POISSON-CHARLIER AND POLY-CAUCHY MIXED TYPE POLYNOMIALS

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Abstract. In this paper, we consider Poisson-Charlier and poly-Cauchy mixed type polynomials and give various identities of those polynomials which are derived from umbral calculus.

1. Introduction and Preliminaries

For $r \in \mathbb{Z}_{\geq 0}$, the Cauchy numbers of the first kind with order r are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{n=0}^{\infty} \mathbb{C}_n^{(r)} \frac{t^n}{n!}, \text{ (see [3, 10, 11, 12])}. \tag{1.1}$$

In particular, when $r = 1$, $\mathbb{C}_n^{(1)} = C_n$ are called Cauchy numbers of the first kind.

The Cauchy numbers of the second kind with order r are defined by

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^r = \sum_{n=0}^{\infty} \hat{\mathbb{C}}_n^{(r)} \frac{t^n}{n!}, \text{ (see [3, 10, 11, 12])}. \tag{1.2}$$

When $r = 1$, $\hat{\mathbb{C}}_n^{(1)} = \hat{C}_n$ are called the Cauchy numbers of the second kind.

As is well known, the generating function for the Poisson-Charlier polynomials is given by

$$e^{-t}\left(1+\frac{t}{a}\right)^x = \sum_{n=0}^{\infty} C_n(x:a) \frac{t^n}{n!}, \quad (a \neq 0), \text{ (see [14, 15])}. \tag{1.3}$$

Recently, Komatsu has considered the poly-Cauchy polynomials of the first kind as follows :

$$\frac{1}{(1+t)^x} \mathrm{Lif}_k(\log(1+t)) = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!}, \tag{1.4}$$

where

$$\mathrm{Lif}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}, \text{ (see [10, 11])}. \tag{1.5}$$

He also introduced the poly-Cauchy polynomials of the second kind by

$$(1+t)^x \mathrm{Lif}_k(-\log(1+t)) = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x) \frac{t^n}{n!}, \text{ (see [10, 11])}. \tag{1.6}$$

In this paper, we consider Poisson-Charlier and poly-Cauchy of the first kind mixed type polynomials as follows :

$$e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-x}=\sum_{n=0}^{\infty}PC_n^{(k)}\left(x:a\right)\frac{t^n}{n!},\ (a\neq 0). \tag{1.7}$$

The Poisson-Charlier and poly-Cauchy of the second kind mixed type polynomials are defined by the generating function to be

$$e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{x}=\sum_{n=0}^{\infty}P\hat{C}_n^{(k)}\left(x:a\right)\frac{t^n}{n!},\ (a\neq 0). \tag{1.8}$$

It is known that the Frobenius-Euler polynomials of order r are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}\left(x|\lambda\right)\frac{t^n}{n!},\ (\text{see [1, 4, 7, 9]}), \tag{1.9}$$

where $r\in\mathbb{Z}_{\geq 0}$, and $\lambda\in\mathbb{C}$ with $\lambda\neq 1$.

The Bernoulli polynomials of order r are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}\left(x\right)\frac{t^n}{n!},\ (\text{see [2, 5, 9, 10, 13]}). \tag{1.10}$$

The Stirling number of the first kind is given by

$$\left(x\right)_n=\left(x\right)\left(x-1\right)\cdots\left(x-n+1\right)=\sum_{l=0}^{n}S_1\left(n,\,l\right)x^l,\ (\text{see [14, 15]}), \tag{1.11}$$

and by (1.11), we get

$$\left(\log\left(1+t\right)\right)^m=m!\sum_{l=m}^{\infty}S_1\left(l,\,m\right)\frac{t^l}{l!},\ (\text{see [8, 9, 14, 15]}). \tag{1.12}$$

From (1.11), we note that

$$x^{(n)}=\left(-1\right)^n\left(-x\right)_n=\sum_{l=0}^{n}\left(-1\right)^{n-l}S_1\left(n,\,l\right)x^l, \tag{1.13}$$

where $x^{(n)}=x\left(x+1\right)\cdots\left(x+n-1\right)$, (see [1-15]).

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t :

$$\mathcal{F}=\left\{f\left(t\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}. \tag{1.14}$$

Let $\mathbb{P}=\mathbb{C}\left[x\right]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

$\left\langle L|p\left(x\right)\right\rangle$ is the action of the linear functional L on the polynomial $p\left(x\right)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\left\langle L+M|p\left(x\right)\right\rangle=\left\langle L|p\left(x\right)\right\rangle+\left\langle M|p\left(x\right)\right\rangle$, $\left\langle cL|p\left(x\right)\right\rangle=c\left\langle L|p\left(x\right)\right\rangle$, where c is complex constant in $\mathbb{C}$. For $f\left(t\right)\in\mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\left\langle f\left(t\right)|x^n\right\rangle=a_n,\ (n\geq 0). \tag{1.15}$$

