Adv. Studies Theor. Phys., Vol. 8, 2014, no. 9, 423 - 445
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2014.4448

Poisson-Charlier and Poly-Cauchy Mixed Type Polynomials

Dae San Kim

Department of Mathematics
Sogang University, Seoul 121-742
Republic of Korea

Dmitry V. Dolgy

Hanrimwon, Kwangwoon University
Seoul 139-701, Republic of Korea

Taekyun Kim

Department of Mathematics
Kwangwoon University, Seoul 139-701
Republic of Korea

Jong-Jin Seo

Department of Applied Mathematics
Pukyong National University, Pusan
Republic of Korea

Abstract

In this paper, we consider Poisson-Charlier and poly-Cauchy mixed type polynomials and give various identities of those polynomials which are derived from umbral calculus.

1. Introduction and Preliminaries

For $r \in \mathbb{Z}_{\geq 0}$, the Cauchy numbers of the first kind with order r are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{n=0}^{\infty} \mathbb{C}_n^{(r)} \frac{t^n}{n!}, \text{ (see [3, 10, 11, 12]).} \tag{1.1}$$

In particular, when $r = 1$, $\mathbb{C}_n^{(1)} = C_n$ are called Cauchy numbers of the first kind.

The Cauchy numbers of the second kind with order r are defined by

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^r = \sum_{n=0}^{\infty} \hat{\mathbb{C}}_n^{(r)} \frac{t^n}{n!}, \text{ (see [3, 10, 11, 12]).} \tag{1.2}$$

When $r = 1$, $\hat{\mathbb{C}}_n^{(1)} = \hat{C}_n$ are called the Cauchy numbers of the second kind.

As is well known, the generating function for the Poisson-Charlier polynomials is given by

$$e^{-t}\left(1+\frac{t}{a}\right)^x = \sum_{n=0}^{\infty} C_n(x:a) \frac{t^n}{n!}, \quad (a \neq 0), \text{ (see [14, 15]).} \tag{1.3}$$

Recently, Komatsu has considered the poly-Cauchy polynomials of the first kind as follows :

$$\frac{1}{(1+t)^x}\mathrm{Lif}_k(\log(1+t)) = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!}, \tag{1.4}$$

where

$$\mathrm{Lif}_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!(n+1)^k}, \text{ (see [10, 11]).} \tag{1.5}$$

He also introduced the poly-Cauchy polynomials of the second kind by

$$(1+t)^x \mathrm{Lif}_k(-\log(1+t)) = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x) \frac{t^n}{n!}, \text{ (see [10, 11]).} \tag{1.6}$$

In this paper, we consider Poisson-Charlier and poly-Cauchy of the first kind mixed type polynomials as follows :

$$e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-x} = \sum_{n=0}^{\infty} PC_n^{(k)}(x:a) \frac{t^n}{n!}, \ (a \neq 0). \tag{1.7}$$

The Poisson-Charlier and poly-Cauchy of the second kind mixed type polynomials are defined by the generating function to be

$$e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^x=\sum_{n=0}^{\infty}P\hat{C}_n^{(k)}\left(x:a\right)\frac{t^n}{n!},\ \left(a\neq 0\right). \quad (1.8)$$

It is known that the Frobenius-Euler polynomials of order r are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}\left(x|\lambda\right)\frac{t^n}{n!},\ \left(\text{see [1, 4, 7, 9]}\right), \quad (1.9)$$

where $r\in\mathbb{Z}_{\geq 0}$, and $\lambda\in\mathbb{C}$ with $\lambda\neq 1$.

