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p-ADIC L-FUNCTIONS OF HILBERT MODULAR FORMS

by Andrzej DABROWSKI

1. Introduction.

Let p be a prime number and F a totally real number field of degree n over $\mathbb{Q}$. Let $\mathcal{O}_F$, $\vartheta \subset \mathcal{O}_F$, $D_F = \mathcal{N}(\vartheta)$ denote, respectively, the maximal order, the different and the discriminant of F. Let J_F be the set of all real embeddings of F, and let $\mathrm{Sgn}_F \subset F_\infty^\times := (F \otimes \mathbb{R})^\times \simeq \mathbb{R}^{\times n}$ denote the group of signs of F (i.e. elements of order 2 in $F_\infty^\times$).

Let $f \in \mathcal{M}_k(\mathfrak{c}, \psi)$ be a primitive Hilbert cusp form of type (k, ψ), where $k = (k_1, ..., k_n)$, $k_1 \equiv ... \equiv k_n (\mathrm{mod}\ 2)$; put $k_0 := \max\limits_i\{k_i\}$. Let $c^\pm(\sigma, f)$, $\sigma \in J_F$ denote the corresponding periods. Let f_χ denote the twist of f with a Hecke character χ of finite order. Let

$$L(s, f_\chi) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) C(\mathfrak{n}, f) \mathcal{N}\mathfrak{n}^{-s}$$
$$= \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p}) C(\mathfrak{p}, f) \mathcal{N}\mathfrak{p}^{-s} + \chi^2(\mathfrak{p}) \psi(\mathfrak{p}) \mathcal{N}\mathfrak{p}^{k_0 - 1 - 2s}\right)^{-1}.$$

Analytic properties of $L(s, f)$ suggest that f should correspond to a certain motive $M(f)$ over F of rank 2 and weight k_0 with coefficients in a field T containing all $C(\mathfrak{n}, \mathfrak{f})$ (see [Pa2; sec.7] for a discussion).

Through the paper we fix embeddings

$$i_\infty; \overline{\mathbb{Q}} \to \mathbb{C}, \qquad i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$$

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where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ is the Tate field (the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$) endowed with a unique norm $|\cdot|_p$ such that $|p|_p = p^{-1}$.

Put

$$1 - C(\mathfrak{p}, f)X + \psi(\mathfrak{p})\mathcal{N}\mathfrak{p}^{k_0-1}X^2 = (1-\alpha(\mathfrak{p})X)\,(1-\alpha'(\mathfrak{p})X) \in \mathbb{C}_p[X]$$

where $\alpha(\mathfrak{p}), \alpha'(\mathfrak{p})$ are the inverse roots of the Hecke polynomial; assume $\mathrm{ord}_p\alpha(\mathfrak{p}) \le \mathrm{ord}_p\alpha'(\mathfrak{p})$. Note, that with an embedding $\sigma_i \in J_F$ one can associate the embeddings $F \hookrightarrow \overline{\mathbb{Q}}$, $F \hookrightarrow \mathbb{C}_p$ and define a prime divisor $\mathfrak{p} = \mathfrak{p}(\sigma_i)$ of p in F attached to σ_i.

Let $[m_*, m^*]$ be the critical strip for $L(s, f_\chi)$, where

$$m_* = \max_i\{(k_0 - k_i)/2\} + 1,\ m^* = \min_i\{(k_0 + k_i)/2\} - 1.$$

Let $\Lambda(s, f_\chi) := \prod_{i=1}^{n} \Gamma_{\mathbb{C}}\left(s - \frac{k_0 - k_i}{2}\right) L(s, f_\chi)$ be the modified L-function of f_χ.

Let Gal_p denote the Galois group of the maximal abelian extension of F unramified outside p and ∞. The domain of definition of the non-archimedean L-function is the p-adic analytic Lie group $\mathfrak{X}_p := \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}_p, \mathbb{C}_p)$ of all continuous p-adic characters of the Galois group Gal_p.

For a Hecke character χ of finite order, with conductor $\mathfrak{c}(\chi)$, let $G(\chi)$ denote the Gauss sum; also define $\mathrm{sgn}(\chi) = \mathrm{sgn}(\chi_\infty) = (\epsilon_\sigma(\chi)) \in \mathrm{Sgn}_F$.

The aim of this paper is to prove the following result (stated in [Pa2; 8.2] without proof).

THEOREM 1. — *Put* $h = [\max_i(\mathrm{ord}_p(\alpha(\mathfrak{p}(\sigma_i)) - (k_0 - k_i)/2))] + 1$. *Then for each sign* $\epsilon_0 = \{\epsilon_{0,\sigma}\} \in \mathrm{Sgn}_F$ *there exists a* $\mathbb{C}_p$*-analytic function* $L_{(p)}^{(\epsilon_0)}$ *on* $\mathfrak{X}_p$ *of type* $o(\log^h)$ *with the properties :*