Thus, by (1.14) and (1.15), we get

$$\left\langle t^k|x^n\right\rangle=n!\,\delta_{n,k},\quad (n,\,k\geq 0),\ (\text{see [4, 8, 14]}), \tag{1.16}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L\left(t\right)=\sum_{k=0}^{\infty}\frac{\left\langle L|x^k\right\rangle}{k!}t^k$. Then, by (1.15), we see that $\left\langle f_L\left(t\right)|\,x^n\right\rangle=\left\langle L|\,x^n\right\rangle$. The map $L\longmapsto f_L\left(t\right)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth,

$\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)\,(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a delta series; if $O(f(t)) = 0$, then $f(t)$ is called an invertible series. For $f(t)\,,\, g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ such that $\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n!\delta_{n,k}$, for $n,\, k \geq 0$. The sequence $s_n(x)$ is called the sheffer sequence for $(g(t)\,,\, f(t))$ which is denoted by $s_n(x) \sim (g(t)\,,\, f(t))$ (see [8, 10, 14, 15]).

Let $f(t)\,,\, g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$\langle f(t) g(t) | p(x) \rangle = \langle f(t) | g(t) p(x) \rangle = \langle g(t) | f(t) p(x) \rangle, \tag{1.17}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \middle| x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \middle| p(x) \right\rangle \frac{x^k}{k!}. \tag{1.18}$$

By (1.18), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \text{ and } e^{yt} p(x) = p(x+y)\,, \text{ (see [14])}. \tag{1.19}$$

For $s_n(x) \sim (g(t)\,,\, f(t))$, we have the generating function of $s_n(x)$ as follows :

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \text{ for all } x \in \mathbb{C}, \tag{1.20}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = t$.

Let $s_n(x) \sim (g(t)\,,\, f(t))$. Then we have the following equations (see [8, 14, 15]):

$$f(t) s_n(x) = n s_{n-1}(x)\,,\ (n \geq 0)\,,\ \frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \overline{f}(t) \middle| x^{n-l} \right\rangle s_l(x)\,, \tag{1.21}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\left(\overline{f}(t)\right)^{-1} \overline{f}(t)^j \middle| x^n \right\rangle x^j,\ \langle f(t) | x p(x) \rangle = \langle \partial_t f(t) | p(x) \rangle\,, \tag{1.22}$$

and

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \text{ where } p_n(x) = g(t) s_n(x)\,. \tag{1.23}$$

For $p_n(x) \sim (1,\, f(t))\,,\, q_n(x) \sim (1,\, g(t))$, it is well known that

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x)\,,\ (n \geq 1)\,, \text{ (see [14, 15])}. \tag{1.24}$$

For $s_n(x) \sim (g(t)\,,\, f(t))\,,\, r_n(x) \sim (h(t)\,,\, l(t))$, let us assume that

$$s_n(x) = \sum_{m=0}^{\infty} C_{n,m} r_n(x)\,,\ (n \geq 0)\,.$$

Then we have

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\left(\overline{f}(t)\right)}{g\left(\overline{f}(t)\right)} l\left(\overline{f}(t)\right)^m \middle| x^n \right\rangle, \quad \text{(see [8, 10, 14])}. \tag{1.25}$$

In this paper, we investigate some identities of Poisson-Charlier and poly-Cauchy mixed type polynomials arising from umbral calculus. That is, we give various identities of the Poisson-Charlier and poly-Cauchy polynomials of the first and second kind mixed type polynomials which are derived from umbral calculus.

2. Poisson-Charlier and poly-Cauchy mixed type polynomials

From (1.6), (1.7) and (1.20), we note that

$$PC_n^{(k)}(x:a) \sim \left(e^{a\left(e^{-t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)},\ a\left(e^{-t}-1\right) \right), \tag{2.1}$$

and

$$P\hat{C}_n^{(k)}(x:a) \sim \left(e^{a\left(e^{t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)},\ a\left(e^{t}-1\right) \right). \tag{2.2}$$

Now, we observe that

$$\begin{aligned} & PC_n^{(k)}(y:a) \\ = & \left\langle \sum_{l=0}^{\infty} PC_l^{(k)}(y:a) \frac{t^l}{l!} \middle| x^n \right\rangle \\ = & \left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y} \middle| x^n \right\rangle \\ = & \left\langle e^{-t} \middle| \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y} x^n \right\rangle \\ = & \sum_{l=0}^{n} C_l^{(k)}(y) \frac{1}{a^l l!}(n)_l \left\langle e^{-t} \middle| x^{n-l} \right\rangle = \sum_{l=0}^{n} C_l^{(k)}(y) \binom{n}{l} \frac{(-1)^{n-l}}{a^l}. \end{aligned} \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$PC_n^{(k)}(x:a) = \sum_{l=0}^{n} C_l^{(k)}(x) \binom{n}{l} \frac{(-1)^{n-l}}{a^l},$$

where $a \neq 0$.

Alternatively,

$$\begin{aligned} PC_n^{(k)}(y:a) &= \left\langle \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \middle| e^{-t}\left(1+\frac{t}{a}\right)^{-y} x^n \right\rangle \\ &= \sum_{l=0}^{n} C_l(-y:a) \frac{1}{l!}(n)_l \left\langle \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \middle| x^{n-l} \right\rangle \\ &= \sum_{l=0}^{n} C_l(-y:a) \binom{n}{l} C_{n-l}^{(k)} \frac{1}{a^{n-l}} \\ &= \sum_{l=0}^{n} \frac{\binom{n}{l} C_{n-l}^{(k)}}{a^{n-l}} C_l(-y:a). \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following proposition.

