TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 346, Number 1, November 1994

ON THE FUNDAMENTAL PERIODS OF HILBERT MODULAR FORMS

ZE-LI DOU

ABSTRACT. The main purpose of this paper is to establish the existence of fundamental periods of primitive cusp forms of Hilbert modular type of several variables, as well as the relationship between those fundamental periods and the special values of the associated L-functions. These results, together with some recent results of Shimura, give us the means of translating with ease results concerning periods of automorphic forms derived from various points of view. We also verify several conjectures of Shimura on the properties of such fundamental periods.

INTRODUCTION

The concept of periods of an automorphic form has been studied from various points of view by many authors. It frequently appears in at least the following three contexts: integrals over cycles of the differential form attached to an automorphic form; special values of the L-function associated to a form; and coefficients of the Fourier expansion of a form. In a recent work [Sh3], Shimura formulated a sequence of very precise conjectures on the properties of the periods of automorphic forms, as well as the relationship among the periods arising from the several contexts mentioned above. Furthermore, he was able to establish a result relating certain special values of the L-function associated to a primitive form to the Fourier coefficients of a suitably defined Hilbert modular form. Shimura then proved some of his own conjectures, in the division algebra case, in a subsequent paper [Sh4]. The purpose of this work, then, is to show the following:

1. The so-called fundamental periods, enjoying the same properties stipulated in [Sh4], can be defined in the Hilbert modular case as well, and
2. A relation between these fundamental periods and the L-values can be established.

Therefore, the fundamental periods can now be defined for primitive forms defined with respect to any quaternion algebra over a totally real algebraic number field. Moreover, properties of those periods, derived from any of the above described viewpoints, can now be translated with ease to any other.

In order to keep a sharp focus on our main ideas, and also to keep this paper

Received by the editors November 8, 1993.
1991 *Mathematics Subject Classification.* Primary 11F67.
Key words and phrases. Periods, automorphic forms, modular forms, L-functions, periods of modular forms, special values of L-functions, cohomology of automorphic forms.

as short as possible, we shall assume on the reader's part certain familiarity with Shimura's paper [Sh4]. We have tried to conform to the notations adopted there. A very brief review of the background material can also be found in the first section of this paper. The main results are then explained in the second section in a more detailed fashion.

The author wishes to thank Professor G. Shimura for suggesting the problem to him, and for the encouragement he received while this work was in progress.

1. Background

1.1. **Cusp forms on $H^{\mathbf{a}}$ and $G_{\mathbf{A}}$.** Throughout this paper we write $B = M_2(F)$ and $G = GL_2(F)$, where F is a totally real algebraic number field of degree n. We denote the archimedean and finite parts of F by $\mathbf{a}$ and $\mathbf{f}$, respectively. Also, $\mathfrak{r}$ and $\mathfrak{d}$ denote the ring of integers and the different of F, respectively. The adelization of B and G, and their respective archimedean and finite parts, are denoted by $B_{\mathbf{A}}$, $B_{\mathbf{a}}$, $B_{\mathbf{f}}$, $G_{\mathbf{A}}$, $G_{\mathbf{a}}$, and $G_{\mathbf{f}}$. We identify $B_{\mathbf{a}}$ with $M_2(\mathbb{R})^{\mathbf{a}}$ by fixing a suitable isomorphism, and then identify $G_{\mathbf{a}}$ with $GL_2(\mathbb{R})^{\mathbf{a}}$. Here the notation $X^{\mathbf{a}}$ means n copies of X indexed by $\mathbf{a}$. Finally we define

$$G_{\mathbf{a}+} = GL_2^+(\mathbb{R})^{\mathbf{a}}, \text{ and } G_{\mathbb{Q}+} = G_{\mathbf{a}+} \cap G. \tag{1.1}$$

The cusp forms can be defined as functions either on $H^{\mathbf{a}}$ or $G_{\mathbf{A}}$. Given a congruence subgroup Γ, a weight $k \in \mathbb{Z}^{\mathbf{a}}$, and $\varepsilon \subset \mathbf{a}$, the space of cusp forms on $H^{\mathbf{a}}$, of weight k with respect to Γ and ε, denoted by $\mathscr{S}_k^{\varepsilon}(\Gamma)$, consists of functions f satisfying the following conditions:

(1.2a) $f\|_k^{\varepsilon}\gamma = f$, $\forall \gamma \in \Gamma$;
(1.2b) $f(z)$ is holomorphic in z_v for every $v \in \mathbf{a} - \varepsilon$ and antiholomorphic in z_v for every $v \in \varepsilon$;
(1.2c) f is fast decreasing at every cusp; i.e., $f\|_k^0\beta$ is a holomorphic Hilbert cusp form for every $\beta \in G \cap G_{\varepsilon}$.

We denote by $\mathscr{S}_k^{\varepsilon}(B)$ the union of $\mathscr{S}_k^{\varepsilon}(\Gamma)$ for all congruence subgroups Γ. (The symbol $f\|_k^{\varepsilon}$ is defined in [Sh4].) We shall assume throughout that $k_v \geq 2$ for all v.

To define cusp forms on $G_{\mathbf{A}}$, we fix notations as follows. Given a finite prime $v \in \mathbf{f}$, we denote $M_2(\mathfrak{r}_v)$ by $\mathfrak{o}_v$ for notational simplicity. For two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{r}$, put $D_v[\mathfrak{a}, \mathfrak{b}] = GL_2\left(\begin{smallmatrix} \mathfrak{r}_v & \mathfrak{a}_v \\ \mathfrak{b}_v & \mathfrak{r}_v \end{smallmatrix}\right)$. Fix an integral ideal $\mathfrak{m}$ in F. We define

$$W = W_{\mathfrak{m}} = G_{\mathbf{a}+} \prod_{v \in \mathbf{f}} D_v[\mathfrak{d}^{-1}, \mathfrak{m}\mathfrak{d}], \tag{1.3}$$

and

$$W^1 = W_{\mathfrak{m}}^1 = \{x \in W_{\mathfrak{m}} | a_v(x) - 1 \in \mathfrak{m}_v, \forall v | \mathfrak{m}\}. \tag{1.4}$$

Here a_v is the first entry of x_v. Writing $h = [G_{\mathbf{a}+}G_{\mathbf{f}} \colon G_{\mathbb{Q}+}W]$, we have the following decompositions:

$$G_{\mathbf{A}} = \coprod_{\lambda=1}^{h} Gx_\lambda W = \coprod_{\lambda=1}^{h} Gx_\lambda^{-*} W, \tag{1.5a}$$

$$G_{\mathbf{A}} = \coprod_{\lambda=1}^{h} Gx_\lambda W^1 = \coprod_{\lambda=1}^{h} Gx_\lambda^{-*} W^1, \tag{1.5b}$$

$$F_{\mathbf{A}}^\times = \coprod_{\lambda=1}^{h} F^\times t_\lambda N(W). \tag{1.5c}$$