(i) *for all* $m \in \mathbb{Z}, m_* \le m \le m^*$, *and for all Hecke characters of finite order* $\chi \in \mathfrak{X}_p^{\mathrm{tors}}$ *with* $(-1)^m\epsilon_\sigma(\chi) = \epsilon_{0,\sigma}$ $(\sigma \in J_F)$ *the following equality holds :*

$$L_{(p)}^{(\epsilon_0)}(\chi\mathcal{N}x_p^m) = \frac{D_F^m i^{nm}}{G(\chi)} \prod_{\mathfrak{p}|p} A_{\mathfrak{p}}(f_\chi, m) \cdot \frac{\Lambda(m, f_\chi)}{\Omega(\epsilon_0, f)}$$

where

$$A_{\mathfrak{p}}(f_\chi, m) = \begin{cases} (1-\alpha'(\mathfrak{p})\mathcal{N}\mathfrak{p}^{-m})(1-\alpha(\mathfrak{p})^{-1}\mathcal{N}\mathfrak{p}^{m-1}) & \text{if } \mathfrak{p} \nmid \mathfrak{c}(\chi) \\ \left[\dfrac{\mathcal{N}\mathfrak{p}^m}{\alpha(\mathfrak{p})}\right]^{ord_{\mathfrak{p}}\mathfrak{c}(\chi)} & \text{if } \mathfrak{p} \mid \mathfrak{c}(\chi) \end{cases}$$

and

$$\Omega(\epsilon_0, f) = (2\pi i)^{-nm_*} \cdot D_F^{\frac{1}{2}} \cdot \prod_{\sigma} c^{\epsilon_{0,\sigma}}(\sigma, f),$$

is a certain constant depending only on ϵ_0 *and* f.

(ii) *If* $h \leq m^* - m_* + 1$ *then the function* $L_{(p)}^{(\epsilon_0)}$ *on* $\mathfrak{X}_{\mathfrak{p}}$ *is uniquely determined by conditions (i).*

(iii) *If* $\max_i \left(\mathrm{ord}_p \alpha(\mathfrak{p}(\sigma_i)) - \dfrac{(k_0 - k_i)}{2}\right) = 0$ *then the function* $L_{(p)}^{(\epsilon_0)}$ *is bounded on* $\mathfrak{X}_p$.

Remarks. — (i) Note that the quantities $c^{\pm}(\sigma, f)$ are defined in terms of the corresponding "motivic data" $M(f)$. In section 4 of his recent paper [Yo] Yoshida explained that the existence of these quantities is in agreement with the Shimura's paper [Shi] : take

$$u(m, f^\rho) := D_F^{\frac{1}{2}} \prod_{\sigma} c^{\epsilon_{0,\sigma}}(\sigma, f)^\rho$$

then the Deligne's period conjecture for $M(f)$ is equivalent to the Shimura's result ([Shi ;Th.4.3.(I)]).

(ii) In Shimura's theory, to deduce an algebraicity theorem for $L(m, f_\chi)$ (m critical) itself, on would like to be able to divide by $L(n, f_\rho)$ for some integer n and some Hecke character of finite order ρ. In fact, by [Shi, Prop.4.1.6] we have $L(s, f) \neq 0$ for $\mathrm{Re}(s) \geq \dfrac{k_0+1}{2}$, so, assuming $k^0 \geq 3$, we can divide by $L\left(\dfrac{k_0+k^0}{2} - 1, f_\rho\right)$ in this case. If $k_0 = 2$, k_0 have to be even, and the only choice for n is $n = \dfrac{k_0}{2}$. By a recent result of Rohrlich [Roh] we have in this case $L\left(\dfrac{k_0}{2}, f_\rho\right) \neq 0$, for some choice of ρ.

Strategy for proving the theorem.

First we give the Rankin type representation for the appropriate complex-valued distributions on the Galois group as a convolution with the Eisenstein series. In order to prove algebraicity properties of special values of distributions, we apply the holomorphic projection operator. In the proof of the growth conditions we use Atkin-Lehner theory and explicit form of Fourier coefficients of the corresponding modular forms. Then the p-adic L-function is constructed as the non-archimedean Mellin transform of the corresponding admissible measure.

We follow the notations and definitions from [Pa1], [Pa2], [Shi] if otherwise is not stated.

2. Eisenstein series for Hilbert modular group.

Notation. For $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and $z = (z_1, ..., z_n) \in \mathbb{C}$, we write

$$z^k := \prod_{\mu=1}^{n} z_\mu^{k_\mu}\ ,\ \{k\} := \sum_{\mu=1}^{n} k_\mu\ ,\ \{z\} := \sum_{\mu}^{n} z_\mu .$$

Let $e_F(z) := e(\{z\}) = \exp\left(2\pi i \sum\limits_{\mu=1}^{n} z_\mu\right)$.

We recall the definition of Eisenstein series in the Hilbert modular case. Let $\mathfrak{a}$, $\mathfrak{b}$ be arbitrary fractional ideals, η be a Hecke character of finite order of conductor $\mathfrak{c} \subset \mathcal{O}_F$ such that $\eta^*((x)) = \operatorname{sgn} x^{m\cdot 1}$ for $x \equiv 1 \bmod \mathfrak{c}$ (here $m \cdot 1 = (m, ..., m)$, for $m > 0$ an integer). Let $q = (q_1, ..., q_n) \in \mathbb{Z}^n$, $q_\nu \geq 0$. Put

$$\begin{aligned} K_m^q(z, s, \mathfrak{a}, \mathfrak{b}, \eta) &= (2\pi i)^{-\{q\}}(z - \overline{z})^{-q} \\ &\times \sum_{c\ ,\ d} \operatorname{sgn} \mathcal{N}(d)^m \eta^*(d\mathfrak{b}^{-1}) \left(\frac{c\overline{z}+d}{cz+d}\right)^q \mathcal{N}(cz+d)^{-m} |\mathcal{N}(cz+d)|^{-2s} \\ L_m^q(z, s, \mathfrak{a}, \mathfrak{b}, \eta) &= (2\pi i)^{-\{q\}}(z - \overline{z})^{-q} \\ &\times \sum_{c\ ,\ d} \operatorname{sgn} \mathcal{N}(c)^m \eta^*(c\mathfrak{a}^{-1}) \left(\frac{c\overline{z}+d}{cz+d}\right)^q \mathcal{N}(cz+d)^{-m} |\mathcal{N}(cz+d)|^{-2s} \end{aligned}$$