Proposition 2. *For $n \geq 0$, $a \neq 0$, we have*

$$PC_n^{(k)}(x:a) = \sum_{l=0}^{n} \frac{\binom{n}{l} C_{n-l}^{(k)}}{a^{n-l}} C_l(-x:a).$$

Remark. By the same method as (2.3) and (2.4), we get

$$PĈ_n^{(k)}(x:a) = \sum_{l=0}^{n} \frac{(-1)^{n-l}\binom{n}{l}}{a^l} \hat{C}_l^{(k)}(x), \tag{2.5}$$

and

$$PĈ_n^{(k)}(x:a) = \sum_{l=0}^{n} \frac{\binom{n}{l} \hat{C}_l^{(k)}}{a^l} C_{n-l}(x:a). \tag{2.6}$$

It is not difficult to show that

$$\left(-\frac{1}{a}\right)^n x^{(n)} = a^{-n} \sum_{l=0}^{n} (-1)^k S_1(n,k) x^k \sim \left(1, a\left(e^{-t}-1\right)\right), \tag{2.7}$$

and

$$a^{-n}(x)_n = a^{-n} \sum_{k=0}^{n} S_1(n,k) x^k \sim \left(1, a\left(e^t-1\right)\right). \tag{2.8}$$

By (2.1), we get

$$e^{a\left(e^{-t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)} PC_n^{(k)}(x:a) \sim \left(1, a\left(e^{-t}-1\right)\right). \tag{2.9}$$

From (2.7), (2.9), we have

$$e^{a\left(e^{-t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)} PC_n^{(k)}(x:a) = \left(-\frac{1}{a}\right)^n x^{(n)}. \tag{2.10}$$

Thus, by (2.10) we get

$$\begin{aligned} PC_n^{(k)}(x:a) &= \mathrm{Lif}_k(-t)\, e^{-a\left(e^{-t}-1\right)} \left(-\frac{1}{a}\right)^n x^{(n)} \\ &= \left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t) \sum_{l=0}^{n} \frac{a^l}{l!} \left(1-e^{-t}\right)^l x^{(n)}. \end{aligned} \tag{2.11}$$

By (1.13), we see that $x^{(n)} \sim (1, 1-e^{-t})$. From (1.21) and (2.11), we have

$$\left(1-e^{-t}\right)^l x^{(n)} = (n)_l\, x^{(n-l)}.$$

and

$$\begin{aligned}
&PC_n^{(k)}(x:a) \\
=&\left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t)\sum_{l=0}^{n}\frac{a^l}{l!}\left(1-e^{-t}\right)^l x^{(n)} \\
=&\left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t)\sum_{l=0}^{n}\binom{n}{l}a^l x^{(n-l)} \\
=&\left(-\frac{1}{a}\right)^n \sum_{l=0}^{n}a^l\binom{n}{l}\sum_{m=0}^{n-l}(-1)^{n-l-m}S_1(n-l,m)\sum_{r=0}^{m}\frac{(-1)^r}{r!\,(r+1)^k}t^r x^m \\
=&a^{-n}\sum_{l=0}^{n}\sum_{m=0}^{n-l}\sum_{r=0}^{m}(-1)^{l+m+r}\binom{n}{l}\binom{m}{r}\frac{a^l}{(r+1)^k}S_1(n-l,m)\,x^{m-r} \\
=&a^{-n}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+j}\binom{n}{l}\binom{m}{j}\frac{a^l}{(m-j+1)^k}S_1(n-l,m)\right\}x^j.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 3. *For $n\ge 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$\begin{aligned}
&PC_n^{(k)}(x:a) \\
=&a^{-n}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+j}\binom{n}{l}\binom{m}{j}\frac{a^l}{(m-j+1)^k}S_1(n-l,m)\right\}x^j.
\end{aligned}$$

Remark. By (2.2) and (2.8), we get

$$\begin{aligned}
P\hat{C}_n^{(k)}(x:a) &= \mathrm{Lif}_k(-t)\,e^{-a\left(e^t-1\right)}a^{-n}(x)_n \\
&= a^{-n}\mathrm{Lif}_k(-t)\sum_{l=0}^{n}\frac{(-a)^l}{l!}\left(e^t-1\right)^l(x)_n.
\end{aligned} \tag{2.13}$$

By the same method as (2.12), we get

$$\begin{aligned}
&P\hat{C}_n^{(k)}(x:a) \\
=&a^{-n}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+m+j}\binom{n}{l}\binom{m}{j}\frac{a^l}{(m-j+1)^k}S_1(n-l,m)\right\}x^j.
\end{aligned}$$