Here we have chosen $x_\lambda \in G_{\mathbf{f}}$, $t_\lambda \in F_{\mathbf{f}}^\times$ for $\lambda = 1, \dots, h$ such that $N(x_\lambda) = t_\lambda$ and also that $x_\lambda = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t_{\lambda v}\end{smallmatrix}\right)$ for $v|\mathfrak{m}$. We further define

$$W_\lambda = x_\lambda W x_\lambda^{-1}, \qquad W_\lambda^1 = x_\lambda W^1 x_\lambda^{-1}, \tag{1.6}$$

$$\Gamma_\lambda = G \cap W_\lambda, \qquad \Gamma_\lambda^1 = G \cap W_\lambda^1. \tag{1.7}$$

We now let Φ be a Hecke character of F (of finite order) such that

$$\mathfrak{c}_\Phi|\mathfrak{m} \quad \text{and} \quad \Phi_{\mathbf{a}}(x) = \operatorname{sgn}(x)^k, \quad \forall x \in F_{\mathbf{a}}^\times.$$

Here $\mathfrak{c}_\Phi$ is the conductor of Φ. Then the set of cusp forms on $G_{\mathbf{A}}$ of weight k and level $\mathfrak{m}$, denoted by $\mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi_\mathfrak{m})$, consists of functions $\mathbf{g}\colon G_{\mathbf{A}} \to \mathbb{C}$ such that the following conditions are satisfied:

(1.8a) $\mathbf{g}(\alpha x u) = \Phi_\mathfrak{m}(d_u)\mathbf{g}(x)$, $\forall \alpha \in G, \forall u \in W_{\mathbf{f}}$, and $\forall x \in G_{\mathbf{A}}$;

(1.8b) for every $x \in G_{\mathbf{f}}$, there is an element g_x of $\mathscr{S}_k^\varepsilon(B)$ such that

$$\mathbf{g}(xy) = (g_x\|_k^\varepsilon y)(\mathbf{i}), \quad \forall y \in G_{\mathbf{a}+}.$$

Here $\mathbf{i} = (i, \dots, i) \in \mathbb{C}^{\mathbf{a}}$, d_u is the element of $F_{\mathbf{f}}^\times$ whose v-component is the last entry of u_v if $v|\mathfrak{m}$, and 1 otherwise.

The cusp forms defined on $H^{\mathbf{a}}$ and $G_{\mathbf{A}}$ can be related as follows. For each Γ_λ we let

$$\mathscr{S}_k^\varepsilon(\Gamma_\lambda, \Phi_\mathfrak{m}) = \{f \in \mathscr{S}_k^\varepsilon(B) \mid f\|_k^\varepsilon \gamma = \Phi_\mathfrak{m}(a_\gamma)f, \forall \gamma \in \Gamma_\lambda\}. \tag{1.9}$$

Given $\mathbf{g} \in \mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi_\mathfrak{m})$, we define $f_\lambda \in \mathscr{S}_k^\varepsilon(\Gamma_\lambda, \Phi_\mathfrak{m})$ for each $\lambda \in \{1, \dots, h\}$ by

$$f_\lambda(z) = \mathbf{g}(x_\lambda^{-*}y) j_k^\varepsilon(y, \mathbf{i}) \Phi_\mathfrak{m}(d_y)^{-1}, \tag{1.10}$$

where $y \in W$ and $y(\mathbf{i}) = z$.

Conversely, given $(f_1, \dots, f_h) \in \prod_{\lambda=1}^{h} \mathscr{S}_k^\varepsilon(\Gamma_\lambda, \Phi_\mathfrak{m})$, we define

$$\mathbf{g}(\alpha x_\lambda^{-*} u) = \Phi_\mathfrak{m}(d_u)(f_\lambda\|_k^\varepsilon u)(\mathbf{i}), \quad \forall \alpha \in G, u \in W. \tag{1.11}$$

Then (1.10) and (1.11) give a canonical isomorphism

$$\mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi_\mathfrak{m}) \cong \prod_{\lambda=1}^{h} \mathscr{S}_k^\varepsilon(\Gamma_\lambda, \Phi_\mathfrak{m}).$$

Finally we define

$$\mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi) = \{\mathbf{g} \in \mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi_\mathfrak{m}) | \mathbf{g}(sx) = \Phi(s)\mathbf{g}(x), \forall s \in F_{\mathbf{A}}^\times\}. \tag{1.12}$$

For the definitions of the Petersson inner product and other details, we refer the reader to [Sh4].

1.2. **Cohomology theories and operators.** The fundamental periods of cusp forms will be defined via cohomology theory. Therefore, we start by recalling the equivalency of three different kinds of cohomology theory under certain conditions to be specified below.

The symbol (ρ, E) will denote a linear representation $\rho\colon G_{\mathbf{a}} \to GL(E)$, where E is a finite-dimensional vector space over $\mathbb{C}$. We note that, if Γ is a congruence subgroup of $G_{\mathbb{Q}+}$, then the restriction of ρ to Γ gives a linear representation of Γ. For every $0 \le q \in \mathbb{Z}$, let $A^q(H^{\mathbf{a}}; E)$ be the space of smooth E-valued differential q-forms on $H^{\mathbf{a}}$. We define

$$(1.13)\qquad A^q(H^{\mathbf{a}}; E)^\Gamma = \{\omega \in A^q(H^{\mathbf{a}}; E) | \omega \circ \gamma = \rho(\gamma)\omega, \forall \gamma \in \Gamma\},$$

where $\omega \circ \gamma$ denotes the transform of ω under the action of γ. This is the space of Γ-invariant q-forms on $H^{\mathbf{a}}$. We set $A(H^{\mathbf{a}}; E)^\Gamma = \sum_{q=0}^\infty A^q(H^{\mathbf{a}}; E)^\Gamma$. This complex, together with the exterior differentiation of differential forms, then gives a cohomology theory which we denote by

$$(1.14)\qquad H^*(A(H^{\mathbf{a}}; E)^\Gamma) = \sum_q H^q(A(H^{\mathbf{a}}; E)^\Gamma).$$

To define the singular cohomology, we assume $\rho(\Gamma \cap F) = 1$. Let $S(H^{\mathbf{a}}) = \sum_q S_q(H^{\mathbf{a}})$ be the complex of singular chains on $H^{\mathbf{a}}$. Then Γ acts naturally on $S(H^{\mathbf{a}})$. We denote by $C_s^q(\Gamma; E)$ the set of all E-valued q-cochains that are Γ-equivariant. Then the complex $C_s^*(\Gamma; E) \overset{\text{def}}{=} \sum_q C_s^q(\Gamma; E)$, together with the usual differentiation δ defined by $\delta\varphi = \varphi\partial$, gives the singular cohomology theory

$$(1.15)\qquad H_s^*(\Gamma; E) = \sum_q H_s^q(\Gamma; E).$$

Finally, we have the group cohomology of Γ with respect to (ρ, E) which we shall denote simply by

$$(1.16)\qquad H^*(\Gamma; E) = \sum_q H^q(\Gamma; E).$$

We now recall that, under the conditions specified above, those three cohomology theories are canonically isomorphic to one another. See the book by Borel and Wallach [B-W] for details.