the summation being taken over a system of representatives (c, d) of equivalence classes with respect to the $\mathcal{O}_F^\times$-equivalence relation for non-zero elements in $\mathfrak{a} \times \mathfrak{b}$ given by $(c, d) \sim (uc, ud)$ with $u \in \mathcal{O}_F^\times$. The above series can be extended to functions on the adelized group G_A so that

$$K_m^q(s, \mathfrak{a}, \mathfrak{b}, \eta)_\lambda = \mathcal{N}(\tilde{t}_\lambda)^{s+\frac{m}{2}} \mathcal{N}(y)^s K_m^q(z, s, \tilde{t}_\lambda \mathfrak{a}\vartheta, \mathfrak{b}, \eta)$$

$$L_m^q(s, \mathfrak{a}, \mathfrak{b}, \eta)_\lambda = \mathcal{N}(\tilde{t}_\lambda)^{-s-\frac{m}{2}} \mathcal{N}(y)^s L_m^q(z, s, \mathfrak{a}, \mathfrak{b}\tilde{t}_\lambda^{-1}\vartheta^{-1}, \eta).$$

We shall need the explicit form of Fourier expansion of the Eisenstein series.

PROPOSITION ([Ka], [Pal ; 4.4.2]). — *For $s \in \mathbb{Z}$ such that $\forall \nu\ s-q_\nu \le 0$ we have the following Fourier expansion :*

$$\frac{D_F^{\frac{1}{2}}\mathcal{N}(\tilde{t}_\lambda)\prod_\nu \Gamma(s+m+q_\nu)}{(-2\pi i)^{n(m+2s)}(-1)^{ns+\{q_\nu\}}} L_m^q(z,s,\mathcal{O}_F,\tilde{t}_\lambda^{-1}\vartheta^{-1},\eta)$$
$$= (4\pi y)^{-q} \sum_{0 \ll \xi \in \tilde{t}_\lambda} a_\lambda(\xi,s,y,\eta)\, e_F(\xi z)$$

where

$$a_\lambda(\xi,s,y,\eta) := \sum_{\tilde{\xi}=\tilde{d}\,\tilde{d}'\ ,\ d\in\tilde{t}_\lambda\ ,\ d'\in\mathcal{O}_F} \operatorname{sgn}\mathcal{N}(d)^{m-1}\mathcal{N}(d)^{m+2s-1}\eta^*(\tilde{d}')$$
$$\times \prod_\nu W(4\pi\xi_\nu y_\nu\ ,\ m+s+q_\nu\ ,\ s-q_\nu)$$

and

$$W(y,\alpha,-r) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}y^{r-i}$$

for $r \in \mathbb{Z}$, $r \gg 0$.

Let χ, ρ be Hecke characters of finite order of conductor $\mathfrak{m}$, $\mathfrak{n}$ respectively, such that $\chi(x_\infty) = \operatorname{sgn}(x_\infty)^q$, $\rho(x_\infty) = \operatorname{sgn}(x_\infty)^t$, where $q = (q_1,\dots,q_n), t = (t_1,\dots,t_n) \in \mathbb{Z}^n$ (parity of χ, ρ respectively). Assume that $q+t \equiv l\cdot 1 (\mathrm{mod}\ 2\mathbb{Z}^n)$ for some $l \in \mathbb{N}$. Then there exist modular forms $g_l(\chi)$, $g_l^{\sim}(\chi) \in \mathcal{M}_{l\cdot 1}(\mathfrak{mn},\chi\rho)$ such that

$$L(s,g_l(\chi)) = L_\mathfrak{n}(s,\rho)\cdot L_\mathfrak{m}(s+1-l,\chi),$$
$$L(s,g_l^{\sim}(\chi)) = L_\mathfrak{n}(s+1-l,\rho)\cdot L_\mathfrak{m}(s,\chi)$$

(see [Shi, p. 660]).

3. Algebraicity theorem.

Let $f \in \mathcal{M}_k(\mathfrak{c},\psi)$ be a primitive cusp form of type (k,ψ), where $k_1 \equiv \dots \equiv k_n (\mathrm{mod}\ 2)$, i.e. f is a "motivic" form. Then

$$\mathfrak{D}_{\mathfrak{cmn}}(s,f,g_l(\chi)) := L_{\mathfrak{cmn}}(2s+2-k_0-l,\psi\chi)\cdot L(s,f,g_l(\chi))$$
$$= L\left(s+1-l,f_\chi\right)\cdot L(s,f_\rho)$$

where $1 \le l \le k^0-1$, $k^0 := \min_\nu\{k_\nu\}$, $k_0 := \max_\nu\{k_\nu\}$.