From (1.22) and (2.1), we note that

(2.14)

$$\begin{aligned}
&PC_n^{(k)}(x:a)\\
=&\sum_{l=0}^{n}\frac{1}{l!}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(-\log\left(1+\frac{t}{a}\right)\right)^l\middle| x^n\right\rangle x^l\\
=&\sum_{l=0}^{n}\frac{(-1)^l}{l!}\sum_{r=0}^{n-l}\frac{l!}{(r+l)!a^{r+l}}S_1(r+l,\,l)\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\middle| t^{r+n}x^n\right\rangle x^l\\
=&\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{r+l}}\binom{n}{r+l}S_1(r+l,\,l)\,PC_{n-r-l}^{(k)}(0:a)\right\}x^l\\
=&\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{n-r}}\binom{n}{r}S_1(n-r,\,l)\,PC_{r}^{(k)}(0:a)\right\}x^l.
\end{aligned}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 4. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$PC_n^{(k)}(x:a)=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{n-r}}\binom{n}{r}S_1(n-r,\,l)\,PC_{r}^{(k)}(0:a)\right\}x^l.$$

Remark. From (1.22) and (2.2), we can also derive the following equation.

$$P\hat{C}_n^{(k)}(x:a)=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{\binom{n}{r}}{a^{n-r}}S_1(n-r,\,l)\,P\hat{C}_r^{(k)}(0:a)\right\}x^l. \tag{2.15}$$

By (2.1), we easily see that

$$e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}PC_n^{(k)}(x:a)\sim\left(1,\,a\left(e^{-t}-1\right)\right),\quad x^n\sim(1,\,t). \tag{2.16}$$

Thus, by (1.24) and (2.16), for $n\geq 1$ we get

(2.17)

$$\begin{aligned}
&e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}PC_n^{(k)}(x:a)\\
=&x\left(\frac{t}{a\left(e^{-t}-1\right)}\right)^n x^{-1}x^n=\left(-a^{-1}\right)^n x\left(\frac{-t}{e^{-t}-1}\right)^n x^{n-1}\\
=&\left(-a^{-1}\right)^n x\sum_{r=0}^{\infty}B_r^{(n)}\frac{(-t)^r}{r!}x^{n-1}\\
=&\left(-a^{-1}\right)^n x\sum_{r=0}^{n-1}B_r^{(n)}(n-1)_r\frac{(-1)^r}{r!}x^{n-r-1}\\
=&\left(-a^{-1}\right)^n\sum_{r=0}^{n-1}B_r^{(n)}(-1)^r\binom{n-1}{r}x^{n-r},
\end{aligned}$$

where $B_r^{(n)}=B_r^{(n)}(0)$ are called the Bernoulli numbers of order n.

From (2.17), we have

$$\begin{aligned}
(2.18)\quad & PC_n^{(k)}(x:a) \\
&= \left(-a^{-1}\right)^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \mathrm{Lif}_k(-t)\, e^{-a\left(e^{-t}-1\right)} x^{n-r} \\
&= \left(-a^{-1}\right)^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \mathrm{Lif}_k(-t) \sum_{l=0}^{n-r} \frac{(-a)^l}{l!} \left(e^{-t}-1\right)^l x^{n-r} \\
&= \left(-a^{-1}\right)^n \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} B_r^{(n)} \sum_{l=0}^{n-r} \left\{ \frac{(-a)^l}{l!} \sum_{j=0}^{n-r-l} \frac{l!}{(j+l)!} S_2(j+l,\, l) \right. \\
&\qquad \left. \times (-1)^{j+l} \mathrm{Lif}_k(-t)\, t^{j+l} x^{n-r} \right\},
\end{aligned}$$

where $S_2(n,\, k)$ is the stirling number of the second kind.

Now, we observe that

$$\begin{aligned}
(2.19)\quad & \mathrm{Lif}_k(-t)\, t^{j+l} x^{n-r} \\
&= (n-r)_{j+l} \mathrm{Lif}_k(-t)\, x^{n-r-j-l} \\
&= (n-r)_{j+l} \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{m!\,(m+1)^k} x^{n-r-j-l} \\
&= (n-r)_{j+l} \sum_{m=0}^{n-r-j-l} \frac{(-1)^m}{m!\,(m+1)^k} (n-r-j-l)_m\, x^{n-r-j-l-m}.
\end{aligned}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 5. *For $n \geq 1$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned}
& PC_n^{(k)}(x:a) \\
&= a^{-n} \sum_{m=0}^{n} \left\{ \sum_{r=0}^{n-m} \sum_{l=0}^{n-m-r} \sum_{j=0}^{n-m-r-l} (-1)^{l+m} \binom{n-1}{r} \binom{n-r}{j+l} \right. \\
&\qquad \left. \times \binom{n-r-j-l}{m} \frac{a^l S_2(j+l,\, l)}{(n-r-j-l-m+1)^k} B_r^{(n)} \right\} x^m.
\end{aligned}$$

Remark. We note that

$$(2.20)\qquad e^{a\left(e^t-1\right)} \frac{1}{\mathrm{Lif}_k(-t)} P\hat{C}_n^{(k)}(x:a) \sim \left(1, a\left(e^t-1\right)\right), \quad x^n \sim (1,\, t).$$

