HIGHER-ORDER CAUCHY OF THE FIRST KIND AND POLY-CAUCHY OF THE FIRST KIND MIXED TYPE POLYNOMIALS

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Abstract. In this paper, we study higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials with viewpoint of umbral calculus and give some interesting identities and formulae of those polynomials which are derived from umbral calculus.

1. Introduction

The polylogarithm factorial function is defined by

$$Lif_k(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k},\ (k\in\mathbb{Z})\ (see\ [13,14,15]).\tag{1.1}$$

The poly-Cauchy polynomials of the first kind (of index k) are defined by the generating function to be

$$\frac{Lif_k(log(1+t))}{(1+t)^x}=\sum_{n=0}^{\infty}C_n^{(k)}(x)\frac{t^n}{n!},\ (see\ [13,14,15]).\tag{1.2}$$

When $x=0$, $C_n^{(k)}=C_n^{(k)}(0)$ are called the poly-Cauchy numbers.

In particular, for $k=1$, we note that

$$Lif_1(log(1+t))=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}C_n\frac{t^n}{n!},\ (see\ [13,14]).\tag{1.3}$$

where $C_n=C_n^{(1)}(0)$ are called the Cauchy numbers of the first kind. The Cauchy numbers of the first kind with order r are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r=\sum_{n=0}^{\infty}\mathbb{C}_n^{(r)}\frac{t^n}{n!},\ (see\ [13,14,15]).\tag{1.4}$$

Note that $\mathbb{C}_n^{(1)}=C_n$. Let us consider higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials as follows:

$$\left(\frac{t}{\log(1+t)}\right)^r\frac{Lif_k(\log(1+t))}{(1+t)^x}=\sum_{n=0}^{\infty}A_n^{(r,k)}(x)\frac{t^n}{n!},\tag{1.5}$$

where $r,k\in\mathbb{Z}$. When $x=0$, $A_n^{(r,k)}=A_n^{(r,k)}(0)$ are called higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type numbers.

For $\lambda\neq 1\in\mathbb{C}$, the Frobenius-Euler polynomials of order α are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^\alpha e^{xt}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!},\quad \text{(see [9-12]).} \tag{1.6}$$

As is well known, the Bernoulli polynomials of order α are given by

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!},\quad \text{(see [1-8]).} \tag{1.7}$$

When $x=0$, $B_n^{(\alpha)}=B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α. The Stirling number of the first kind is defined by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,\quad \text{(see [16]).} \tag{1.8}$$

For $m\in\mathbb{Z}_{\geq 0}$, the generating function of the Stirling number of the first kind is given by

$$(\log(1+t))^m=m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}=\sum_{l=0}^{\infty}\frac{m!S_1(l+m,m)}{(l+m)!}t^{l+m}. \tag{1.9}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\;\middle|\; a_k\in\mathbb{C}\right\}. \tag{1.10}$$

Let $\mathbb{P}=\mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L\mid p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$ with $\langle L+M\mid p(x)\rangle=\langle L\mid p(x)\rangle+\langle M\mid p(x)\rangle$, and $\langle cL\mid p(x)\rangle=c\,\langle L\mid p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t)\in\mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle=a_k,\ (n\geq 0),\text{(see [1,7,16])}. \tag{1.11}$$

From (1.10) and (1.11), we note that

$$\left\langle t^k|x^n\right\rangle=n!\delta_{n,k},\ (n,k\geq 0), \tag{1.12}$$

where $\delta_{n,k}$ is the Kronecker symbol. (see [16, 17]).

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{\langle L|x^k\rangle}{k!}t^k$. Then by (1.12), we get $\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle$. So, $L=f_L(t)$. The map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. (see [16]). The order $o(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t))=1$, then $f(t)$ is called a delta series; if $o(f(t))=0$, then $g(t)$ is said to be an invertible seires. For $f(t),g(t)\in\mathcal{F}$, let us assume that $f(t)$ is a delta series and $g(t)$ is an invertible series. Then there exists a unique sequence $S_n(x)$ ($\deg S_n(x)=n$) such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle=n!\delta_{n,k}\ for\ n,k\geq 0$.