Let

$$\Psi(s, f, g_l(\chi)) := \gamma_n(s) \cdot \mathfrak{D}_{\mathrm{cm}}(s, f, g_l(\chi)),$$

where

$$\gamma_n(s) := (2\pi)^{-2ns} \cdot \prod_{\nu=1}^{n} \Gamma\left(s + 1 + \frac{k_\nu - k_0}{2} - l\right) \cdot \Gamma\left(s + \frac{k_0 - k_\nu}{2}\right).$$

We have the following fundamental result of Shimura (see [Shi, p. 661-662]) :

$$\frac{\Psi(h_* + r, f, g_l(\chi))}{\pi^{\{k\}-n(k_0+l-1)}\langle f, f\rangle} \in \overline{\mathbb{Q}}, \qquad 0 \le r \le h^* - h_* - 1 \tag{1}$$

where

$$h_* = h_*(l) := \max_\nu \left\{\frac{k_0 - k_\nu}{2}\right\} + l,$$

$$h^* = h^*(l) := \min_\nu \left\{\frac{k_0 + k_\nu}{2}\right\}.$$

Let $r(l) := h^*(l) - h_*(l) - 1 = k^0 - l - 1$. We have a sequence of modular forms

$$g_{l(r)}(\chi), \quad l(r) = k^0 - r - 1, \ r = 0, 1, ..., k^0 - 2, \ l(r) \cdot 1 \equiv q + t (\mathrm{mod}\ 2\mathbb{Z}^n).$$

Now, fixing the sign ϵ_0 as in Theorem 1, we choose $\rho = \rho(\epsilon_0)$ so that

$$L\left(\frac{k_0 + k^0}{2} - 1, f_\rho\right) \neq 0.$$

Then (1) is equivalent to the following :

$$\frac{L\left(r + 1 + \frac{k_0 - k^0}{2}, f_\chi\right)}{(-2\pi i)^{n(r+1)} \cdot \Omega(f)} \in \overline{\mathbb{Q}}$$

for $0 \le r \le k^0 - 2$, $(k^0 - r - 1) \cdot 1 \equiv q + t (\mathrm{mod}\ 2\mathbb{Z}^n)$,

where

$$\Omega(\epsilon_0, f) := L\left(\frac{k_0 + k^0}{2} - 1, f_\rho\right)^{-1} \cdot G(\rho)^{-1} \cdot \langle f, f\rangle_c \cdot (-2\pi i)^{\{k\}-n(k_0+k^0-1)}$$

(compare [Shi, 4.34]).

4. Complex valued distributions on Gal_p associated with the Hilbert modular form.

We define a family $\mu_r^{\sim} = \mu_r^{\sim\,(\epsilon_0)}$, $r = 0, 1, ..., k^0 - 2$ of complex-valued distributions in the following way :

$$\mu_{r,\mathfrak{m}}^{\sim}(\chi_\mathfrak{m})$$

$$:= \frac{\mathcal{N}(\mathfrak{c}\vartheta)^{\frac{k_0+k^0}{2}-1} \cdot \mathcal{N}(\mathfrak{m})^{\frac{k_0+r-1}{2}}}{\alpha(\mathfrak{m}) \cdot 2^{2\{k\}-n(k_0+k^0-r-1)}} \cdot \frac{\Psi\left(\frac{k_0+k^0}{2} - 1\ ,\ f_0\ ,\ g_{l(r)}^{\sim}(\chi_\mathfrak{m})|_{J_{\mathfrak{cmn}}}\right)}{\pi^{\{k\}-n(k_0+l(r)-1)} \cdot i^{\{k+(k^0-r+1)1\}} \cdot \langle f, f\rangle_\mathfrak{c}},$$

where $\mathfrak{m}$ is arbitrary ideal with the condition $\mathfrak{m}_0\mathfrak{c}(\chi)|\mathfrak{m}$, $\mathfrak{m}_0 := \prod_{\mathfrak{p}|p} \mathfrak{p}$ and

$$f_0 := \sum_{\mathfrak{a}|\mathfrak{m}_0} \mu(\mathfrak{a})\alpha'(\mathfrak{a})f|\mathfrak{a}.$$

The operators $F|_\mathfrak{q}$, $f|_{U(\mathfrak{q})}$, $f|_{J_\mathfrak{c}}$ are defined in [Pal, p.124], [Shi, p.655].

5. The Rankin integral representation.

First let us specialize the integral representation of Rankin type (see [Shi, 4.32]) to our situation :

$$\begin{aligned}\Psi\left(s, f_0, g_{l(r)}^{\sim}(\chi_\mathfrak{m})|_{J_{\mathfrak{cmn}}}\right) &= D_F^{\frac{1}{2}}\,(-1)^{\{\frac{k-k^0\cdot 1}{2}\}}\,2^{-n(k_0+k^0)+\{k\}}\,\pi^{-n(\frac{k_0+k^0}{2}+s)+\{k\}} \\ &\times \prod_\nu \Gamma\left(s+1+\frac{k_\nu - l_\nu - k_0 - l_0}{2}\right) \\ &\times \langle\ f_0^\rho\ ,\ g_{l(r)}^{\sim}(\chi_\mathfrak{m})\ K_{r+1}^q(0, \mathfrak{cmn}, \psi\chi^{-1})\rangle_{\mathfrak{cmn}}.\end{aligned}$$

The method by which the right hand side can be explicitely calculated is based on an application of the trace operator (see [Pal, p.136]). Actually, we obtain the following representation for the values of the distributions :

$$\mu_{r,\mathfrak{m}}^{\sim}(\chi_\mathfrak{m}) = \frac{\gamma(\mathfrak{m})}{\langle f, f\rangle} \cdot \langle f_0^\rho, g_{l(r)}^{\sim}(\chi_{\mathfrak{m}_0}) \cdot E_{r+1}^q(0, \mathcal{O}_F, \mathcal{O}_F, \psi\chi^{-1})|_{U(\mathfrak{m})J_{\mathfrak{cm}_0\mathfrak{n}}}\rangle_{\mathfrak{cm}_0\mathfrak{n}}$$

where

$$\gamma(\mathfrak{m}) := \alpha(\mathfrak{m})^{-1} \cdot \mathcal{N}(\mathfrak{m}_0)^{\frac{k_0}{2}-1}$$

and

$$E_m^q(s,\mathcal{O}_F,\mathcal{O}_F,\eta) := D_F^{1/2}\prod_\nu \Gamma(s+m+q_\nu)(-2\pi i)^{-n(m+s)}$$
$$\times (-4\pi)^{ns} L_m^q(s,\mathcal{O}_F,\mathcal{O}_F,\eta).$$

