two forms $f, g \in S_{2k}(\Gamma)$ is defined by

$$\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}^n} f(z) \overline{g(z)} d\mu(z),$$

where for $z = x + iy = (x_1 + iy_1, \ldots, x_n + iy_n)$ we have

$$d\mu(z) = \prod_{i=1}^{n} \frac{y_i^{2k} dx_i dy_i}{y_i^2}.$$

Let B_{2k} be an orthogonal basis for $S_{2k}(\Gamma)$ consisting of arithmetically normalized Hecke eigenforms.

The Rankin-Selberg L-function $L(f \times g, s)$ of two forms $f, g \in S_{2k}(\Gamma)$ has an analytic continuation to $\mathbb{C}$ and satisfies a functional equation under $s \mapsto 1 - s$. We will establish the following nonvanishing theorem for the central values of these Rankin-Selberg L-functions.

Theorem 1.1. *For any fixed Hecke eigenform $f \in S_{2k}(\Gamma)$ and any $\varepsilon > 0$ we have*

$$\#\{g \in B_{2k} : L(f \times g, \tfrac{1}{2}) \neq 0\} \gg k^{n-\varepsilon}.$$

To prove Theorem 1.1 we will use the asymptotic formula for the first moment in Theorem 3.1, the upper bound for the second moment in Theorem 4.1, and Cauchy's inequality.

2. Preliminaries

2.1. **The Petersson trace formula.** For each Hecke eigenform $f \in S_{2k}(\Gamma)$ one has the Fourier expansion

$$f(z) = \sum_{\substack{\nu \in \mathcal{O} \\ \nu \gg 0}} \lambda_f(\nu) N(\nu)^{\frac{2k-1}{2}} e^{2\pi i \operatorname{Tr}(\nu \delta^{-1} z)},$$

where $\mathfrak{d} = (\delta)$ with $\delta \gg 0$ is the different of F and

$$\operatorname{Tr}(\nu \delta^{-1} z) = \sum_{i=1}^{n} \sigma_i(\nu \delta^{-1} z).$$

Note that the Fourier coefficient $\lambda_f(\nu)$ depend only on the ideal $\mathfrak{m} = (\nu)$. The Ramanujan conjecture asserts that (see e.g. Blasius [B])

$$\lambda_f(\mathfrak{m}) \ll N_F(\mathfrak{m})^{\varepsilon}, \tag{2.1}$$

where $N_F(\mathfrak{m})$ is the norm of $\mathfrak{m}$.

We will need the following Petersson trace formula due to Luo [Lu].

Proposition 2.1. *For any $\nu \gg 0$ and $\mu \gg 0$ in $\mathcal{O}$,*

$$\sum_{f\in B_{2k}} w_f^{-1}\overline{\lambda_f(\nu)}\lambda_f(\mu) =$$

$$\chi_\nu(\mu) + \frac{(2\pi)^n(-1)^{nk}}{D^{1/2}} \sum_{\epsilon\in U} \sum_{c\in\mathcal{O}^*/U} \frac{S(\nu, \mu\epsilon^2; c)}{|N(c)|} \prod_{i=1}^{n} J_{2k-1}\left(\frac{4\pi\sqrt{\nu^{(i)}\mu^{(i)}}|\epsilon^{(i)}|}{|c^{(i)}|}\right),$$

where

$$w_f = \frac{(4\pi)^{n(2k-1)}}{\Gamma^n(2k-1)D^{2k-1/2}}\|f\|^2,$$

χ_ν is the characteristic function of the set $\{\nu\epsilon^2 : \epsilon \in U\}$, and

$$S(\nu, \mu; c) = \sum_{a \,(\mathrm{mod}\ c)}^{*} e\left(\mathrm{Tr}(\frac{\nu a + \mu\bar{a}}{c})\right)$$

is the generalized Kloosterman sum.