Let $E = \bigotimes_{v\in\mathbf{a}} \mathbb{C}^{k_v - 1}$. We recall that to each cusp form $f \in \mathscr{S}_k^\varepsilon(\Gamma)$ an element of $A^n(H^{\mathbf{a}}; E)^\Gamma$ can be attached. We recall the definition as follows. For $z \in \mathbb{C}^{\mathbf{a}}$ and $\varepsilon \subset \mathbf{a}$, define $z' \in \mathbb{C}^{\mathbf{a}}$ to be the element such that $z'_v = \overline{z}_v$ for all $v \in \varepsilon$ and $z'_v = z_v$ otherwise. Put $[z]_k^\varepsilon = \bigotimes_{v\in\mathbf{a}} \binom{z'_v}{1}^{k_v-2}$ and $d_\varepsilon z = \bigwedge_{v\in\mathbf{a}} dz'_v$. (Fix an arbitrary order among the places $v \in \mathbf{a}$.) Then an E-valued differential n-form $[f]$ is given by

$$(1.17)\qquad [f] = [z]_k^\varepsilon \otimes f d_\varepsilon z.$$

We also recall that a linear representation ρ_k can be defined as in [Sh4]. The differential form $[f]$ is square integrable, harmonic, and hence also closed with respect to the exterior differentiation. (See [M-S] and also a correction in [Sh4, pp. 411–412].) Therefore we have a natural mapping

$$(1.18)\qquad \prod \mathscr{S}_k^\varepsilon(\Gamma) \to H^n(\Gamma; E).$$

In the division algebra case, the well-known Eichler-Shimura theorem implies that (1.18) is an embedding. However, the following results, due to Borel [B], show that we have an embedding in our case as well.

Proposition 1.1. *The cohomology $H^*(A(H^{\mathbf{a}}; E)^{\Gamma})$ is generated by closed forms of moderate growth.*

Let $\mathscr{H}_{fd}$ denote the space of harmonic fast decreasing differential forms contained in $A(H^{\mathbf{a}}; E)^{\Gamma}$. Then the natural mapping $\mathscr{H}_{fd} \to H^*(\Gamma; E)$ is injective. Furthermore, if $\omega \in \mathscr{H}_{fd}$, then ω can be written in the form $\omega = \mu + d\nu$, where μ has compact support modulo Γ and ν is fast decreasing.

The injectivity of (1.18) follows from the fact that the cusp forms are fast decreasing.

Let us now state the following structure theorem for $H^n(\Gamma; E)$, which captures the image of (1.18) precisely. See the paper by Harder [Ha] for a proof.

Proposition 1.2. *We have*

$$H^n(\Gamma; E) = H^n_{\mathrm{sq}}(\Gamma; E) \oplus H^n_{\mathrm{Eis}}(\Gamma; E), \tag{1.19}$$

where $H^n_{\mathrm{sq}}(\Gamma; E)$ is the space of cohomology classes which can be represented by square integrable differential forms, and $H^n_{\mathrm{Eis}}(\Gamma; E)$ can be constructed by means of Eisenstein series. $H^n_{\mathrm{Eis}}(\Gamma; E) = 0$ when Γ is co-compact. The image of (1.18) is a subspace of $H^n_{\mathrm{sq}}(\Gamma; E)$ and we have

$$H^n_{\mathrm{sq}}(\Gamma; E) \cong \begin{cases} \prod_{\varepsilon \subset \mathbf{a}} \mathscr{S}^{\varepsilon}_k(\Gamma), & \text{if } k \neq 2 \cdot \mathbf{1}, \\ \prod_{\varepsilon \subset \mathbf{a}} \mathscr{S}^{\varepsilon}_k(\Gamma) \oplus \left(\sum_{\substack{\zeta \subset \mathbf{a} \\ 2\|\zeta\| = n}} \mathbb{C}\omega_\zeta \right), & \text{if } k = 2 \cdot \mathbf{1}, \end{cases} \tag{1.20}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^{\mathbf{a}}$ and $\omega_\zeta = \bigwedge_{v \in \zeta} \mathrm{Im}(z_v)^{-2}(dz_v \wedge d\bar{z}_v)$.

We note that if a cusp form f belongs to $\mathscr{S}^{\zeta}_k(\Gamma_\lambda, \Phi_{\mathrm{m}})$, then obviously $f \in \mathscr{S}^{\zeta}_k(\Gamma^1_\lambda)$ also. Therefore we have an injection

$$\mathscr{S}^{\zeta}_k(\mathrm{m}, \Phi_{\mathrm{m}}) \to \prod_{\lambda=1}^{h} \mathscr{S}^{\zeta}_k(\Gamma^1_\lambda). \tag{1.21}$$

We check easily that the requisite conditions in the discussion of the cohomology theories are all satisfied, if we take the Γ there to be Γ^1_λ, $\lambda = 1, 2, \dots, h$. Thus the injectivity of (1.18) and (1.21) yields an embedding

$$\prod_{\zeta \subset \mathbf{a}} \mathscr{S}^{\zeta}_k(\mathrm{m}, \Phi_{\mathrm{m}}) \to \prod_{\zeta \subset \mathbf{a}} \prod_{\lambda=1}^{h} \mathscr{S}^{\zeta}_k(\Gamma^1_\lambda) \to \prod_{\lambda=1}^{h} H^n(\Gamma^1_\lambda; E). \tag{1.22}$$

The product of cohomology groups on the right-hand side of (1.22) can also be replaced by $\prod_{\lambda=1}^{h} H^n_s(\Gamma^1_\lambda; E)$ or $\prod_{\lambda=1}^{h} H^n(A(H^{\mathbf{a}}; E)^{\Gamma^1_\lambda})$, because of the equivalency we pointed out before. We shall use these cohomology theories interchangeably without further remarks. If $\mathbf{g} = (f_1, \dots, f_h) \in \mathscr{S}^{\zeta}_k(\mathrm{m}, \Phi_{\mathrm{m}})$, then the image of $\mathbf{g}$ in $\prod_{\lambda=1}^{h} H^n(A(H^{\mathbf{a}}; E)^{\Gamma^1_\lambda})$ will be written as $[\mathbf{g}] = ([f_1], \dots, [f_h])$. That is, we shall identify $[f_\lambda]$ with the cohomology class represented by it.