6. Application of the holomorphic projection operator.

In order to prove algebraicity properties of special values of distributions, we apply a certain holomorphic projection operator to the Hilbert automorphic form

$$g_{\widetilde{l(r)}}(\chi_{\mathfrak{m}}) \cdot E_{r+1}^q(0,\mathcal{O}_F,\mathcal{O}_F,\psi\chi^{-1}).$$

DEFINITION. — *Let $\overline{\mathcal{M}}_k(\mathfrak{c},\psi)$ denote the $\mathbb{C}$-linear space of $\mathcal{C}^\infty$-Hilbert automorphic forms consisting of Hilbert automorphic forms of weight $k\in\mathbb{Z}^n$, level $\mathfrak{c}\subset\mathcal{O}_F$, and Hecke character ψ satisfying the following condition :*

(*) for each $x\in G_A$ with $x_\infty=1$ there exists a $\mathcal{C}^\infty$- function G_x : $H^n\to\mathbb{C}$ such that $F(x\,y)=(G_x|_k\,y)(i,\dots,i)$ for all $y=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in G_\infty$.

For $\lambda=1,\dots,k$, $F_\lambda := G_{x'_\lambda}$ belongs to the vector space $\overline{\mathcal{M}}(\Gamma_\lambda,\psi)$ of all $\mathcal{C}^\infty$-modular forms on H^n relative to the congruence subgroup Γ_λ :

$$(F_\lambda|_k\,\gamma)\,(x)=\psi(\gamma)\;F_\lambda(z)\qquad\forall\gamma\in\Gamma_\lambda$$

where $a_\lambda(\xi,y)\in\mathcal{C}^\infty((\mathbb{R}_+)^n)$.

The map

$$F\longmapsto(F_1,\dots,F_\lambda)$$

defines a vector space isomorphism

$$\overline{\mathcal{M}}_k(\mathfrak{c},\psi)\cong\oplus_{\lambda=1}^h\overline{\mathcal{M}}_k(\Gamma_\lambda,\psi).$$

DEFINITION. — *A function $F\in\overline{\mathcal{M}}_k(\mathfrak{c},\psi)$ is called a function of moderate growth if for all $\lambda=1,\dots,h$, $z\in H^n$, $s\in\mathbb{C}$, $\mathrm{Re}(s)\gg 0$ the integral*

$$\int_{H^n} F(w)(\overline{w}-z)^{-k-|2s|}\,\mathrm{Im}(w)^{k+s}\,d^\times w$$

is absolutely convergent and admits an analytic continuation over s to the point $s = 0$.

Here we use the notation :

$$u^{-k-|2s|} := \mathcal{N}u^{-k}|\mathcal{N}u|^{-2s} = \prod_{\nu} \left(u_\nu^{-k_\nu}|u_\nu|^{-2s}\right).$$

PROPOSITION. — *Let $F \in \overline{\mathcal{M}}_k(\mathfrak{c}, \psi)$ be a function of moderate growth such that its Fourier expansion contains only terms $a_\lambda(\xi, y)$ with totally positive $\xi \in \tilde{t}_\lambda$. Set for $\xi \gg 0$, $\xi \in \tilde{t}_\lambda$*

$$a_\lambda(\xi) := (4\pi\xi)^{k-1} \prod_{\nu} \Gamma(k_\nu - 1)^{-1} \int_{(\mathbb{R}^+)^n} a_\lambda(\xi, y) e_F(i\xi y) y^{k-2} dy$$

and suppose that the integral is absolutely convergent.

Put

$$\mathrm{Hol}(F) = (\mathrm{Hol}\,(F_1)\,, ..., \mathrm{Hol}\,(F_k))\,,$$

$$\mathrm{Hol}\,(F_\lambda(z)) = \sum_{\xi \in t_\lambda^\sim} a_\lambda(\xi) e_F(\xi z).$$

Then $\mathrm{Hol}(F) \in \mathcal{M}_k(\mathfrak{c}, \psi)$and for any cusp form $g \in \mathcal{S}_k(\mathfrak{c}, \psi)$ we have that

$$< g, F >_{\mathfrak{c}} = < g, \mathrm{Hol}(F) >_{\mathfrak{c}} .$$

The proof of the proposition is carried out in a similar way to that of the Proposition 4.7 in [Pa1], where the case of scalar vector weight was treated (see also [Ka]), so we omit the proof.

Now we put

$$V_{r+1}^q(\chi) := \mathrm{Hol}\left(g_{\tilde{l}(r)}(\chi_{\mathfrak{m}_0}) \cdot E_{r+1}^q(0, \mathcal{O}_F, \mathcal{O}_F, \psi\chi^{-1})\right).$$

Then

$$\mu_{r,\mathfrak{m}}^\sim(\chi_{\mathfrak{m}}) = \frac{\gamma(\mathfrak{m})}{\langle f, f \rangle} \cdot \left\langle f_0^\rho, V_{r+1}^q(\chi)|_{U(\mathfrak{m})J_{\mathfrak{c}\mathfrak{m}_0\mathfrak{n}}} \right\rangle_{\mathfrak{c}\mathfrak{m}_0\mathfrak{n}}.$$

7. Explicit formulae for the Fourier coefficients.

From Proposition in Section 2, we plainly obtain (see [Pa1, pp.142-143], for a similar calculation) the following