Thus, for $n \geq 1$ we have

$$(2.21)\qquad e^{a\left(e^t-1\right)} \frac{1}{\mathrm{Lif}_k(-t)} P\hat{C}_n^{(k)}(x:a) = a^{-n} \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(n)} x^{n-l}.$$

From (2.21), for $n \geq 1$ we can derive

(2.22)

$$\begin{aligned} & P\hat{C}_n^{(k)}(x:a) \\ &= \left(-a^{-1}\right)^n \sum_{m=0}^{n} \left[\sum_{r=0}^{n-m} \sum_{l=0}^{n-m-r} \sum_{j=0}^{n-m-r-l} \left\{ (-1)^{r+j+m} \frac{\binom{n-1}{r}\binom{n-r}{j+l}}{(n-r-j-l-m+1)^k} \right. \right. \\ & \qquad \left. \left. \times \binom{n-r-j-l}{m} a^l S_2(j+l,l) B_r^{(n)} \right\} \right] x^m . \end{aligned}$$

By (1.23), (2.1) and (2.2), we get

$$\begin{aligned} PC_n^{(k)}(x+y:a) &= \sum_{j=0}^{n} \binom{n}{j} PC_j^{(k)}(x:a) \left(-a^{-1}\right)^{n-j} y^{(n-j)} \\ &= \sum_{j=0}^{n} \binom{n}{j} PC_{n-j}^{(k)}(x:a) \left(-a^{-1}\right)^{j} y^{(j)} \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} P\hat{C}_n^{(k)}(x+y:a) &= \sum_{j=0}^{n} \binom{n}{j} P\hat{C}_j^{(k)}(x:a) a^{-(n-j)} (y)_{n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j} P\hat{C}_{n-j}^{(k)}(x:a) a^{-j} (y)_{j} . \end{aligned} \tag{2.24}$$

From (1.21), (2.1) and (2.2), we have

$$PC_n^{(k)}(x-1:a) - PC_n^{(k)}(x:a) = a^{-1} n PC_{n-1}^{(k)}(x:a), \tag{2.25}$$

and

$$P\hat{C}_n^{(k)}(x+1:a) - PC_n^{(k)}(x:a) = a^{-1} n P\hat{C}_{n-1}^{(k)}(x:a). \tag{2.26}$$

For $s_n(x) \sim (g(t), f(t))$, we note that recurrence formula for $s_n(x)$ is given by

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x). \tag{2.27}$$

Thus, by (2.1), (2.2) and (2.27), we get

(2.28)

$$\begin{aligned} & PC_{n+1}^{(k)}(x:a) \\ &= -\frac{1}{a} x PC_n^{(k)}(x+1:a) - PC_n^{(k)}(x:a) \\ & \quad + a^{-(n+1)} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+2)^k} S_1(n-l,m) \right\} (x+1)^j , \end{aligned}$$

and

$$
\begin{aligned}
(2.29)\qquad & P\hat{C}_{n+1}^{(k)}(x:a) \\
=&\frac{1}{a}xP\hat{C}_{n}^{(k)}(x-1:a)-P\hat{C}_{n}^{(k)}(x:a) \\
&-a^{-(n+1)}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+m+j}\binom{n}{l}\binom{m}{j}\right. \\
&\left.\times\frac{a^{l}}{(m-j+2)^{k}}S_1(n-l,m)\right\}(x-1)^{j}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(2.30)\quad & PC_{n}^{(k)}(y:a) \\
=&\left\langle\sum_{l=0}^{\infty}PC_{l}^{(k)}(y:a)\frac{t^{l}}{l!}\middle|x^{n}\right\rangle=\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n}\right\rangle \\
=&\left\langle\partial_t\left(e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\right)\middle|x^{n-1}\right\rangle \\
=&\left\langle\left(\partial_t e^{-t}\right)\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n-1}\right\rangle \\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n-1}\right\rangle \\
&+\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(\partial_t\left(1+\frac{t}{a}\right)^{-y}\right)\middle|x^{n-1}\right\rangle \\
=&-\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n-1}\right\rangle \\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n-1}\right\rangle \\
&-\frac{y}{a}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\middle|x^{n-1}\right\rangle \\
=&-PC_{n-1}^{(k)}(y:a)-\frac{1}{a}yPC_{n-1}^{(k)}(y+1:a) \\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle|x^{n-1}\right\rangle.
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
(2.31)\quad & \partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \\
=&\frac{1}{a\left(1+\frac{t}{a}\right)\log\left(1+\frac{t}{a}\right)}\left\{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right\}.
\end{aligned}
$$

From (2.31), we have

$$
\begin{aligned}
&\left\langle e^{-t}\left(\partial_t \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle \\
=&\frac{1}{a}\left\langle e^{-t}\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)}{\left(1+\frac{t}{a}\right)\log\left(1+\frac{t}{a}\right)}\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle \\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{\hat{C}_l}{a^l}\left\{PC_{n-l}^{(k-1)}(y:a)-PC_{n-l}^{(k)}(y:a)\right\}.
\end{aligned}
\tag{2.32}
$$

Therefore, by (2.30) and (2.32), we obtain the following theorem.