The Hecke operators on $\mathscr{S}_k^\zeta(\mathfrak{m}, \Phi_\mathfrak{m})$ can be defined as usual and will be denoted by T_v and S_v (which are generalizations of the operators $T(p)$ and $T(p, p)$ in [Sh1]). Hecke operators can also be defined on the cohomology groups via (1.22), and are denoted by the same symbols. We refer to [Sh4] for the details. The following equality holds:

$$\mathscr{S}_k^\zeta(\mathfrak{m}, \Phi) = \{\mathbf{g} \in \mathscr{S}_k^\zeta(\mathfrak{m}, \Phi_\mathfrak{m}) |\ \mathbf{g}|S_v = \Phi(\pi_v)\mathbf{g}, \forall v \in \mathbf{f}, v \nmid \mathfrak{m}\}. \tag{1.23}$$

Finally, an operator sending $\mathscr{S}_k^\zeta(\mathfrak{m}, \Phi_\mathfrak{m})$ onto $\mathscr{S}_k^{\zeta+\varepsilon}(\mathfrak{m}, \Phi_\mathfrak{m})$ can be defined in the same way as in [Sh4]. The image of $\mathbf{g}$ under this operator will be denoted simply by $\mathbf{g}^\varepsilon$. Its analogue in the cohomology groups will be denoted by $R(\varepsilon)$. Again we omit the details for the economy of space.

2. The fundamental periods of cusp forms

2.1. The definition of fundamental periods. Consider a primitive form $\mathbf{h} \in \mathscr{S}_k^0(\mathfrak{m}, \Phi)$. That is, $\mathbf{h}$ is a common eigenform of the Hecke operators T_v, and a nonzero form with the same eigenvalues cannot appear at a lower level. We denote the system of eigenvalues by χ. The set of all primitive forms is one-dimensional. Therefore, $\mathbf{h}$ is uniquely determined up to a constant factor (in $\mathbb{C}$) and hence is also uniquely determined, up to a constant factor, as an element of $\mathscr{S}_k^0(\mathfrak{m}, \Phi_\mathfrak{m})$ such that $\mathbf{h}|T_v = \chi(v)\mathbf{h}$ for all $v \in \mathbf{f}$ and $\mathbf{h}|S_v = \Phi(\pi_v)\mathbf{h}$ for $v \nmid \mathfrak{m}$, where $S_v = W\pi_v W$. Therefore we have, for every $\varepsilon \subset \mathbf{a}$,

$$\begin{aligned} \mathbb{C}\mathbf{h}^\varepsilon = \{\mathbf{g} \in \mathscr{S}_k^\varepsilon(\mathfrak{m}, \Phi_\mathfrak{m}) |\ \mathbf{g}|T_v = \chi(v)\mathbf{g}, \forall v \in \mathbf{f},\ \text{and}& \\ \mathbf{g}|S_v = \Phi(\pi_v)\mathbf{g}, \forall v \in \mathbf{f}, v \nmid \mathfrak{m}\}.& \end{aligned} \tag{2.1}$$

To avoid notational confusion, let us use the symbol Λ to denote the index set $\{1, \dots, h\}$. Now if we write $\mathbf{h} = (h_\lambda)_{\lambda\in\Lambda}$, and $\mathbf{h}^\varepsilon = (h_{\varepsilon,\lambda})_{\lambda\in\Lambda}$, then $h_{\varepsilon,\lambda} \in \mathscr{S}_k^\varepsilon(\Gamma_\lambda^1)$. Therefore, by (1.22), the images of $\mathbf{h}^\varepsilon$ in, say, $\prod_{\lambda\in\Lambda} H_s^n(\Gamma_\lambda^1; E)$, are linearly independent. Thus they form a basis of the image space, which we naturally denote by $\sum_{\varepsilon\subset\mathbf{a}} \mathbb{C}[\mathbf{h}^\varepsilon]$. In particular,

$$\dim\left(\sum_{\varepsilon\subset\mathbf{a}} \mathbb{C}[\mathbf{h}^\varepsilon]\right) = 2^n. \tag{2.2}$$

Denote the space of $\overline{\mathbb{Q}}$-rational elements in $\mathscr{S}_k^0(\mathfrak{m}, \Phi_\mathfrak{m})$ by $\mathscr{S}_k^0(\mathfrak{m}, \Phi_\mathfrak{m}, \overline{\mathbb{Q}})$. Then it is well known that

$$\mathscr{S}_k^0(\mathfrak{m}, \Phi_\mathfrak{m}) = \mathscr{S}_k^0(\mathfrak{m}, \Phi_\mathfrak{m}, \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}. \tag{2.3}$$

In particular, we may now assume that $\mathbf{h}$ is $\overline{\mathbb{Q}}$-rational.

As for the $\overline{\mathbb{Q}}$-rational structure of the cohomology group, we note that $E = \bigotimes_{v\in\mathbf{a}} \mathbb{C}^{k_v-1}$ obviously has a $\overline{\mathbb{Q}}$-rational structure $E = E(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. Therefore, for any congruence subgroup Γ, we may consider $E(\overline{\mathbb{Q}})$-valued elements of $C_s^q(\Gamma; E)$ and the resulting cohomology groups $H_s^q(\Gamma; E; \overline{\mathbb{Q}})$. We then have

$$H_s^q(\Gamma; E) = H_s^q(\Gamma; E; \overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, \quad \forall 0 \le q \in \mathbb{Z}. \tag{2.4}$$

Let $f \in \mathscr{S}_k^\varepsilon(\Gamma_\lambda, \Phi_\mathfrak{m})$ for any $\lambda \in \Lambda$. A simple computation shows that $\rho_k(\gamma)^{-1}[f] \circ \gamma = \Phi_\mathfrak{m}(a_\gamma)[f]$, for all $\gamma \in \Gamma_\lambda$. Therefore, given $q \in \sum_{\varepsilon\subset\mathbf{a}} \mathbb{C}[\mathbf{h}^\varepsilon]$ and writing $q = (q_\lambda)_{\lambda=1}^h \in \prod_{\lambda=1}^h H_s^n(\Gamma_\lambda^1; E)$, the following properties hold:

$$q|T_v = \chi(v)q, \forall v \in \mathbf{f}, \quad \text{and} \quad q|S_v = \Phi(\pi_v)q, \forall v \in \mathbf{f}, v \nmid \mathfrak{m}, \tag{2.5a}$$

$$\rho_k(\gamma)^{-1}q_\lambda \circ \gamma = \Phi_\mathfrak{m}(a_\gamma)q_\lambda, \quad \forall \gamma \in \Gamma_\lambda. \tag{2.5b}$$

We observe that the eigenvalues χ certainly do not occur in $H^n_{\mathrm{Eis}}(\Gamma^1_\lambda; E)$. They also do not occur in the space spanned by the ω_ζ in (1.20) (when $k = 2\cdot\mathbf{1}$). Therefore, by Proposition 1.2, $\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]$ is actually characterized by the properties (2.5a,b). Now these properties define a $\overline{\mathbb{Q}}$-rational structure on $\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]$. (Recall that the eigenvalues $\chi(v)$ are algebraic numbers.) Therefore, it is meaningful to take the intersection

$$K_\chi = \left(\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]\right)\cap\prod_{\lambda=1}^{h}H^n_s(\Gamma^1_\lambda; E; \overline{\mathbb{Q}}), \tag{2.6}$$

and we have

$$\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon] = K_\chi\otimes_{\overline{\mathbb{Q}}}\mathbb{C}. \tag{2.7}$$