The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$. Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle , \tag{1.13}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!},\ p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!},\ (see [16]). \tag{1.14}$$

From (1.14), we note that

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k},\ and\ e^{yt}p(x) = p(x+y). \tag{1.15}$$

For $S_n(x) \sim (g(t), f(t))$, the generating function of $S_n(x)$ is given by

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!},\ for\ all\ x \in \mathbb{C}, \tag{1.16}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$.

From (1.5), we observe that $A_n^{(r,k)}(x)$ is the Sheffer sequence for the pair $\left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^{-t}-1\right)$. That is,

$$A_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^{-t}-1\right). \tag{1.17}$$

In [13], Komatsu considered the numbers $A_n^{(r,k)}$, which were denoted by $T_{r-1}^{(k)}(n)$.

Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$f(t)S_n(x) = nS_{n-1}(x),\ (n \geq 0),\ S_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle x^j, \tag{1.18}$$

$$S_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} S_j(x) P_{n-j}(y),\ where\ p_n(x) = g(t)S_n(x), \tag{1.19}$$

and

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} S_n(x),\ (see\ [16]). \tag{1.20}$$

The transfer formula for $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$ is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x),\ (n \geq 0)\ (see\ [16]). \tag{1.21}$$

For $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

$$S_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x),\ (n \geq 0), \tag{1.22}$$

where

$$C_{n,m} = \frac{1}{m!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \;\middle|\; x^n \right\rangle, \ (see\ [16]). \tag{1.23}$$

In this paper, we study higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials with viewpoint of umbral calculus. The purpose of this paper is to give some interesting identities and formulae of those polynomials which are derived from umbral calculus.

2. Poly-Cauchy polynomials and Higher-order Cauchy polynomials

By (1.17), we see that

$$\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)} A_n^{(r,k)}(x) \sim (e^{-t}-1), \tag{2.1}$$

and

$$(-1)^n x^{(n)} = \sum_{m=0}^{n} (-1)^m S_1(n,m) x^m \sim (1, e^{-t}-1), \tag{2.2}$$

where $x^{(n)} = x(x+1)\cdots(x+n-1)$.

Thus, from (2.1) and (2.2), we have

$$\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)} A_n^{(r,k)}(x) = (-1)^n x^{(n)} = \sum_{m=0}^{n} (-1)^m S_1(n,m) x^m. \tag{2.3}$$

By (2.3), we get

$$\begin{aligned} A_n^{(r,k)}(x) &= \left(\frac{e^t-1}{te^t}\right)^r Lif_k(-t)(-1)^n x^{(n)} \\ &= \sum_{m=0}^{n} (-1)^m S_1(n,m) \left(\frac{e^t-1}{te^t}\right)^r Lif_k(-t) x^m \\ &= \sum_{m=0}^{n} (-1)^m S_1(n,m) \sum_{l=0}^{m} \frac{(-1)^l (m)_l}{l!(l+1)^k} \left(\frac{e^{-t}-1}{-t}\right)^r x^{m-l}. \end{aligned} \tag{2.4}$$

It is well known that

$$(e^t-1)^m = \sum_{l=0}^{\infty} S_2(l+m,m) \frac{m!}{(l+m)!} t^{l+m}, \tag{2.5}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

From (2.4) and (2.5), we have

$$
\begin{aligned}
A_n^{(r,k)}(x) &= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=0}^{m-l}\frac{\binom{m}{l}\binom{m-l}{a}}{\binom{a+r}{r}(l+1)^k}S_1(n,m)S_2(a+r,r)(-x)^{m-l-a}\\
&= \sum_{m=0}^{n}\left\{\sum_{l=0}^{m}\sum_{j=0}^{m-l}\frac{\binom{m}{l}\binom{m-l}{j}}{\binom{m-l-j+r}{r}(l+1)^k}S_1(n,m)S_2(m-l-j+r,r)\right\}(-x)^j\\
&= \sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{m-j}\frac{\binom{m}{l}\binom{m-l}{j}}{\binom{m-l-j+r}{r}(l+1)^k}S_1(n,m)S_2(m-l-j+r,r)\right\}(-x)^j
\end{aligned}
\tag{2.6}
$$

where $r \in \mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$.

Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *For $n, r \geq 0$, we have*

$$
A_n^{(r,k)}(x) = \sum_{0\leq j\leq n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{m-j}\frac{\binom{m}{l}\binom{m-l}{j}}{\binom{m-l-j+r}{r}(l+1)^k}S_1(n,m)S_2(m-l-j+r,r)\right\}(-x)^j.
$$

From (1.17) and (1.18), we have

$$
\begin{aligned}
&A_n^{(r,k)}(x)\\
&= \sum_{j=0}^{n}\frac{1}{j!}\left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(-\log(1+t))^j \,\middle|\, x^n\right\rangle x^j\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}(-1)^j\binom{n}{l+j}S_1(l+j,j)\left\langle Lif_k(\log(1+t))\left(\frac{t}{\log(1+t)}\right)^r \,\middle|\, x^{n-l-j}\right\rangle x^j\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}(-1)^j\binom{n}{l+j}S_1(l+j,j)\sum_{a=0}^{n-l-j}B_a^{(a-r+1)}(1)\binom{n-l-j}{a}\\
&\qquad\qquad \times \left\langle Lif_k(\log(1+t)) \mid x^{n-l-j-a}\right\rangle x^j\\
&= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-l-j}(-1)^j\binom{n}{l+j}\binom{n-l-j}{a}S_1(l+j,j)B_a^{(a-r+1)}(1)C_{n-j-l-a}^{(k)}\right\}x^j.
\end{aligned}
\tag{2.7}
$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $r, k \in \mathbb{Z}$, and $n \in \mathbb{Z}_{\geq 0}$, we have*

$$
\begin{aligned}
&A_n^{(r,k)}(x)\\
&= \sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-l-j}(-1)^j\binom{n}{l+j}\binom{n-l-j}{a}S_1(l+j,j)B_a^{(a-r+1)}(1)C_{n-j-l-a}^{(k)}\right\}x^j.
\end{aligned}
$$

As is knwon, the Narumi polynomials of order r are given by $N_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, (e^t-1)\right)$. Thus, we note that

$$\left(\frac{t}{\log(1+t)}\right)^{-r}(1+t)^x=\sum_{n=0}^{\infty}N_n^{(r)}(x)\frac{t^n}{n!}. \tag{2.8}$$

Indeed, we see that $N_n^{(r)}(x)=B_n^{(n+r+1)}(x+1)$.

From (2.7) and (2.8), we can derive the following equation:

$$A_n^{(r,k)}(x)=\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-l-j}(-1)^j\binom{n}{l+j}\binom{n-l-j}{a}S_1(l+j,j)N_a^{(-r)}C_{n-j-l-a}^{(k)}\right\}x^j, \tag{2.9}$$

where $N_a^{(r)}=N_a^{(r)}(0)$ are called the Narumi numbers of order r.

The Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!},\ (see\ [16]). \tag{2.10}$$

From (2.7) and (2.10), we note that

$$\begin{aligned}A_n^{(r,k)}(x)=&\sum_{j=0}^{n}\left\{\sum_{l=0}^{n-j}\sum_{a=0}^{n-l-j}\sum_{a_1+\cdots+a_r=a}(-1)^j\binom{n}{l+j}\binom{n-l-j}{a}\binom{a}{a_1,\cdots,a_r}\right.\\&\left.\times S_1(l+j,j)\left(\Pi_{i=1}^{r}b_{a_i}\right)C_{n-l-j-a}^{(k)}\right\}x^j.\end{aligned} \tag{2.11}$$