Theorem 6. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$
\begin{aligned}
PC_n^{(k)}(x:a) &= -PC_{n-1}^{(k)}(x:a)-\frac{1}{a}xPC_{n-1}^{(k)}(x+1:a) \\
&+ \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{\hat{C}_l}{a^l}\left\{PC_{n-l}^{(k-1)}(x:a)-PC_{n-l}^{(k)}(x:a)\right\}.
\end{aligned}
$$

Remark. Note that

$$
\begin{aligned}
&\frac{1}{a}\left\langle e^{-t}\left(\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)}{\frac{t}{a}}\right)\right. \\
&\qquad\qquad \left.\times\left(1+\frac{t}{a}\right)^{-y-1}\middle|\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}x^{n-1}\right\rangle \\
=&\sum_{l=0}^{n-1}\frac{C_l}{a^l}\binom{n-1}{l} \\
&\times\left\langle e^{-t}\left(\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)}{t}\right)\left(1+\frac{t}{a}\right)^{-y-1}\middle| t\frac{x^{n-l}}{n-l}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{1}{n-l}\frac{C_l}{a^l}\left\langle e^{-t}\left(\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)\right.\right. \\
&\qquad\left.\left.-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\middle| x^{n-l}\right\rangle \\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l} \\
&\times\left\langle e^{-t}\left(\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\middle| x^{n-l}\right\rangle \\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{PC_{n-l}^{(k-1)}(y+1:a)-PC_{n-l}^{(k)}(y+1:a)\right\}.
\end{aligned}
\tag{2.33}
$$

By (2.30) and (2.33), we also get the following equation :

$$
\begin{aligned}
(2.34)\qquad & PC_n^{(k)}(x:a) \\
&= -PC_{n-1}^{(k)}(x:a) - \frac{1}{a}xPC_{n-1}^{(k)}(x+1:a) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{PC_{n-l}^{(k-1)}(x+1:a) - PC_{n-l}^{(k)}(x+1:a)\right\}.
\end{aligned}
$$

By the same method as Theorem 6, we see that

$$
\begin{aligned}
(2.35)\qquad & P\hat{C}_n^{(k)}(x:a) \\
&= -P\hat{C}_{n-1}^{(k)}(x:a) + \frac{1}{a}xP\hat{C}_{n-1}^{(k)}(x-1:a) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n-1}\frac{\hat{C}_l}{a^l}\binom{n}{l}\left\{P\hat{C}_{n-l}^{(k-1)}(x:a) - P\hat{C}_{n-l}^{(k)}(x:a)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
(2.36)\qquad & P\hat{C}_n^{(k)}(x:a) \\
&= -P\hat{C}_{n-1}^{(k)}(x:a) + \frac{1}{a}xP\hat{C}_{n-1}^{(k)}(x-1:a) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{P\hat{C}_{n-1}^{(k-1)}(x-1:a) - P\hat{C}_{n-l}^{(k)}(x-1:a)\right\}.
\end{aligned}
$$

Here, we compute

$$
\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n\right\rangle
$$

in two different ways.

On the one hand,

$$
\begin{aligned}
(2.37)\qquad & \left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)^m\right) \middle| x^n\right\rangle \\
&= \sum_{l=0}^{n-m}\frac{m!}{a^{l+m}}\binom{n}{l+m}S_1(l+m,m)\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\middle| x^{n-l-m}\right\rangle \\
&= \sum_{l=0}^{n-m}\frac{m!}{a^{l+m}}\binom{n}{l+m}S_1(l+m,m)\,P\hat{C}_{n-l-m}^{(k)}(0:a) \\
&= \sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1(n-l,m)\,P\hat{C}_l^{(k)}(0:a).
\end{aligned}
$$

On the other hand,

(2.38)
$$
\begin{aligned}
&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n\right\rangle \\
=&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x\cdot x^{n-1}\right\rangle \\
=&\left\langle \partial_t\left\{e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m\right\} \middle| x^{n-1}\right\rangle \\
=&\left\langle \left(\partial_t e^{-t}\right)\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^{n-1}\right\rangle \\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^{n-1}\right\rangle \\
&+\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\partial_t\left(\log\left(1+\frac{t}{a}\right)\right)^m\right) \middle| x^{n-1}\right\rangle \\
=&-\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^{n-1}\right\rangle \\
&+\frac{m-1}{a}\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \middle| \left(\log\left(1+\frac{t}{a}\right)\right)^{m-1} x^{n-1}\right\rangle \\
&+\frac{1}{a}\left\langle e^{-t}\mathrm{Lif}_{k-1}\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \middle| \left(\log\left(1+\frac{t}{a}\right)\right)^{m-1} x^{n-1}\right\rangle .
\end{aligned}
$$