We define an element of $GL(E)$ by $\mathbf{\Theta}_k = \bigotimes_{v\in\mathbf{a}}P_{k_v-2}$. (For the definition of P_m see [Sh4].) If $\omega\in A^q(H^\mathbf{a}; E)^\Gamma$ and $\nu\in A^r(H^\mathbf{a}; E)^\Gamma$, then ${}^t\omega\wedge\mathbf{\Theta}_k\nu$ is easily seen to be meaningful as an element of $A^{q+r}(H^\mathbf{a};\mathbb{C})^\Gamma$. In particular, if we take $\omega = [f]$ and $\nu = [g]$ where $f\in\mathscr{S}^\varepsilon_k(\Gamma)$ and $g\in\mathscr{S}^\zeta_k(\Gamma)$, respectively, then ${}^t[f]\wedge\mathbf{\Theta}_k[g]\in A^{2n}(H^\mathbf{a};\mathbb{C})^\Gamma$. When $\varepsilon+\zeta\neq\mathbf{a}$, it is obviously 0. In the remaining case, we compute easily that for $f, g\in\mathscr{S}^\varepsilon_k(B)$ (and hence $\overline{f}\in\mathscr{S}^{\mathbf{a}-\varepsilon}_k(B)$), the following formula holds:

$${}^t[\overline{f}]\wedge\mathbf{\Theta}_k[g] = (-1)^{\|k-k\varepsilon+\varepsilon\|+\frac{1}{2}n(n-1)}(2i)^{\|k\|-n}\cdot\overline{f}g\operatorname{Im}(z)^k d^\mathbf{a}_H z, \tag{2.8}$$

where

$$d^\mathbf{a}_H z = (2i)^{-n}\prod_{v\in\mathbf{a}}\operatorname{Im}(z_v)^{-2}d\overline{z}_v\wedge dz_v.$$

Consider a pairing of coefficients $P\colon E\times E\to\mathbb{C}$ defined by $P(a, b) = {}^ta\mathbf{\Theta}_k b$. Since Γ^1_λ acts on E via ρ_k and on $\mathbb{C}$ via $\rho_{2\cdot\mathbf{1}}$, which is the trivial representation, we have $P(\gamma\cdot a, \gamma\cdot b) = \gamma\cdot P(a, b)$. Therefore we have a cup product

$$\smile\colon H^q_s(\Gamma^1_\lambda; E)\times H^r_s(\Gamma^1_\lambda; E)\to H^{q+r}_s(\Gamma^1_\lambda;\mathbb{C}).$$

If $\omega\in A^q(H^\mathbf{a}; E)^{\Gamma^1_\lambda}$ and $\sigma\in A^r(H^\mathbf{a}; E)^{\Gamma^1_\lambda}$ are both closed, then $[\omega]\smile[\sigma]$ corresponds to the exterior product ${}^t\omega\wedge\mathbf{\Theta}_k\sigma$ under the de Rham isomorphism.

We now define the fundamental periods of $\mathbf{h}$ as follows. Since K_χ is stable under the operators $R(\zeta)$ for all $\zeta\subset\mathbf{a}$, we can define a regular representation of the additive group $(\mathbb{Z}/2\mathbb{Z})^\mathbf{a}$ on the space K_χ (and also on $\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]$, of course) by sending $\zeta\subset\mathbf{a}$ to $R(\zeta)$. Moreover, we have a nondegenerate pairing $(\mathbb{Z}/2\mathbb{Z})^\mathbf{a}\times(\mathbb{Z}/2\mathbb{Z})^\mathbf{a}\to\mathbb{Z}/2\mathbb{Z}$ defined by $\langle\varepsilon, \zeta\rangle = (-1)^{\|\varepsilon\cdot\zeta\|}$. Therefore, we can find a basis of K_χ over $\overline{\mathbb{Q}}$, denoted by $\{y_\varepsilon\}_{\varepsilon\subset\mathbf{a}}$, such that

$$y_\varepsilon|R(\zeta) = \langle\varepsilon, \zeta\rangle y_\varepsilon, \quad \forall\varepsilon, \zeta\subset\mathbf{a}. \tag{2.9}$$

These y_ε are uniquely determined up to factors in $\overline{\mathbb{Q}}$. Of course, they also form a basis of $\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]$ over $\mathbb{C}$. However, $\{[\mathbf{h}^\varepsilon]\}_{\varepsilon\subset\mathbf{a}}$ is also a basis of $\sum_{\varepsilon\subset\mathbf{a}}\mathbb{C}[\mathbf{h}^\varepsilon]$. Thus we may write

$$[\mathbf{h}] = \sum p(\chi, \varepsilon; B)y_\varepsilon, \tag{2.10}$$

where $p(\chi, \varepsilon; B) \in \mathbb{C}$. The coefficients of $[\mathbf{h}^\zeta]$ as a linear combination of the y_ε are given by

$$[\mathbf{h}^\zeta] = \sum_{\varepsilon \subset \mathbf{a}} \langle \varepsilon, \zeta \rangle p(\chi, \varepsilon; B) y_\varepsilon, \quad \forall \varepsilon \subset \mathbf{a}. \tag{2.11}$$

The complex numbers $p(\chi, \varepsilon; B)$ are called the fundamental periods of $\mathbf{h}$. They are uniquely determined up to algebraic factors since the y_ε are. From now on we shall regard them as elements of $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$. Clearly $p(\chi, \varepsilon; B) \neq 0$ for every $\varepsilon \subset \mathbf{a}$, since $\{y_\varepsilon\}_\varepsilon$ and $\{[\mathbf{h}^\varepsilon]\}_\varepsilon$ are both bases.

Theorem 2.1. *For each* $\varepsilon \subset \mathbf{a}$, *we have*

$$\overline{p(\chi, \varepsilon; B)} = p(\overline{\chi}, \varepsilon; B). \tag{2.12}$$

The proof of Theorem 4.4, (1) in [Sh4] goes through here without change.