From (1.17),(1.19) and (2.3), we note that

$$A_n^{(r,k)}(x+y)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}A_j^{(r,k)}(x)y^{(n-j)}, \tag{2.12}$$

and, by (1.15) and (1.18), we get

$$nA_{n-1}^{(r,k)}(x)=(e^{-t}-1)A_n^{(r,k)}(x)=A_n^{(r,k)}(x-1)-A_n^{(r,k)}(x). \tag{2.13}$$

By (1.17) and (1.20), we get

$$\begin{aligned}A_{n+1}^{(r,k)}(x)&=\left(\frac{g'(t)}{g(t)}-x\right)e^tA_n^{(r,k)}(x)\\&=e^t\frac{g'(t)}{g(t)}A_n^{(r,k)}(x)-xA_n^{(r,k)}(x+1)\\&=r\frac{e^t-1-t}{t^2}\frac{te^t}{e^t-1}A_n^{(r,k)}(x)+e^t\frac{Lif_k'(-t)}{Lif_k(-t)}A_n^{(r,k)}(x)-xA_n^{(r,k)}(x+1).\end{aligned} \tag{2.14}$$

From (2.3), we note that

$$A_n^{(r,k)}(x)=\sum_{m=0}^{n}(-1)^mS_1(n,m)\left(\frac{e^t-1}{te^t}\right)^rLif_k(-t)x^m, \tag{2.15}$$

$$\frac{1}{Lif_k(-t)}A_n^{(r,k)}(x)=\sum_{m=0}^{n}(-1)^m S_1(n,m)\left(\frac{e^t-1}{te^t}\right)^r x^m. \tag{2.16}$$

By (2.15), we get

$$\begin{aligned}
&r\left(\frac{e^t-1-t}{t^2}\right)\left(\frac{te^t}{e^t-1}\right)A_n^{(r,k)}(x)\\
&=r\sum_{m=0}^{n}(-1)^m S_1(n,m)\frac{e^t-1-t}{t^2}\left(\frac{te^t}{e^t-1}\right)^{1-r}Lif_k(-t)x^m\\
&=r\sum_{m=0}^{n}(-1)^m S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l(m)_l}{l!(l+1)^k}\left(\frac{te^t}{e^t-1}\right)^{1-r}\sum_{a=0}^{m-l}\frac{t^a x^{m-l}}{(a+2)!}\\
&=r\sum_{m=0}^{n}(-1)^m S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l(m)_l}{l!(l+1)^k}\sum_{a=0}^{m-l}\frac{(m-l)_a}{(a+2)!}\left(\frac{-t}{e^{-t}-1}\right)^{1-r}x^{m-l-a}\\
&=r\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=0}^{m-l}\frac{(-1)^a\binom{m}{l}\binom{m-l}{a}}{(a+2)(a+1)(l+1)^k}S_1(n,m)B_{m-l-a}^{(1-r)}(-x),
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&e^t\frac{Lif_k'(-t)}{Lif_k(-t)}A_n^{(r,k)}(x)=e^tLif_k'(-t)\left(\frac{1}{Lif_k(-t)}A_n^{(r,k)}(x)\right)\\
&=e^tLif_k'(-t)\sum_{m=0}^{n}(-1)^m S_1(n,m)\left(\frac{e^t-1}{te^t}\right)^r x^m\\
&=\sum_{m=0}^{n}(-1)^m S_1(n,m)e^t\left(\frac{e^t-1}{te^t}\right)^r\sum_{a=0}^{m}\frac{(-1)^a}{a!(a+2)^k}t^a x^m\\
&=\sum_{m=0}^{n}(-1)^m S_1(n,m)\sum_{a=0}^{m}\frac{(-1)^a}{a!(a+2)^k}(m)_a e^t\left(\frac{-t}{e^{-t}-1}\right)^{-r}x^{m-a}\\
&=\sum_{m=0}^{n}\sum_{a=0}^{m}\binom{m}{a}\frac{S_1(n,m)}{(a+2)^k}B_{m-a}^{(-r)}(-x-1).
\end{aligned} \tag{2.18}$$

Therefore, by (2.14), (2.17) and (2.18), we obtain the following theorem.