PROPOSITION. — *$V_{r+1}^q(\chi)$ has the following Fourier expansion :*

$$V_{r+1}^q(\chi)_\lambda(z) = \sum_{0 \ll \xi \in t_\lambda^\sim} U_\lambda^q(\xi, r, \chi) e_F(\xi z)$$

where

$$\begin{aligned} U_\lambda^q(\xi, r, \chi) := \mathcal{N}(\tilde{t}_\lambda)^{-\frac{r+1}{2}} \cdot & \sum_{\xi = \xi_1 + \xi_2, \xi_1 \gg 0, \xi_2 \gg 0} b_\lambda(r, \xi_1, \chi) \\ \times & \sum_{(\xi_2) = (d)(d'), d \in \tilde{t}_\lambda, d' \in \mathcal{O}_F} \operatorname{sgn} \mathcal{N}((d)) \cdot \mathcal{N}((d))^r (\psi\chi^{-1})((d')) \\ \times & \prod_{\nu=1}^{n} P_\nu^q(\xi_{2,\nu}, \xi_\nu, r) \end{aligned}$$

and

$$\begin{aligned} & P_\nu^q(\xi_{2,\nu}, \xi_\nu, r) \\ & := \sum_{i=0}^{q_\nu} (-1)^i \binom{q_\nu}{i} \frac{\Gamma(r+1+q_\nu)}{\Gamma(r+1+q_\nu-i)} \frac{\Gamma(k_\nu - 1 - i)}{\Gamma(k_\nu - 1)} \cdot \xi_{2,\nu}^{q_\nu - i} \xi_\nu^i, \end{aligned}$$

and

$$g_{\widetilde{l(r)}}(\chi_{\mathfrak{m}_0})(z)_\lambda = \sum_{0 \ll \xi \in \tilde{t}_\lambda} b_\lambda(r, \xi, \chi) e_F(\xi z) + b_0^\lambda(r, \chi).$$

8. Application of the Atkin-Lehner theory.

Now let us consider the linear form given by

$$\mathbf{L} : \Phi \mapsto \frac{\langle f_0^\rho, \Phi|_{J_{\mathfrak{c}\mathfrak{m}_0\mathfrak{n}}} \rangle}{\langle f, f \rangle_{\mathfrak{c}}}$$

on the complex linear space $\mathcal{M}_k(\mathfrak{c}\mathfrak{m}_0, \psi)$. From the Atkin-Lehner theory (in Miyake's form [Miy]) it follows that $\mathbf{L}$ is defined over $\overline{\mathbb{Q}}$, i.e. for a finite number of ideals $\mathfrak{m}_i$ and fixed algebraic numbers $l(\mathfrak{m}_i) \in \overline{\mathbb{Q}}$ we have the equality :

$$\mathbf{L}(\Phi) = \sum_i C(\mathfrak{m}_i, \Phi) l(\mathfrak{m}_i).$$

Therefore the distributions $\mu_r^{\sim}$ can be written in the form

$$\mu_{r,\mathfrak{m}}^{\sim}(\chi_{\mathfrak{m}}) = \gamma(\mathfrak{m}) \cdot \mathbf{L}(\Phi_{r,\mathfrak{m}}^{q}(\chi)),$$

where

$$\Phi_{r,\mathfrak{m}}^{q}(\chi) := V_{r+1}^{q}(\chi)|_{U(\mathfrak{m})}.$$

9. The growth conditions.

Given an integral ideal $\mathfrak{m} \subset \mathcal{O}_F$, let $I(\mathfrak{m})$ denote the group of all fractional ideals in F, prime to $\mathfrak{m}$. Also let

$$P(\mathfrak{m}) := \{(\alpha) |\ \alpha \in F_+^{\times},\ \alpha \equiv 1(\mathrm{mod}^{\times}\mathfrak{m})\},\ \ H(\mathfrak{m}) := I(\mathfrak{m})/P(\mathfrak{m}),$$

$$h(\mathfrak{m}) := \mathrm{card}\ H(\mathfrak{m}).$$

Then $\mathrm{Gal}_p = \varprojlim H(\mathfrak{m})$ (where limit is over $\mathfrak{m}$ with the condition $S(\mathfrak{m}) \subset S(\mathfrak{m}_0)$).

Let $\pi_{\mathfrak{m}} : \mathrm{Gal}_p \to H(\mathfrak{m})$ be the natural projection; put $(\mathfrak{m}) := \ker \pi_{\mathfrak{m}}$.

Let $\mathcal{C}^m(\mathrm{Gal}_p)$ denote the space of $\mathbb{C}_p$-valued functions, which locally can be represented by polynomials of degree less than m in variable $\mathcal{N}x_p$.

THEOREM 2. — *There exist $\mathbb{C}_p$ - linear forms*

$$\mu^{\sim} = \mu^{\sim (\epsilon_0)} : \mathcal{C}^{k^0-1}(Gal_p) \to \mathbb{C}_p$$

such that

$$\int\limits_{a+(\mathfrak{m})} \mathcal{N}x^r d\mu^{\sim} = (-1)^{nr} \int\limits_{a+(\mathfrak{m})} d\mu_r^{\sim}, \quad r = 0, 1, ..., k^0 - 2.$$

For $\mu^{\sim}$ the following growth condition is satisfied :

$$\sup_{a \in \mathrm{Gal}_p} \left| \int\limits_{a+(\mathfrak{m})} (\mathcal{N}x_p - \mathcal{N}a_p)^r d\mu^{\sim} \right|_p = O\left(|\mathfrak{m}|_p^{r-\mathrm{ord}_p\alpha(\mathfrak{p})}\right).$$