Theorem 2.2. *For every* $\varepsilon \subset \mathbf{a}$, *we have*

$$p(\chi, \varepsilon; B) p(\overline{\chi}, k + \mathbf{a} + \varepsilon; B) \sim \pi^n \langle \mathbf{h}, \mathbf{h} \rangle. \tag{2.13}$$

Proof. The proof is a modification of that of Theorem 4.4, (2) of [Sh4]. Namely, writing $\mathbf{h}^\eta = (h_{\eta,\lambda})_{\lambda \in \Lambda}$ for every $\eta \subset \mathbf{a}$, we have by (2.8)

$${}^t[\overline{h_{\eta,\lambda}}] \wedge \Theta_k[h_{\eta,\lambda}] = (-1)^{b(\eta)} (2i)^{\|k\|-n} \cdot \overline{h_{\eta,\lambda}} \cdot h_{\eta,\lambda} \mathrm{Im}(z)^k d_H^{\mathbf{a}} z,$$

where we have written for notational simplicity $b(\eta) = \|k - k\eta + \eta\| + n(n-1)/2$. Integrating both sides over $F_\lambda \overset{\text{def}}{=} \Gamma_\lambda^1 \backslash H^{\mathbf{a}}$, we obtain on the right-hand side (by definition of the Petersson inner product)

$$(-1)^{b(\eta)} (2i)^{\|k\mathbf{a}\|-n} \cdot \mathrm{vol}(F_\lambda) \cdot \langle h_{\eta,\lambda}, h_{\eta,\lambda} \rangle,$$

and on the left-hand side

$$\int_{F_\lambda} {}^t[\overline{h_{\eta,\lambda}}] \wedge \Theta_k[h_{\eta,\lambda}].$$

We explain this integral as follows. Recall that $[h_{\eta,\lambda}]$ and $[\overline{h_{\eta,\lambda}}]$ are both fast decreasing harmonic forms; i.e., they belong to $\mathscr{H}_{fd}$ of Proposition 1.1. Thus we can write $[h_{\eta,\lambda}] = \mu + d\nu$, where μ has compact support $\bmod \Gamma_\lambda^1$ and ν is fast decreasing, and similarly for $[\overline{h_{\eta,\lambda}}]$. Also they are closed forms. Therefore we may approximate F_λ by cycles and hence it is meaningful to speak of the integral $\int_{F_\lambda} {}^t[\overline{h_{\eta,\lambda}}] \wedge \Theta_k[h_{\eta,\lambda}]$. Furthermore, its value depends only on the cohomology classes of $[\overline{h_{\eta,\lambda}}]$ and $[h_{\eta,\lambda}]$. Therefore we may denote this integral by $([\overline{h_{\eta,\lambda}}] \smile [h_{\eta,\lambda}])(F_\lambda)$. The rest of the computation is the same as in [Sh4]. $\square$

Finally we recall the concept of equivariant cycles. Given Γ and (ρ_k, E), an equivariant q-cycle is an element $u \in E \otimes_{\mathbb{Z}} S_q(H^{\mathbf{a}})$ such that ∂u is a finite sum of the form $\partial u = \sum [v \otimes \gamma(c) - {}^t\rho_k(\gamma) v \otimes c]$, where $u \in E$, $\gamma \in \Gamma$, and $c \in S_{q-1}(H^{\mathbf{a}})$. Here ∂ acts trivially on E. If $u \in E(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} S_q(H^{\mathbf{a}})$, then u is called $\overline{\mathbb{Q}}$-rational. For $\varphi \in C_s^q(\Gamma; E)$ and $w = \sum v \otimes c \in E \otimes_{\mathbb{Z}} S_q(H^{\mathbf{a}})$, we define $\varphi(w) = \sum {}^t v \varphi(c)$. Now we can easily check that if φ is a cocycle and u is an equivariant cycle, then $\varphi(u)$ depends only on the cohomology class of φ. This applies in particular to a closed form $\omega \in A^q(H^{\mathbf{a}}; E)^\Gamma$, and in this case $\varphi(u)$ is called a period of ω.

Theorem 2.3. *Write* $\mathbf{h}^\zeta = (h_{\zeta,\lambda})_{\lambda\in\Lambda}$ *for every* $\zeta \subset \mathbf{a}$. *If* u *is a* $\overline{\mathbb{Q}}$*-rational equivariant* n*-cycle with respect to* Γ_λ^1 *and* (ρ_k, E), *then the period* $[h_{\zeta,\lambda}](u)$ *is a* $\overline{\mathbb{Q}}$*-linear combination of the fundamental periods* $p(\chi, \varepsilon; B)$.

This follows immediately from (2.11).

2.2. **Relation to the special values of L-functions.** For the rest of this paper we assume that $k \in 2\mathbb{Z}^{\mathbf{a}}$. In order to establish the relationship between the fundamental periods and the special values of L-functions, we first recall two theorems, which are due to Shimura and Hida, respectively.

Let $\mathbf{g} = (f_\lambda)_{\lambda\in\Lambda} \in \mathscr{S}_k^0(\mathfrak{m}, \Phi_{\mathfrak{m}})$. Then f_λ is a Hilbert cusp form for every λ. Recall that f_λ has a Fourier expansion $f_\lambda(z) = \sum_\xi a_\lambda(\xi)\mathbf{e}(\xi z)$, where ξ runs through all the positive definite elements in $\mathfrak{t}_\lambda = t_\lambda \mathfrak{r}$, with the t_λ defined in (1.5c). Recall that every integral ideal can be written as $\xi \mathfrak{t}_\lambda^{-1}$ with a unique λ and a totally positive element $\xi \in \mathfrak{t}_\lambda$. So we may define, for every fractional ideal $\mathfrak{a}$,

$$c(\mathfrak{a}, \mathbf{g}) = \begin{cases} a_\lambda(\xi)\xi^{-k/2}, & \text{if } \mathfrak{a} = \xi\mathfrak{t}_\lambda^{-1} \text{ is integral;} \\ 0, & \text{if } \mathfrak{a} \text{ is not integral.} \end{cases} \tag{2.14}$$

Let ω be a Hecke character of finite order defined on $F_{\mathbf{A}}^\times$. We define an L-function associated to $\mathbf{g}$ and ω by

$$L(s, \mathbf{g}, \omega) = \sum_{\mathfrak{a}} c(\mathfrak{a}, \mathbf{g})\omega(\mathfrak{a})N(\mathfrak{a})^{-s}. \tag{2.15}$$

Now let $\mathbf{h}$ be a normalized $\overline{\mathbb{Q}}$-rational primitive form. (By normalized we mean $c(\mathfrak{r}, \mathbf{h}) = 1$.) Then we have $c(\mathfrak{a}, \mathbf{h})N(\mathfrak{a}) = \chi(\mathfrak{a})$. Therefore, we may define

$$L(s, \chi, \omega) = \sum_{\mathfrak{a}} \chi(\mathfrak{a})\omega(\mathfrak{a})N(\mathfrak{a})^{-s-1}, \tag{2.16}$$

and the following equality holds:

$$L(s, \mathbf{h}, \omega) = L(s, \chi, \omega). \tag{2.17}$$

We impose here one last condition.

(2.18) If $k_v = 2$ for some $v \in \mathbf{a}$ and $F \neq \mathbb{Q}$, then for every $r \in (\mathbb{Z}/2\mathbb{Z})^{\mathbf{a}}$ and every integral ideal $\mathfrak{n}$, there exists a Hecke character η such that $\eta_{\mathbf{a}}(x) = \operatorname{sgn}(x_{\mathbf{a}})^r$, $\mathfrak{n}|\mathfrak{c}_\eta$, and such that $L(0, \chi, \eta) \neq 0$.