Theorem 2.3. *For $r,k\in\mathbb{Z}$, and $n\geq 0$, we have*

$$\begin{aligned}
A_{n+1}^{(r,k)}(x)=-xA_n^{(r,k)}(x+1)+r\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{a=0}^{m-l}\frac{(-1)^a\binom{m}{l}\binom{m-l}{a}}{(a+2)(a+1)(l+1)^k}\\
\times S_1(n,m)B_{m-l-a}^{(1-r)}(-x)+\sum_{m=0}^{n}\sum_{a=0}^{m}\binom{m}{a}\frac{S_1(n,m)}{(a+2)^k}B_{m-a}^{(-r)}(-x-1).
\end{aligned}$$

By (1.12), we easily see that

$$
\begin{aligned}
A_n^{(r,k)}(y) &= \left\langle \sum_{l=0}^{\infty} A_l^{(r,k)}(y)\frac{t^l}{l!} \;\middle|\; x^n \right\rangle \\
&= \left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(1+t)^{-y} \;\middle|\; x^n \right\rangle \\
&= \left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(1+t)^{-y} \;\middle|\; xx^{n-1} \right\rangle \\
&= \left\langle \partial_t \left\{\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(1+t)^{-y}\right\} \;\middle|\; x^{n-1} \right\rangle \\
&= \left\langle \partial_t \left(\left(\frac{t}{\log(1+t)}\right)^r\right) Lif_k(\log(1+t))(1+t)^{-y} \;\middle|\; x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{t}{\log(1+t)}\right)^r (\partial_t Lif_k(\log(1+t)))(1+t)^{-y} \;\middle|\; x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\partial_t(1+t)^{-y}) \;\middle|\; x^{n-1} \right\rangle \\
&= -yA_{n-1}^{(r,k)}(y+1) + \left\langle \left(\partial_t \left(\frac{t}{\log(1+t)}\right)^r\right) Lif_k(\log(1+t))(1+t)^{-y} \;\middle|\; x^{n-1} \right\rangle \\
&\quad + \left\langle \left(\frac{t}{\log(1+t)}\right)^r (\partial_t Lif_k(\log(1+t)))(1+t)^{-y} \;\middle|\; x^{n-1} \right\rangle .
\end{aligned} \tag{2.19}
$$

Now, we observe that

$$
\begin{aligned}
&\left\langle \left(\partial_t \left(\frac{t}{\log(1+t)}\right)^r\right) Lif_k(\log(1+t))(1+t)^{-y} \;\middle|\; x^{n-1} \right\rangle \\
&= r\sum_{l=0}^{n-1}\sum_{a=0}^{l}(-1)^{n-a}\frac{(n-1-l)!(l-a)!}{l-a+2}\binom{n-1}{l}\binom{l}{a}A_a^{(r+1,k)}(y) \\
&+ r\sum_{l=0}^{n-1}(-1)^{n-1-l}(n-1-l)!\binom{n-1}{l}A_l^{(r,k)}(y),
\end{aligned} \tag{2.20}
$$

and

$$
\begin{aligned}
&\left\langle \left(\frac{t}{\log(1+t)}\right)^r (\partial_t Lif_k(\log(1+t))(1+t)^{-y} | x^{n-1} \right\rangle \\
&= \frac{1}{n}(A_n^{(r+1,k-1)}(y+1) - A_n^{(r+1,k)}(y+1)).
\end{aligned} \tag{2.21}
$$

Therefore, by (2.19),(2.20) and (2.21), we obtain the following theorem.