Proof. — The existence follows from the definition of $\mu_r^{\sim}$. We now check the growth condition; we can suppose that $a \in \mathcal{O}_F$. From the section 8 we obtain :

$$\begin{aligned}\int_{a+(\mathfrak{m})} (\mathcal{N}x - \mathcal{N}a)^r d\mu^{\sim} &= \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(a))^{r-j} (-1)^{nj} \int_{a+(\mathfrak{m})} d\mu_j^{\sim} \\ &= (-1)^{rn} \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(-a))^{r-j} \\ &\quad \times \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \mu_j^{\sim}(\chi) \\ &= (-1)^{rn} \gamma(\mathfrak{m}) \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(-a))^{r-j} \\ &\quad \times \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \mathbf{L}(\Phi_j^q(\chi)).\end{aligned}$$

From the Atkin - Lehner theory it follows that the congruences in the above theorem are sufficient to check for the following number $\mathbf{A}$ (for $\xi \tilde{t}_\lambda^{-1} \equiv 0 \bmod \mathfrak{m}$) :

$$\begin{aligned}\mathbf{A} := (-1)^{nr} \gamma(\mathfrak{m}) \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(-a))^{r-j} \frac{1}{h(\mathfrak{m})} \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) U_\lambda^q(\xi, j, \chi) \\ = (-1)^{nr} \gamma(\mathfrak{m}) \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(-a))^{r-j} \frac{1}{h(\mathfrak{m})} \\ \times \sum_{\chi \bmod \mathfrak{m}} \chi^{-1}(a) \sum_{\xi = \xi_1 + \xi_2, \xi_1 \gg 0, \xi_2 \gg 0} b_\lambda(j, \xi_1, \chi) \\ \times \sum_{(\xi_2) = (d)(d'), d \in \tilde{t}_\lambda, d' \in \mathcal{O}_F} \operatorname{sgn} \mathcal{N}(d) \cdot \mathcal{N}(d)^j (\psi \chi^{-1})(d') \prod_{\nu=1}^{n} P_\nu^q(\xi_{2,\nu}, \xi_\nu, j).\end{aligned}$$

Now, taking into account the explicit form of the Fourier coefficients for $g_{l(r)}^{\sim}(\chi_{\mathfrak{m}_0})$ (see [Shi], p 660) and taking summation over all χ, we obtain :

$$\mathbf{A} = (-1)^{nr} \cdot \alpha(\mathfrak{m}\mathfrak{m}_0)^{-1} \cdot \mathcal{N}(\mathfrak{m}_0)^{\frac{k_0}{2}-1}$$

$$\times \sum_{\substack{\xi = \xi_1 + \xi_2, \\ \xi_1 \gg 0, \xi_2 \gg 0\,, \\ (\xi_1) = (e)(e')\,, \\ (\xi_2) = (d)(d')\,, \\ -ae \equiv d\tilde{\mathfrak{t}}_\lambda^{-1} (\mathrm{mod}\ \mathfrak{m})}} \rho(e')\psi(d')\mathrm{sgn}\mathcal{N}(de) \cdot \sum_{j=0}^{r} \binom{r}{j} (-\mathcal{N}(-a))^{r-j} \cdot \mathcal{N}(d)^j$$

$$\times \mathcal{N}(e)^{k^0-j-1} \cdot \mathcal{N}(\tilde{\mathfrak{t}}_\lambda^{-1})^j \cdot \prod_{\nu=1}^{n} P_\nu^q(\xi_{2,\nu}, \xi_\nu, j).$$

Lemma. — *Let $h \geq q$ be positive rational integers, and $\alpha, \beta \in \mathcal{O}_F$, $\alpha \equiv \beta \bmod \mathfrak{m}$. Then*

$$\sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j} \cdot (-\beta)^j \cdot j^q$$

belongs to $\mathfrak{m}^{h-q}$.

Proof (of the Lemma). — Induction with respect to q. The case $q = 0$ is trivial.

Now

$$\sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j}(-\beta)^j j^q = \sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j}(-\beta)^j \left[j \cdot \ldots \cdot (j-q+1) + P_{q-1}(j)\right]$$

$$= h \cdot \ldots \cdot (h-q+1)(-b)^q(\alpha-\beta)^{h-q} + \sum_{j=0}^{h} \binom{h}{j} \alpha^{h-j}(-\beta)^j P_{q-1}(j)$$

where $P_{q-1}(j)$ is a polynomial of degree $q-1$ in j.

The assertion follows.

Let us continue the proof of the theorem. Firstly, $\prod_{\nu=1}^{n} P_\nu^q(\xi_{2,\nu}, \xi_\nu, j)$ is the polynomial of degree $\sum_{\nu=1}^{n} q_\nu$ in variable j; note also that $P_\nu^q(\xi_{2,\nu}, \xi_\nu, j)$ is homogeneous of degree q_ν in variables $\xi_{2,\nu}$ and ξ_ν. On the other hand,

ξ is divisible by $\mathfrak{m}$ and $|\xi_\nu|_p = |\xi|_p$; consequently $\prod_{\nu=1}^{n} \xi_\nu^{q_\nu}$ is divisible by $\prod_\nu \mathfrak{m}^{q_\nu}$. Now, if $r \geq \sum_\nu q_\nu$, then the theorem follows from the above lemma : take $h = r$, $\alpha = \mathcal{N}(-ae)$, $\beta = \mathcal{N}\left(d\tilde{\mathfrak{t}}_\lambda\right)$. The case $r < \sum_\nu q_\nu$ is trivial.