2.2. **Eisenstein series.** Define the Eisenstein series

$$E(z, s) = \sum_{\substack{(c,d)\in\mathcal{O}^2/U\\(c,d)=1}} \frac{N(y)^s}{|N(cz+d)|^{2s}} \qquad \text{for} \quad \mathrm{Re}(s) > 1.$$

Let

$$\zeta_F^*(s) := \pi^{-\frac{ns}{2}} D^{s/2}\Gamma^n(\frac{s}{2})\zeta_F(s)$$

be the completed Dedekind zeta function of F. Then the completed Eisentein series

$$E^*(z, s) := \zeta_F^*(2s)E(z, s)$$

satisfies the functional equation $E^*(z, s) = E^*(z, 1-s)$ and has the Fourier expansion (see [vG, Proposition 6.9])

$$E^*(z, s) = \zeta_F^*(2s)N(y)^s + \zeta_F^*(2s-1)N(y)^{1-s} +$$

$$2^n N(y)^{1/2} \sum_{\substack{\nu\in\mathfrak{d}^{-1}/U\\ \nu\neq 0}} N_F((\nu)\mathfrak{d})^{s-1/2}\sigma_{1-2s}((\nu)\mathfrak{d}) \prod_{i=1}^{n} K_{s-1/2}(2\pi|\nu^{(i)}|y_i)e^{2\pi i\mathrm{Tr}(\nu x)},$$

where

$$\sigma_r(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} N_F(\mathfrak{b})^r.$$

Moreover, $E^*(z, s)$ has meromorphic continuation to all $s \in \mathbb{C}$, with only simple poles at $s = 0$ and $s = 1$ with residue $\frac{2^{n-1}R_F}{W_F}$. From the Fourier expansion, we have

$$E^*(z, \tfrac{1}{2}) \ll N(y)^{1/2+\varepsilon} \tag{2.2}$$

for any z in the Siegel domain of Γ.

2.3. **Rankin-Selberg L-functions.** For two forms $f, g \in S_{2k}(\Gamma)$, the Rankin-Selberg L-function is defined by

$$L(f \times g, s) = \zeta_F(2s) \sum_{\substack{\mathfrak{m} \subset \mathcal{O} \\ \mathfrak{m} \neq (0)}} \frac{\lambda_f(\mathfrak{m})\overline{\lambda_g(\mathfrak{m})}}{N_F(\mathfrak{m})^s} \qquad \text{for} \quad \mathrm{Re}(s) > 1.$$

The completed L-function

$$\Lambda(f \times g, s) := (2\pi)^{-2n(s+2k-1)} D^{2(s+2k-1)} \Gamma^n(s) \Gamma^n(s + 2k - 1) L(f \times g, s)$$

satisfies the functional equation $\Lambda(f \times g, s) = \Lambda(f \times g, 1 - s)$. Moreover, it has the integral representation

$$D^{-2k+3/2} \pi^{n(2k-1)} \Lambda(f \times g, s) = \int_{\Gamma \backslash \mathbb{H}^n} E^*(z, s) f(z) \overline{g(z)} d\mu(z). \tag{2.3}$$

3. Asymptotic formula for the first moment

In this section we will establish the following asymptotic formula for the first moment of the Rankin-Selberg L-functions.

Theorem 3.1. *For any fixed Hecke eigenform $f \in S_{2k}(\Gamma)$ and any $\varepsilon > 0$ we have*

$$\sum_{g \in B_{2k}} \omega_g^{-1} L(f \times g, \tfrac{1}{2}) = 2\gamma_F + \kappa_F \left(n \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + n \frac{\Gamma'(2k - \frac{1}{2})}{\Gamma(2k - \frac{1}{2})} - \log\Big(\frac{(2\pi)^n}{D^2} \Big) \right) + O\left(k^{-\frac{n}{2}+\varepsilon}\right),$$

where κ_F and γ_F are the residue and constant term in the Laurent expansion of $\zeta_F(s)$ at $s = 1$, respectively.

For the proof we will need the following approximate functional equation.

Proposition 3.2. *Let $G(u)$ be an even, holomorphic function which satisfies $G(0) = 1$ and decays rapidly as $|Im(u)| \to \infty$. Then we have*

$$L(f \times g, \tfrac{1}{2}) = 2 \sum_{\substack{\mathfrak{m} \subset \mathcal{O} \\ \mathfrak{m} \neq (0)}} \frac{\lambda_f(\mathfrak{m})\overline{\lambda_g(\mathfrak{m})}}{N_F(\mathfrak{m})^{1/2}} V(N_F(\mathfrak{m})),$$

where

$$V(y) = \frac{1}{2\pi i} \int_{(3)} y^{-u} \frac{\gamma(\frac{1}{2} + u)}{\gamma(\frac{1}{2})} \zeta_F(1 + 2u) G(u) \frac{du}{u}$$

and

$$\gamma(s) = (2\pi)^{-2n(s+k-1)} D^{2(s+k-1)} \Gamma^n(s) \Gamma^n(s + 2k - 1).$$