Theorem 2.4. *For $r, k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$\begin{aligned}A_n^{(r,k)}(x) &= -xA_{n-1}^{(r,k)}(x+1) + r\sum_{l=0}^{n-1}\sum_{a=0}^{l}(-1)^{n-a}\frac{(n-1-l)!(l-a)!}{l-a+2}\binom{n-1}{l} \\ &\quad\times \binom{l}{a}A_n^{(r+1,k)}(x) + r\sum_{l=0}^{n-1}(-1)^{n-l-1}(n-l-1)!\binom{n-1}{l}A_l^{(r,k)}(x) \\ &\quad+ \frac{1}{n}(A_n^{(r+1,k-1)}(x+1) - A_n^{(r+1,k)}(x+1)).\end{aligned}$$

Here we compute $\left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\log(1+t))^m | x^n \right\rangle$ in two different ways.

On the one hand, we have

$$\begin{aligned}&\left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\log(1+t))^m | x^n \right\rangle \\ &= \sum_{l=0}^{n-m}\frac{m!}{(l+m)!}S_1(l+m,m)(n)_{l+m}\left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t)) | x^{n-l-m} \right\rangle \\ &= \sum_{l=0}^{n-m} m!\binom{n}{l+m}S_1(l+m,m)A_{n-l-m}^{(r,k)} \\ &= \sum_{l=0}^{n-m} m!\binom{n}{l}S_1(n-l,m)A_l^{(r,k)}.\end{aligned} \tag{2.22}$$

On the other hand, we get

$$\begin{aligned}&\left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\log(1+t))^m | x^n \right\rangle \\ &= \left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\log(1+t))^m | x x^{n-1} \right\rangle \\ &= \left\langle \partial_t \left\{ \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\log(1+t))^m \right\} | x^{n-1} \right\rangle \\ &= \left\langle \left(\partial_t \left(\frac{t}{\log(1+t)}\right)^r\right) Lif_k(\log(1+t))(\log(1+t))^m | x^{n-1} \right\rangle \\ &\quad+ \left\langle \left(\frac{t}{\log(1+t)}\right)^r (\partial_t Lif_k(\log(1+t)))(\log(1+t))^m | x^{n-1} \right\rangle \\ &\quad+ \left\langle \left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(\partial_t(\log(1+t))^m) | x^{n-1} \right\rangle.\end{aligned} \tag{2.23}$$

Now, we observe that

$$\left\langle \left(\partial_t \left(\frac{t}{\log(1+t)} \right)^r \right) Lif_k(\log(1+t))(\log(1+t))^m | x^{n-1} \right\rangle$$
$$= r \sum_{l=0}^{n-1-m} m! \binom{n-1}{l} S_1(n-l-1,m) A_l^{(r,k)}(1)$$
$$+ r \sum_{l=0}^{n-1-m} \sum_{a=0}^{l} (-1)^{l-a+1} \frac{m!(l-a)!}{l-a+2} \binom{n-1}{l} \binom{l}{a} S_1(n-1-l,m) A_a^{(r+1,k)}(1), \tag{2.24}$$

$$\left\langle \left(\frac{t}{\log(1+t)} \right)^r (\partial_t Lif_k(\log(1+t))(\log(1+t))^m) \, | x^{n-1} \right\rangle$$
$$= \sum_{l=0}^{n-m} (m-1)! \binom{n-1}{l} S_1(n-l-1,m-1) \{ A_l^{(r,k-1)}(1) - A_l^{(r,k)}(1) \}, \tag{2.25}$$

and

$$\left\langle \left(\frac{t}{\log(1+t)} \right)^r Lif_k(\log(1+t)) \, (\partial_t(\log(1+t))^m) \, | x^{n-1} \right\rangle$$
$$= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) A_l^{(r,k)}(1), \tag{2.26}$$

where $n-1 \geq m \geq 1$.