Proposition 2.4 ([Sh2]). *Under the above conditions, there exists, for every* $r \in (\mathbb{Z}/2\mathbb{Z})^{\mathbf{a}}$, *a complex number* $V(\chi, r)$, *such that the following property holds:*

If ω *is a Hecke character on* $F_{\mathbf{A}}^\times$ *such that* $\omega_{\mathbf{a}}(x) = \operatorname{sgn}(x_{\mathbf{a}})^{t\cdot\mathbf{1}+r}$ *with* $t \in \mathbb{Z}$ *and* $|t| < k_v/2$, $\forall v \in \mathbf{a}$, *then*

$$L(t, \chi, \omega) \sim \pi^{tn} V(\chi, r). \tag{2.19}$$

We shall now briefly explain Hida's work [Hi]. For ease of reference, we shall first follow some notations and conventions adopted in Hida's paper and then explain how they correspond to our notations. Let a cusp form $f \in \mathscr{S}_k^\varepsilon(\Gamma)$ be given. We may attach a vector-valued differential form to f in the following manner. For every $v \in \mathbf{a}$, we define

$$\mathbf{x}_v = \begin{pmatrix} X_v \\ Y_v \end{pmatrix},$$

where X_v, Y_v are two indeterminate variables. Furthermore, put $n' = k - 2 \cdot \mathbf{1}$. We then denote by $L(n'; \mathbb{C})$ the module generated by homogeneous polynomials in (X_v, Y_v) of degree n'_v at every $v \in \mathbf{a}$. $L(n'; \mathbb{C})$ becomes a $GL_2(F)$-module via the action

$$\alpha \cdot P((\mathbf{x}_v)_{v\in\mathbf{a}}) = P((|\det(\alpha_v)|^{-1/2} \cdot \alpha_v^* \mathbf{x}_v)_{v\in\mathbf{a}}).$$

For $\mathbf{x} = (\mathbf{x}_v)_{v\in\mathbf{a}}$, we define furthermore

$$\psi_{n'}(\mathbf{x}) = \prod_{v\in\varepsilon} (X_v + iY_v)^{n'_v} \cdot \prod_{v\in\mathbf{a}-\varepsilon} (-X_v + iY_v)^{n'_v}.$$

Recall that, given f, we may consider the corresponding function f_1 defined on $SL_2(\mathbb{R})^{\mathbf{a}}$ given by

$$f_1(x) = f(x(\mathbf{i})) \cdot (j_k^\varepsilon(x, \mathbf{i}))^{-1}, \quad \forall x \in SL_2(\mathbb{R})^{\mathbf{a}}. \tag{2.20}$$

Then the differential form is defined by

$$[f] = j_{2\cdot\mathbf{1}}^\varepsilon(x, \mathbf{i}) \psi_{n'}((x_v^* \mathbf{x}_v)_{v\in\mathbf{a}}) \cdot f_1(x) d_\varepsilon z. \tag{2.21}$$

This differential form is related to the one defined in §1 by the following equation:

$$[f] = \prod_{v\in\mathbf{a}} (X_v - z'_v Y_v)^{k_v - 2} \cdot f(z) d_\varepsilon z, \tag{2.22}$$

where $z \stackrel{\text{def}}{=} x(\mathbf{i})$. Thus a correspondence between (2.21) and (1.17) can obviously be found. It is for this reason that we have denoted the two differential forms by the same symbol $[f]$. We introduce the following notation:

$$[f] = \sum_{\substack{0\le m\le n' \\ m\in\mathbb{Z}^{\mathbf{a}}}} [f]^m \binom{n'}{m} X^{n'-m} Y^m, \tag{2.23}$$

where

$$\binom{n'}{m} \stackrel{\text{def}}{=} \prod_{v\in\mathbf{a}} \binom{n'_v}{m_v}, \quad \text{and} \quad X^{n'-m} Y^m \stackrel{\text{def}}{=} \prod_{v\in\mathbf{a}} X_v^{n'_v - m_v} Y_v^{m_v}.$$

Then $[f]^m$ can be considered as a component of $[f]$ with values in $\mathbb{C}$. For every $\zeta \subset \mathbf{a}$, we define an element of $\{\pm 1\}^{\mathbf{a}}$ by letting the v-component be 1 or -1 according as $v \in \zeta$ or $v \in \mathbf{a} - \zeta$. Denote it again by ζ. Now consider a $j \in \mathbb{Z}^{\mathbf{a}}$ such that $0 \le j \le n'$ and $n' - 2j \in \mathbb{Z} \cdot \mathbf{1}$. Since $n' \in 2\mathbb{Z}^{\mathbf{a}}$, we may write

$$j - \frac{n'}{2} = \left[j - \frac{n'}{2}\right] \cdot \mathbf{1}, \quad \text{where } \left[j - \frac{n'}{2}\right] \in \mathbb{Z}.$$

For such a j and a Hecke character ω of finite order, we define, for our primitive form $\mathbf{h}$,

$$\varphi_\zeta^j = \sum_{\lambda\in\Lambda} \omega(\mathfrak{t}_\lambda \mathfrak{d}) N(\mathfrak{t}_\lambda \mathfrak{d})^{[n'/2-j]} \omega_{\mathbf{a}}(\mathbf{a} - \zeta) \cdot (\mathbf{a} - \zeta)^{n'+j+\mathbf{1}} [h_{\zeta,\lambda}]^j. \tag{2.24}$$

By taking a special cycle C, Hida proved that

Proposition 2.5 ([Hi]). *The integral of φ_ζ^j over C satisfies*

$$\int_C \varphi_\zeta^j \sim \pi^{-\|j+\mathbf{1}\|} \cdot L\left(\left[j - \frac{n'}{2}\right], \mathbf{h}, \omega\right). \tag{2.25}$$

We shall now apply Hida's theorem to relate the fundamental periods to L-values. By straightforward computations, we see that for $\varepsilon \subset \mathbf{a}$ and $m \in \mathbb{Z}^{\mathbf{a}}$, $(\mathbf{a} - \varepsilon)^m = \langle \varepsilon, m \rangle$. Note that the $\mathbf{a} - \varepsilon$ on the left-hand side is identified with an element belonging to $\{\pm 1\}^{\mathbf{a}}$, and the ε on the right-hand side is identified with an element of $(\mathbb{Z}/2\mathbb{Z})^{\mathbf{a}}$. We also have, for any given $m \in (\mathbb{Z}/2\mathbb{Z})^{\mathbf{a}}$,

$$\sum_{\zeta \subset \mathbf{a}} \langle \zeta, m \rangle [h_{\zeta, \lambda}] = 2^n \cdot p_m y_{m, \lambda}. \tag{2.26}$$

Indeed, by (2.11) we have

$$\begin{aligned} \sum_{\zeta \subset \mathbf{a}} \langle \zeta, m \rangle [h_{\zeta, \lambda}] &= \sum_{\zeta} \langle \zeta, m \rangle \sum_{\varepsilon} \langle \varepsilon, \zeta \rangle p_\varepsilon y_{\varepsilon, \lambda} \\ &= \sum_{\varepsilon} \left(\sum_{\zeta} \langle \zeta, m + \varepsilon \rangle p_\varepsilon y_{\varepsilon, \lambda} \right). \end{aligned}$$

But $\sum_\zeta \langle \zeta, m + \varepsilon \rangle = 2^n$ only when $m + \varepsilon = 0$ and is 0 otherwise. Thus (2.26) follows.