Moreover,

$$y^a V^{(a)}(y) \ll (1 + \frac{y}{k^n})^{-A} \tag{3.1}$$

and

$$V(y) = \frac{1}{2} \left\{ 2\gamma_F + \kappa_F \left(n \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + n \frac{\Gamma'(2k - \frac{1}{2})}{\Gamma(2k - \frac{1}{2})} - \log\Big(\frac{(2\pi)^n y}{D^2} \Big) \right) \right\} + O\left((\frac{y}{k^n})^{1/2-\varepsilon} \right). \tag{3.2}$$

Proof. This follows from [IK, Theorem 5.3 and Proposition 5.4], where for the estimate (3.2) we shift the contour to $(-\frac{1}{2}+\varepsilon)$ and pass a double pole at $u=0$. $\square$

Proof of Theorem 3.1. By Propositions 3.2 and 2.1 we have

$$\sum_{g\in B_{2k}} w_g^{-1}L(f\times g,\tfrac{1}{2}) = 2\sum_{\substack{\mathfrak{m}=(\nu)\subset\mathcal{O}\\ \mathfrak{m}\neq(0)}} \frac{\lambda_f(\mathfrak{m})}{N_F(\mathfrak{m})^{1/2}}V(N_F(\mathfrak{m}))\sum_{g\in B_{2k}} w_g^{-1}\overline{\lambda_g(\mathfrak{m})} = 2V(1)+E,$$

where

$$E := 2\frac{(2\pi)^n(-1)^{nk}}{D^{1/2}}\sum_{\substack{\mathfrak{m}=(\nu)\subset\mathcal{O}\\ \mathfrak{m}\neq(0)}} \frac{\lambda_f(\mathfrak{m})}{N_F(\mathfrak{m})^{1/2}}V(N_F(\mathfrak{m}))\sum_{\epsilon\in U}\sum_{c\in\mathcal{O}^*/U}\frac{S(\nu,\epsilon^2;c)}{|N(c)|}\prod_{i=1}^{n} J_{2k-1}\left(\frac{4\pi\sqrt{\nu^{(i)}}|\epsilon^{(i)}|}{|c^{(i)}|}\right).$$

By (3.2), the diagonal term $2V(1)$ contributes

$$2\gamma_F+\kappa_F\left(n\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})}+n\frac{\Gamma'(2k-\frac{1}{2})}{\Gamma(2k-\frac{1}{2})}-\log\Big(\frac{(2\pi)^n}{D^2}\Big)\right)+O\left(k^{-\frac{n}{2}+\varepsilon}\right).$$

Using the estimate (3.1), we may truncate the off-diagonal term E at $N_F(\mathfrak{m})\le k^{n+\varepsilon}$ with a very small error. We denote the truncated sum by E_1. In order to estimate E_1, we first recall some estimates for J-Bessel functions. For all $x>0$, we have $J_{2k-1}(x)\ll 1$. Moreover, from the integral representation (see [GR, 8.411.10])

$$J_{2k-1}(x)=\frac{1}{\Gamma(2k+\frac{1}{2})\Gamma(\frac{1}{2})}\left(\frac{x}{2}\right)^{2k-1}\int_{-1}^{1}e^{ixt}(1-t^2)^{2k-1/2}dt$$

and Stirling's formula, we find that

$$J_{2k-1}(x)\ll\left(\frac{ex}{4k}\right)^{2k-1}.$$

Hence

$$J_{2k-1}(x)\ll\min\left\{1,\left(\frac{ex}{4k}\right)^{2k-1}\right\}\ll\left(\frac{ex}{4k}\right)^{2k-1-\eta}\quad\text{for}\quad 0\le\eta<1.$$