It is easy to show that

$$
\begin{aligned}
(2.39)\qquad &\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \middle| \log\left(1+\frac{t}{a}\right)^{m-1} x^{n-1}\right\rangle \\
&=\sum_{l=0}^{n-m}\frac{(m-1)!}{a^{l+m-1}}\binom{n-1}{l+m-1}S_1\left(l+m-1,m-1\right)P\hat{C}_{n-l-m}^{(k)}\left(-1:a\right) \\
&=\sum_{l=0}^{n-m}\frac{(m-1)!}{a^{n-l-1}}\binom{n-1}{l}S_1\left(n-1-l,m-1\right)P\hat{C}_{l}^{(k)}\left(-1:a\right).
\end{aligned}
$$

Thus, by (2.38) and (2.39), we get

$$
\begin{aligned}
(2.40)\qquad &\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n\right\rangle \\
&=-\sum_{l=0}^{n-m-1}\frac{m!}{a^{n-l-1}}\binom{n-1}{l}S_1\left(n-1-l,m\right)P\hat{C}_{l}^{(k)}\left(0:a\right) \\
&+\left(\frac{m-1}{m}\right)\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n-1}{l}S_1\left(n-l-1,m-1\right)P\hat{C}_{l}^{(k)}\left(-1:a\right) \\
&+\frac{1}{m}\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n-1}{l}S_1\left(n-l-1,m-1\right)P\hat{C}_{l}^{(k-1)}\left(-1:a\right).
\end{aligned}
$$

Therefore, by (2.37) and (2.40), we obtain the following theorem.

Theorem 7. *For n, $m \geq 0$ with $n-m \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n-m} \frac{\binom{n}{l}}{a^{n-l}} S_1(n-l,m)\, P\hat{C}_l^{(k)}(0:a) \\
&+ \sum_{l=0}^{n-1-m} \frac{\binom{n-1}{l}}{a^{n-1-l}} S_1(n-1-l,m)\, P\hat{C}_l^{(k)}(0:a) \\
&= \left(1-\frac{1}{m}\right) \sum_{l=0}^{n-m} \frac{\binom{n-1}{l}}{a^{n-l}} S_1(n-1-l,m-1)\, P\hat{C}_l^{(k)}(-1:a) \\
&+ \frac{1}{m} \sum_{l=0}^{n-m} \frac{\binom{n-1}{l}}{a^{n-l}} S_1(n-1-l,m-1)\, P\hat{C}_l^{(k)}(-1:a).
\end{aligned}$$

Remark. From the computation of

$$\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n \right\rangle$$

in two different ways, we can also derive the following equation :

$$\begin{aligned}
&\sum_{l=0}^{n-m} \frac{\binom{n}{l}}{a^{n-l}} S_1(n-l,m)\, PC_l^{(k)}(0:a) \\
&+ \sum_{l=0}^{n-1-m} \frac{\binom{n-1}{l}}{a^{n-l-1}} S_1(n-l-1,m)\, PC_l^{(k)}(0:a) \\
&= \left(1-\frac{1}{m}\right) \sum_{l=0}^{n-m} \frac{\binom{n-1}{l}}{a^{n-l}} S_1(n-1-l,m-1)\, PC_l^{(k)}(1:a) \\
&+ \frac{1}{m} \sum_{l=0}^{n-m} \frac{\binom{n-1}{l}}{a^{n-l}} S_1(n-1-l,m-1)\, PC_l^{(k-1)}(1:a).
\end{aligned} \tag{2.41}$$

By (1.21), (2.1) and (2.2), we easily see that

$$\frac{d}{dx} PC_n^{(k)}(x:a) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^l}{(n-l)\,l!a^{n-l}} PC_l^{(k)}(x:a), \tag{2.42}$$

and

$$\frac{d}{dx} P\hat{C}_n^{(k)}(x:a) = (-1)^n n! \sum_{l=0}^{n-1} \frac{(-1)^{l-1}}{(n-l)\,l!a^{n-l}} P\hat{C}_l^{(k)}(x:a). \tag{2.43}$$

For

$$PC_n^{(k)}(x:a) \sim \left(e^{a\left(e^{-t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)}, a\left(e^{-t}-1\right)\right),\ (a \neq 0)$$

and

$$B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right),\ (s \in \mathbb{Z}_{\geq 0}),$$

let us assume that

$$PC_n^{(k)}(x:a)=\sum_{m=0}^{n}C_{n,m}B_m^{(s)}(x). \tag{2.44}$$

From (1.25), we note that

(2.45)

$$\begin{aligned}&C_{n,m}\\=&\frac{(-1)^m}{m!}\\&\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\left(\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}\right)^s\middle|\left(\log\left(1+\frac{t}{a}\right)\right)^m x^n\right\rangle\end{aligned}$$

Now, we observe that

$$\left(\log\left(1+\frac{t}{a}\right)\right)^m x^n=\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1(n-l,m)x^l. \tag{2.46}$$

By (2.45) and (2.46), we get

$$\begin{aligned}C_{n,m}&\\=&\frac{(-1)^m}{m!}\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1(n-l,m)\\&\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\middle|\left(\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}\right)^s x^l\right\rangle\\=&(-1)^m\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1(n-l,m)\sum_{i=0}^{l}\frac{\binom{l}{i}\mathbb{C}_i^{(s)}}{a^i}\\&\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\middle|x^{l-i}\right\rangle\\=&(-1)^m\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1(n-l,m)\sum_{i=0}^{l}\frac{\binom{l}{i}\mathbb{C}_i^{(s)}}{a^i}PC_{l-i}^{(k)}(s:a)\\=&(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1(n-l,m)\mathbb{C}_i^{(s)}PC_{l-i}^{(k)}(s:a).\end{aligned} \tag{2.47}$$

Therefore, by (2.44) and (2.47), we obtain the following theorem.