From (2.23),(2.24),(2.25) and (2.26), we have

$$\left\langle \left(\frac{t}{\log(1+t)} \right)^r Lif_k(\log(1+t))(\log(1+t))^m | x^n \right\rangle$$
$$= r \sum_{l=0}^{n-1-m} \sum_{a=0}^{l} (-1)^{l-a+1} \frac{m!(l-a)!}{l-a+2} \binom{n-1}{l} \binom{l}{a} S_1(n-1-l,m) A_a^{(r+1,k)}(1)$$
$$+ r \sum_{l=0}^{n-1-m} m! \binom{n-1}{l} S_1(n-l-1,m) A_l^{(r,k)}(1) + \sum_{l=0}^{n-m} (m-1)! \binom{n-1}{l}$$
$$\times S_1(n-l-1,m-1) A_l^{(r,k-1)}(1) - \sum_{l=0}^{n-m} (m-1)! \binom{m-1}{l} S_1(n-l-1,m-1)$$
$$\times A_l^{(r,k)}(1) + m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) A_l^{(r,k)}(1). \tag{2.27}$$

Therefore, by (2.22) and (2.27), we obtain the following theorem.

Theorem 2.5. *For $n-1 \geq m \geq 1$, we have*

$$\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)A_l^{(r,k)}$$
$$= r\sum_{l=0}^{n-1-m}\sum_{a=0}^{l}(-1)^{l-a+1}\frac{(l-a)!}{(l-a+2)}\binom{n-1}{l}\binom{l}{a}S_1(n-1-l,m)A_a^{(r+1,k)}(1)$$
$$+ r\sum_{l=0}^{n-1-m}\binom{n-1}{l}S_1(n-l-1,m)A_l^{(r,k)}(1)+\frac{1}{m}\sum_{l=0}^{n-m}\binom{n-1}{l}$$
$$\times S_1(n-l-1,m-1)A_l^{(r,k)}(1)+\left(1-\frac{1}{m}\right)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)$$
$$\times A_l^{(r,k)}(1).$$

Remark 1. It is known that

$$\frac{d}{dx}S_n(x)=\sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t) \mid x^{n-l}\right\rangle S_l(x),\ (see[16]), \tag{2.28}$$

where $S_n(x) \sim (g(t), f(t))$.
From (1.17) and (2.28), we have

$$\begin{aligned}\frac{d}{dx}A_n^{(r,k)}(x) &= \sum_{l=0}^{n-1}\binom{n}{l}\left\langle -\log(1+t) \mid x^{n-l}\right\rangle A_l^{(r,k)}(x)\\ &= -\sum_{l=0}^{n-1}\binom{n}{l}\left\langle \sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}t^{m+1} \,\middle|\, x^{n-l}\right\rangle A_l^{(r,k)}(x)\\ &= -\sum_{l=0}^{n-1}\binom{n}{l}\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left\langle t^{m+1} \mid x^{n-l}\right\rangle A_l^{(r,k)}(x)\\ &= (-1)^{n+1}n!\sum_{l=0}^{n-1}\frac{(-1)^{l+1}}{(n-l)l!}A_l^{(r,k)}(x).\end{aligned} \tag{2.29}$$

For $A_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^{-t}-1\right)$ and $B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right)$, $(s \geq 0)$, let us assume that

$$A_n^{(r,k)}(x)=\sum_{m=0}^{n}C_{n,m}B_m^{(s)}(x). \tag{2.30}$$

Then, by (1.23), we get

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{t}{(1+t)\log(1+t)}\right)^{r+s}(1+t)^r Lif_k(\log(1+t))(-\log(1+t))^m \middle| x^n \right\rangle \\
&= \frac{(-1)^m}{m!}\left\langle \left(\frac{t}{\log(1+t)}\right)^{r+s} Lif_k(\log(1+t))(1+t)^{-s} \middle| (\log(1+t))^m x^n \right\rangle \\
&= (-1)^m \sum_{l=0}^{n-m}\binom{n}{l+m} S_1(l+m,m) \\
&\quad \times \left\langle \left(\frac{t}{\log(1+t)}\right)^{r+s} Lif_k(\log(1+t))(1+t)^{-s} \middle| x^{n-l-m} \right\rangle \\
&= (-1)^m \sum_{l=0}^{n-m}\binom{n}{l+m} S_1(l+m,m) A_{n-l-m}^{(r+s,k)}(s) \\
&= (-1)^m \sum_{l=0}^{n-m}\binom{n}{l} S_1(n-l,m) A_{l}^{(r+s,k)}(s).
\end{aligned} \tag{2.31}
$$

Therefore, by (2.30) and (2.31), we obtain the following theorem.