For $x=\dfrac{4\pi\sqrt{\nu^{(i)}}|\epsilon^{(i)}|}{|c^{(i)}|}$, we choose $\eta=0$ if $|\epsilon^{(i)}|\le 1$ and $0<\eta<1$ if $|\epsilon^{(i)}|>1$ to deduce that

$$\prod_{i=1}^{n}J_{2k-1}\left(\frac{4\pi\sqrt{\nu^{(i)}}|\epsilon^{(i)}|}{|c^{(i)}|}\right)\ll\left(\frac{(e\pi)^n\sqrt{N(\nu)}}{k^n|N(c)|}\right)^{2k-1}|N(c)|^{\eta}\prod_{|\epsilon^{(i)}|>1}|\epsilon^{(i)}|^{-\eta}. \tag{3.3}$$

Then using (2.1), (3.3), and the trivial bound

$$|S(\nu,\epsilon^2;c)|\le|N(c)|,$$

we have

$$\begin{aligned}
E_1 &\ll \sum_{\substack{\mathfrak{m}=(\nu)\subset\mathcal{O}\\ \mathfrak{m}\neq(0)\\ N_F(\mathfrak{m})\le k^{n+\varepsilon}}} \frac{1}{N_F(\mathfrak{m})^{1/2-\varepsilon}} \sum_{\epsilon\in U}\sum_{c\in\mathcal{O}^*/U} \left(\frac{(e\pi)^n\sqrt{N(\nu)}}{k^n|N(c)|}\right)^{2k-1} |N(c)|^{\eta} \prod_{|\epsilon^{(i)}|>1} |\epsilon^{(i)}|^{-\eta}\\
&\ll \sum_{\substack{\mathfrak{m}=(\nu)\subset\mathcal{O}\\ \mathfrak{m}\neq(0)\\ N_F(\mathfrak{m})\le k^{n+\varepsilon}}} \frac{1}{N_F(\mathfrak{m})^{1/2-\varepsilon}} N_F(\mathfrak{m}) \sum_{c\in\mathcal{O}^*/U} \frac{1}{|N(c)|^2}|N(c)|^{\eta} \left(\frac{(e\pi)^n\sqrt{N(\nu)}}{k^n|N(c)|}\right)^{2k-3} \sum_{\epsilon\in U}\prod_{|\epsilon^{(i)}|>1} |\epsilon^{(i)}|^{-\eta}\\
&\ll e^{-k},
\end{aligned}$$

where for the last inequality we used (see [Lu, p. 136])

$$\sum_{\epsilon\in U}\prod_{|\epsilon^{(i)}|>1} |\epsilon^{(i)}|^{-\eta} < \infty$$

and

$$\frac{(e\pi)^n\sqrt{N(\nu)}}{k^n|N(c)|} \ll k^{-\frac{n}{2}+\varepsilon}.$$

This completes the proof. □

4. Upper bound for the Second moment

In this section we will establish the following upper bound for the second moment of the Rankin-Selberg L–functions by adapting a method of Blomer [Bl].

Theorem 4.1. *For any fixed Hecke eigenform $f \in S_{2k}(\Gamma)$ and any $\varepsilon > 0$ we have*

$$\sum_{g\in B_{2k}} |L(f\times g, \tfrac{1}{2})|^2 \ll k^{n+\varepsilon}.$$

Proof. By (2.3), for $f \in B_{2k}$ we have

$$\begin{aligned}
\|f\|^2 &= \int_{\Gamma\backslash\mathbb{H}^n} |f(z)|^2 d\mu(z)\\
&= \frac{W_F}{2^{n-1}R_F}\operatorname*{Res}_{s=1}\int_{\Gamma\backslash\mathbb{H}^n} |f(z)|^2 E^*(z,s)d\mu(z)\\
&= \frac{W_F D^{2k+3/2}\Gamma^n(2k)}{R_F\pi^{n(2k+1)}2^{n(4k+1)-1}}\operatorname*{Res}_{s=1} L(f\times f, s)\\
&\ll_F \frac{D^{2k}\Gamma^n(2k)}{\pi^{2nk}2^{4nk}}k^{\varepsilon}, \qquad (4.1)
\end{aligned}$$

where for the last inequality we used (see [Li])

$$\operatorname*{Res}_{s=1} L(f\times f, s) \ll k^{\varepsilon}.$$

By Parseval's identity and (2.3), we have for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\|fE^*(\cdot,s)\|^2 &= \sum_{g\in B_{2k}} \frac{1}{\|g\|^2} |\langle fE^*(\cdot,s),g\rangle|^2 \\ &= \sum_{g\in B_{2k}} \frac{1}{\|g\|^2} |D^{-2k+3/2}\pi^{n(2k-1)}\Lambda(f\times g,s)|^2.\end{aligned}$$