Theorem 8. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$\begin{aligned}&PC_n^{(k)}(x:a)\\=&\sum_{m=0}^{n}\left\{(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1(n-l,m)\mathbb{C}_i^{(s)}PC_{l-i}^{(k)}(s:a)\right\}B_m^{(s)}(x).\end{aligned}$$

Remark. By the same method as Theorem 8, we get

$$(2.48)\quad P\hat{C}_n^{(k)}(x:a) = \sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1(n-l,m)\,\hat{\mathbb{C}}_i^{(s)}P\hat{C}_{l-i}^{(k)}(s:a)\right\}B_m^{(s)}(x).$$

For

$$PC_n^{(k)}(x:a)\sim\left(e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)},a\left(e^{-t}-1\right)\right),\ (a\neq 0),$$

and

$$H_n^{(s)}(x|\lambda)\sim\left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s,t\right),\ (s\in\mathbb{Z}_{\geq 0}),$$

let us assume that

$$(2.49)\quad PC_n^{(k)}(x:a)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\lambda).$$

From (1.25), we have

(2.50)

$$\begin{aligned} &C_{n,m}\\ =&\frac{(-1)^m}{m!(1-\lambda)^s}\\ &\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\left(1-\lambda-\frac{\lambda t}{a}\right)^{-s}\middle|\left(\log\left(1+\frac{t}{a}\right)\right)^m x^n\right\rangle\\ =&\frac{(-1)^m}{(1-\lambda)^s}\sum_{l=0}^{n-m}\sum_{i=0}^{s}\frac{\binom{n}{l}\binom{s}{i}(l)_i}{a^{n-l+i}}(1-\lambda)^{s-i}(-\lambda)^i S_1(n-l,m)PC_{l-i}^{(k)}(s:a).\end{aligned}$$

Therefore, by (2.49) and (2.50), we obtain the following theorem.

Theorem 9. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$\begin{aligned} &PC_n^{(k)}(x:a)\\ =&\frac{1}{(1-\lambda)^s}\sum_{m=0}^{n}\left\{(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{s}\frac{\binom{n}{l}\binom{l}{i}(l)_i}{a^{n-l+i}}\right.\\ &\left.\times(1-\lambda)^{s-i}(-\lambda)^i S_1(n-l,\,m)\,PC_{l-i}^{(k)}(s:a)\right\}H_m^{(s)}(x\mid\lambda).\end{aligned}$$

By the same method as Theorem 9, we get

(2.51)

$$\begin{aligned} & P\hat{C}_n^{(k)}(x:a) \\ = & \frac{1}{(1-\lambda)^s} \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{i=0}^{s} \frac{\binom{n}{l}\binom{s}{i}}{a^{n-l}} (-\lambda)^{s-i} S_1(n-l,m) P\hat{C}_l^{(k)}(i:a) \right\} H_m^{(s)}(x \mid \lambda). \end{aligned}$$

For

$$\begin{aligned} PC_n^{(k)}(x:a) &\sim \left(e^{a\left(e^{-t}-1\right)} \frac{1}{\mathrm{Lif}_k(-t)}, a\left(e^{-t}-1\right) \right), \\ x^{(n)} &= x(x+1)\cdots(x+n-1) \sim \left(1, 1-e^{-t}\right), \end{aligned}$$

let us assume that

$$PC_n^{(k)}(x:a) = \sum_{m=0}^{n} C_{n,m} x^{(m)}. \tag{2.52}$$

From (1.25), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!(-a)^m} \left\langle e^{-t} \mathrm{Lif}_k \left(\log\left(1+\frac{t}{a}\right)\right) \middle| t^m x^n \right\rangle \\ &= \frac{1}{(-a)^m} \binom{n}{m} \left\langle e^{-t} \mathrm{Lif}_k \left(\log\left(1+\frac{t}{a}\right)\right) \middle| x^{n-m} \right\rangle \\ &= \frac{1}{(-a)^m} \binom{n}{m} PC_{n-m}^{(k)}(0:a). \end{aligned} \tag{2.53}$$

Therefore, by (2.52) and (2.53), we obtain the following theorem.

Theorem 10. *For $n \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$PC_n^{(k)}(x:a) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{(-a)^m} PC_{n-m}^{(k)}(0:a) x^{(m)},$$

where $x^{(m)} = x(x+1)\cdots(x+m-1)$.

Remark. By the same method as Theorem 10, we get

$$P\hat{C}_n^{(k)}(x:a) = \sum_{m=0}^{n} \frac{\binom{n}{m}}{a^m} P\hat{C}_{n-m}^{(k)}(0:a)(x)_m,$$

where $(x)_m = x(x-1)\cdots(x-m+1)$.

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