We now consider

$$\Phi^j \overset{\text{def}}{=} \sum_{\zeta \subset \mathbf{a}} \varphi_\zeta^j = \sum_{\lambda \in \Lambda} \omega(\mathfrak{t}_\lambda \mathfrak{d}) N(\mathfrak{t}_\lambda \mathfrak{d})^{[n'/2 - j]} \sum_{\zeta \subset \mathbf{a}} \omega_{\mathbf{a}}(\mathbf{a} - \zeta) \cdot (\mathbf{a} - \zeta)^{n' + j + \mathbf{1}} [h_{\zeta, \lambda}]^j. \tag{2.27}$$

By Proposition 2.5, we again have

$$\int_C \Phi^j \sim \pi^{-\|j+\mathbf{1}\|} \cdot L\left(\left[j - \frac{n'}{2}\right], \mathbf{h}, \omega\right).$$

Since ω is of finite order, we may assume that $\omega_{\mathbf{a}}(x) = \operatorname{sgn}(x_{\mathbf{a}})^l$ for some $l \in \mathbb{Z}^{\mathbf{a}}$. Then

$$\omega_{\mathbf{a}}(\mathbf{a} - \zeta) \cdot (\mathbf{a} - \zeta)^{n' + j + \mathbf{1}} = (\mathbf{a} - \zeta)^l \cdot (\mathbf{a} - \zeta)^{n' + j + \mathbf{1}} = \langle \zeta, n' + j + l + \mathbf{1} \rangle.$$

Hence we have

$$\begin{aligned} \Phi^j &= \sum_{\lambda \in \Lambda} \omega(\mathfrak{t}_\lambda \mathfrak{d}) N(\mathfrak{t}_\lambda \mathfrak{d})^{[n'/2 - j]} \cdot \sum_{\zeta \subset \mathbf{a}} \langle \zeta, n' + j + l + \mathbf{1} \rangle [h_{\zeta, \lambda}]^j \\ &= p_\varepsilon \cdot \sum_{\lambda \in \Lambda} 2^n \cdot \omega(\mathfrak{t}_\lambda \mathfrak{d}) N(\mathfrak{t}_\lambda \mathfrak{d})^{[n'/2 - j]} y_{\varepsilon, \lambda}^j, \end{aligned} \tag{2.28}$$

where $\varepsilon = n' + j + l + \mathbf{1}$. Since $y_{\varepsilon, \lambda}^j$ takes values in $\overline{\mathbb{Q}}$ for any cycle, we obtain $\int_C \Phi^j \sim p_\varepsilon$. Therefore we have the following proposition:

Proposition 2.6. *We adopt the same notations as above. Then*

$$L\left(\left[j - \frac{n'}{2}\right], \mathbf{h}, \omega\right) \sim \pi^{\|j+\mathbf{1}\|} p_\varepsilon. \tag{2.29}$$

It is now an easy matter to relate the p_ε to the $V(\chi, \zeta)$.

Theorem 2.7. *For every* $\zeta \subset \mathbf{a}$, *we have*

$$V(\chi, \zeta) \sim \pi^{\|k/2\|} \cdot p\left(\chi, \zeta + \frac{k}{2}, M_2(F)\right). \tag{2.30}$$

Proof. Let us start with a Hecke character ω such that $\omega_{\mathbf{a}}(x) = (x_{\mathbf{a}})^{\zeta + s \cdot \mathbf{1}}$, where $\zeta \subset \mathbf{a}$, $s \in \mathbb{Z}$, and $s < k_v/2$ for every $v \in \mathbf{a}$. Then we can find a unique $0 \leq j \leq n' = k - 2 \cdot \mathbf{1}$ such that $j - (k - 2 \cdot \mathbf{1})/2 = s \cdot \mathbf{1}$. Namely, $j = k/2 + (s-1) \cdot \mathbf{1}$. Now the ε in Proposition 2.6 is defined by

$$n' + j + l + \mathbf{1} = (k - 2 \cdot \mathbf{1}) + (k/2 + (s-1) \cdot \mathbf{1}) + (\zeta + s \cdot \mathbf{1}) + \mathbf{1} = k/2 + \zeta.$$

Here we recall that the element belongs to $(\mathbb{Z}/2\mathbb{Z})^{\mathbf{a}}$. By Propositions 2.4 and 2.6,

$$\pi^{sn} \cdot V(\chi, \zeta) \sim L(s, \chi, \omega) \sim \pi^{\|k/2 + s \cdot \mathbf{1}\|} \cdot p_{\varepsilon}.$$

Therefore $V(\chi, \zeta) \sim \pi^{\|k/2\|} \cdot p_{\zeta + k/2}$, as desired. □

To relate the fundamental periods to the Fourier coefficients, we recall that Shimura [Sh3, Theorem 9.4] has already established the precise relationship between the V and the Fourier coefficients. This together with Theorem 2.7 then settles the problem.

We conclude this paper by remarking that the fundamental periods p in our work are related to the periods P in [Sh4, §6] by the equation

$$P(\chi, \varepsilon; B) = \pi^{-n} p\left(\chi, \varepsilon + \frac{k}{2}; B\right). \tag{2.31}$$

References

[B] A. Borel, *Stable real cohomology of arithmetic groups.* II, Progr. in Math., vol. 14, Birkhäuser, Boston, MA, 1981, pp. 21–55.

[B-W] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Ann. of Math. Stud., no. 94, Princeton Univ. Press, Princeton, NJ, 1980.

[D] Z.-L. Dou, *Fundamental periods of certain arithmetic cusp forms*, Thesis, Princeton Univ., 1993.

[Ha] G. Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math. **89** (1987), 37–118.

[Hi] H. Hida, *On the critical values of L-functions of* $GL(2)$ *and* $GL(2) \times GL(2)$, Duke Math. J. **74** (1994), 431–529.

[M-S] Y. Matsushima and G. Shimura, *On the cohomology groups attached to certain vector-valued differential forms on the product of the upper half planes*, Ann. of Math. (2) **78** (1963), 417–449.

[Sh1] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Tokyo; Princeton Univ. Press, Princeton, NJ, 1971.

[Sh2] ——, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), 637–679.

[Sh3] ——, *On the critical values of certain Dirichlet series and the periods of automorphic forms*, Invent. Math. **94** (1988), 245–305.

[Sh4] ——, *On the fundamental periods of automorphic forms of arithmetic type*, Invent. Math. **102** (1990), 399–428.

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129
E-mail address: zdou@gamma.is.tcu.edu