Theorem 2.6. *For $n, s \geq 0$, we have*

$$
A_n^{(r,k)}(x) = \sum_{m=0}^{n}\left\{(-1)^m \sum_{l=0}^{n-m}\binom{n}{l} S_1(n-l,m) A_l^{(r+s,k)}(s)\right\} B_m^{(s)}(x).
$$

Let us consider the following two Sheffer sequences:

$$
A_n^{(r,k)}(x) \sim \left(\left(\frac{te^t}{e^t-1}\right)^r \frac{1}{Lif_k(-t)}, e^{-t}-1\right), \tag{2.32}
$$

and

$$
H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \ (s \geq 0). \tag{2.33}
$$

Suppose that

$$
A_n^{(r,k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda) \ (s \geq 0). \tag{2.34}
$$

By (1.23), we get

$$\begin{aligned}
&C_{n,m}\\
&=\frac{(-1)^m}{m!(1-\lambda)^s}\\
&\quad\times\left\langle\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))(1+t)^{-s}(1-\lambda(1+t))^s\middle|(\log(1+t))^m x^n\right\rangle\\
&=\frac{(-1)^m}{(1-\lambda)^s}\sum_{l=0}^{n-m}\sum_{a=0}^{s}(-\lambda)^a\binom{n}{l+m}\binom{s}{a}S_1(l+m,m)A_{n-l-m}^{(r,k)}(s-a)\\
&=\frac{(-1)^m}{(1-\lambda)^s}\sum_{l=0}^{n-m}\sum_{a=0}^{s}(-\lambda)^a\binom{n}{l}\binom{s}{a}S_1(n-l,m)A_l^{(r,k)}(s-a).
\end{aligned}\tag{2.35}$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.7. *For $n,s\geq 0$, $r,k\in\mathbb{Z}$, we have*

$$\begin{aligned}
A_n^{(r,k)}(x)=\frac{1}{(1-\lambda)^s}\sum_{m=0}^{n}&\left\{(-1)^m\sum_{l=0}^{n-m}\sum_{a=0}^{s}(-\lambda)^a\binom{n}{l}\binom{s}{a}S_1(n-l,m)A_l^{(r,k)}(s-a)\right\}\\
&\times H_m^{(s)}(x|\lambda).
\end{aligned}$$

Finally, we consider

$$A_n^{(r,k)}(x)\sim\left(\left(\frac{te^t}{e^t-1}\right)^r\frac{1}{Lif_k(-t)},e^{-t}-1\right),\tag{2.36}$$

and

$$x^{(n)}\sim(1,e^{-t}-1).\tag{2.37}$$

Let us assume that

$$A_n^{(r,k)}(x)=\sum_{m=0}^{n}C_{n,m}x^{(m)}.\tag{2.38}$$

Then, by (1.23), we get

$$\begin{aligned}
C_{n,m}&=\frac{(-1)^m}{m!}\left\langle\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))\middle|t^m x^n\right\rangle\\
&=(-1)^m\frac{(n)_m}{m!}\left\langle\left(\frac{t}{\log(1+t)}\right)^r Lif_k(\log(1+t))\middle|x^{n-m}\right\rangle\\
&=(-1)^m\binom{n}{m}A_{n-m}^{(r,k)}.
\end{aligned}\tag{2.39}$$

Therefore, by (2.38) and (2.39), we obtain the following theorem.

Theorem 2.8. *For $n\geq 0$, $r,k\in\mathbb{Z}$, we have*

$$A_n^{(r,k)}(x)=\sum_{m=0}^{n}(-1)^m\binom{n}{m}A_{n-m}^{(r,k)}x^{(m)},$$

where $x^{(m)} = x(x+1)\cdots(x+m-1)$.

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