By analytic continuation, this holds for all $s\in\mathbb{C}$. Take $s=1/2$ and use (4.1) to obtain

$$\begin{aligned}\|fE^*(\cdot,\tfrac12)\|^2 &= \sum_{g\in B_{2k}} \frac{1}{\|g\|^2} |D^{2k+1/2}\pi^{-2nk}2^{-2n(2k-1/2)}\Gamma^n(\tfrac12)\Gamma^n(2k-\tfrac12)L(f\times g,\tfrac12)|^2 \\ &\gg_F k^{-\varepsilon}\frac{\Gamma^{2n}(2k-\frac12)D^{2k}}{\Gamma^n(2k)2^{4nk}\pi^{2nk}} \sum_{g\in B_{2k}} |L(f\times g,\tfrac12)|^2. \qquad (4.2)\end{aligned}$$

On the other hand, by (2.2) and (2.3) we have

$$\begin{aligned}\|fE^*(\cdot,\tfrac12)\|^2 &= \int_{\Gamma\backslash\mathbb{H}^n} |f(z)|^2|E^*(z,\tfrac12)|^2 d\mu(z) \\ &\ll \int_{\Gamma\backslash\mathbb{H}^n} |f(z)|^2 N(y)^{1+\varepsilon} d\mu(z) \\ &\ll \int_{\Gamma\backslash\mathbb{H}^n} |f(z)|^2 E(z,1+\varepsilon) d\mu(z) \\ &= \frac{D^{-2k+3/2}\pi^{n(2k-1)}}{\zeta_F^*(1+\varepsilon)}\Lambda(f\times f,1+\varepsilon) \\ &\ll_F 2^{-4nk}D^{2k}\pi^{-2nk}\Gamma^n(2k+\varepsilon)k^{\varepsilon}, \qquad (4.3)\end{aligned}$$

where for the last inequality we used (see [Li])

$$L(f\times f,1+\varepsilon)\ll k^{\varepsilon}.$$

Using (4.2), (4.3), and Stirling's formula, we complete the proof. $\square$

5. Proof of Theorem 1.1

By (2.3), Stirling's formula, and the bound (see [HL])

$$\operatorname*{Res}_{s=1} L(f\times f,s) \gg k^{-\varepsilon},$$

we have

$$\begin{aligned}w_g &= \frac{(4\pi)^{n(2k-1)}}{\Gamma^n(2k-1)D^{2k-1/2}}\|g\|^2 \\ &\gg_F \frac{\Gamma^n(2k)}{\Gamma^n(2k-1)} \operatorname*{Res}_{s=1} L(f\times f,s) \\ &\gg_F k^{n-\varepsilon}.\end{aligned}$$

Then it follows from the bound

$$\sum_{g\in B_{2k}} w_g^{-1} L(f\times g,\tfrac12) \ll_F k^{-n+\varepsilon} \sum_{g\in B_{2k}} |L(f\times g,\tfrac12)|$$

and Theorem 3.1 that

$$\sum_{g\in B_{2k}} |L(f\times g,\tfrac{1}{2})| \gg_F k^{n-\varepsilon}. \tag{5.1}$$

On the other hand, by Cauchy's inequality and Theorem 4.1, we have

$$\begin{aligned}\sum_{g\in B_{2k}} |L(f\times g,\tfrac{1}{2})| &\le \Big(\sum_{\substack{g\in B_{2k}\\ L(f\times g,1/2)\neq 0}} 1\Big)^{1/2}\Big(\sum_{g\in B_{2k}} |L(f\times g,\tfrac{1}{2})|^2\Big)^{1/2}\\ &\ll_F \Big(\sum_{\substack{g\in B_{2k}\\ L(f\times g,1/2)\neq 0}} 1\Big)^{1/2}(k^{n+\varepsilon})^{1/2}. \end{aligned}\tag{5.2}$$

Using (5.1) and (5.2), we complete the proof. □

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Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, U.S.A.

E-mail address: scliu@math.tamu.edu

E-mail address: masri@math.tamu.edu