10. End of the proof of Theorem 1.

First, we can explicitely determine Euler factors of the Rankin convolution. Proceeding similarly as in ([Pa1], p.130-132; or [Pa3], p.135-140) we obtain

$$\Psi\left(\frac{k_0+k^0}{2}-1, f_0, g_{l(r)}\left(\chi_{\mathfrak{m}_0}\right)|_{J_{\mathfrak{c}\mathfrak{m}\mathfrak{m}_0\mathfrak{n}}}\right)$$
$$= P(f,\rho)\mathcal{N}(\mathfrak{c}\mathfrak{m}_0\mathfrak{n})^{\frac{l(r)-k_0+k^0}{2}}\mathcal{N}(\mathfrak{n})^{l(r)-k_0}$$
$$\times G(\chi)\prod_{\mathfrak{p}|p} A_\mathfrak{p}\left(f_\chi, l-\frac{k_0+k^0}{2}+1\right)$$
$$\times L\left(\frac{k_0+k^0}{2}-1, f_\rho\right) L\left(\frac{k_0+k^0}{2}-l, f_{\overline{\chi}}\right)$$

where

$$P(f,\rho) = \alpha(\mathfrak{c}\mathfrak{m}_0\mathfrak{n})G(\rho)\prod_{\mathfrak{q}|\mathfrak{n}}\left(1-\mathcal{N}\mathfrak{q}^{-1}\right)\prod_{\mathfrak{p}|\mathfrak{c}} A_\mathfrak{p}\left(f_\rho, \frac{k_0+k^0}{2}-1\right).$$

In order to finish the proof of Theorem 1, we put

$$L_{(p)}^{(\epsilon_0)}(x) := \int_{\mathrm{Gal}_p} x d\mu^{(\epsilon_0)},$$

where

$$\mu^{(\epsilon_0)} := \frac{1}{P(f,\rho)} \cdot \mu^{\sim(\epsilon_0)}|_{C^h(\mathrm{Gal}_p)}.$$

It is well known (due to Amice-Vélu and Vishik), that such non-archimedean Mellin transform is $\mathbb{C}_p$-analytic function of the type $o(\log^h)$, and $\mu^{(\epsilon_0)}$ is uniquely determined by $L_{(p)}^{(\epsilon_0)}\left(\chi\mathcal{N}x_p^m\right)$, $\chi \in \mathfrak{X}_p^{tors}$, $m = 0, 1, ..., h-1$.

11. p-adic functional equation.

THEOREM 3.

$$L_{(p)}^{(\epsilon_0)}(x) = i^{\{k\}+nk_0} \cdot \mathcal{N}(\mathfrak{c})^{\frac{k_0}{2}} x^{-1}(\mathfrak{c}) L_{(p)}^{\rho^{(\epsilon_0)}}(\mathcal{N} x_p^{k_0} x^{-1}).$$

Proof. — Let we use the archimedean functional equation :

$$\mathcal{N}(\mathfrak{c}\vartheta^2)^{\frac{s}{2}} \cdot \Lambda(s,f) = i^{\{k\}} \cdot \mathcal{N}(\mathfrak{c}\vartheta^2)^{\frac{k_0-s}{2}} \cdot \Lambda(k_0 - s, f|_{J_\mathfrak{c}})$$

and the following properties :

$$f|_{J_\mathfrak{c}} = \Lambda(f) \cdot f^\rho, \quad \Lambda(f_\chi) = (\psi^* \chi^*)(\mathfrak{c}) \cdot G(\chi)^2 \cdot \mathcal{N}(\mathfrak{m})^{-1} \cdot \Lambda(f).$$

On the other hand, we remark that the product $\prod_{\mathfrak{p}|p} A_\mathfrak{p}(f_\chi, m)$ is invariant under change of the type

$$(\chi, \alpha(\mathfrak{p}), \beta(\mathfrak{p}), m) \longmapsto (\chi^{-1}, \tilde{\alpha}(\mathfrak{p}), \tilde{\beta}(\mathfrak{p}), k_0 - m),$$

where $\tilde{\alpha}(\mathfrak{q}) := \overline{\psi}(\mathfrak{q}) \cdot \alpha(\mathfrak{q})$, $\tilde{\beta}(\mathfrak{q}) := \overline{\psi}(\mathfrak{q}) \cdot \beta(\mathfrak{q})$. Here $L_{(p)}^{\rho^{(\epsilon_0)}}$ denotes the p-adic L-function, which correspond to the Mellin transform of a measure associated to the form f^ρ.

12. Remarks.

In the p-ordinary case Theorem 1 was established by Yu.I. Manin (see[Ma]) using the theory of generalized modular symbols on some Hilbert-Blumenthal modular varieties. Another approaches (still in the p-ordinary case) are given by the work of H. Hida (see[Hi]), and in CM-case by N.M. Katz (see[Ka]), although they obtain p-adic L- functions of several variables and from their theory we can obtain some p-adic analytic families of Hilbert modular forms. The non-p-ordinary case was treated only for $F = \mathbb{Q}$ by M.M. Vishik [Vi]. Conjectural generalization of the Hida's construction to arbitrary motives has been formulated in the recent paper of A.A. Panchishkin ([Pa4]). In general (including non-critical case) all this ought to be deeply related to Mazur's theory of deformations of representations, Fontaine's theory, p-adic Hodge structures and the like. We hope to treat all this in a separate article. On the other hand it would be interesting to generalize construction of our article to $\mathrm{Sym}^r(f)$, $r = 2, 3, \ldots$(see [Dab] for the case of elliptic modular forms).

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