The Bernoulli polynomials of order r are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}\left(x\right)\frac{t^n}{n!},\ \left(\text{see [2, 5, 9, 10, 13]}\right). \quad (1.10)$$

The Stirling number of the first kind is given by

$$\left(x\right)_n=\left(x\right)\left(x-1\right)\cdots\left(x-n+1\right)=\sum_{l=0}^{n}S_1\left(n,\,l\right)x^l,\ \left(\text{see [14, 15]}\right), \quad (1.11)$$

and by (1.11), we get

$$\left(\log\left(1+t\right)\right)^m=m!\sum_{l=m}^{\infty}S_1\left(l,\,m\right)\frac{t^l}{l!},\ \left(\text{see [8, 9, 14, 15]}\right). \quad (1.12)$$

From (1.11), we note that

$$x^{(n)}=\left(-1\right)^n\left(-x\right)_n=\sum_{l=0}^{n}\left(-1\right)^{n-l}S_1\left(n,\,l\right)x^l, \quad (1.13)$$

where $x^{(n)}=x\left(x+1\right)\cdots\left(x+n-1\right)$, (see [1-15]).

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t :

$$\mathcal{F}=\left\{f\left(t\right)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\,\middle|\,a_k\in\mathbb{C}\right\}. \quad (1.14)$$

Let $\mathbb{P}=\mathbb{C}\left[x\right]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$.

$\left\langle L|p\left(x\right)\right\rangle$ is the action of the linear functional L on the polynomial $p\left(x\right)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\left\langle L+M|p\left(x\right)\right\rangle=\left\langle L|p\left(x\right)\right\rangle+\left\langle M|p\left(x\right)\right\rangle$, $\left\langle cL|p\left(x\right)\right\rangle=c\left\langle L|p\left(x\right)\right\rangle$, where c is complex constant in $\mathbb{C}$. For $f\left(t\right)\in\mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\left\langle f\left(t\right)|x^n\right\rangle=a_n,\ \left(n\geq 0\right). \quad (1.15)$$

Thus, by (1.14) and (1.15), we get

$$\left\langle t^k|x^n\right\rangle=n!\,\delta_{n,k},\quad \left(n,\,k\geq 0\right),\ \left(\text{see [4, 8, 14]}\right), \quad (1.16)$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. Then, by (1.15), we see that $\langle f_L(t)|\, x^n \rangle = \langle L|\, x^n \rangle$. The map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a delta series; if $O(f(t)) = 0$, then $f(t)$ is called an invertible series. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ such that $\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n! \delta_{n,k}$, for $n, k \geq 0$. The sequence $s_n(x)$ is called the sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [8, 10, 14, 15]).

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$\langle f(t) g(t)|\, p(x) \rangle = \langle f(t)|\, g(t) p(x) \rangle = \langle g(t)|\, f(t) p(x) \rangle, \tag{1.17}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \middle| x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \middle| p(x) \right\rangle \frac{x^k}{k!}. \tag{1.18}$$

By (1.18), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \text{ and } e^{yt} p(x) = p(x+y), \text{ (see [14])}. \tag{1.19}$$

For $s_n(x) \sim (g(t), f(t))$, we have the generating function of $s_n(x)$ as follows :

$$\frac{1}{g\left(\overline{f}(t)\right)} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \text{ for all } x \in \mathbb{C}, \tag{1.20}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = t$.

Let $s_n(x) \sim (g(t), f(t))$. Then we have the following equations (see [8, 14, 15]):

$$f(t) s_n(x) = n s_{n-1}(x), \ (n \geq 0), \ \frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \overline{f}(t) \middle| x^{n-l} \right\rangle s_l(x), \tag{1.21}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\left(\overline{f}(t)\right)^{-1} \overline{f}(t)^j \middle| x^n \right\rangle x^j, \ \langle f(t)|\, x p(x) \rangle = \langle \partial_t f(t)|\, p(x) \rangle, \tag{1.22}$$

and

$$s_n(x+y)=\sum_{j=0}^{n}\binom{n}{j}s_j(x)\,p_{n-j}(y)\,,\ \text{where } p_n(x)=g(t)\,s_n(x)\,. \tag{1.23}$$

For $p_n(x)\sim(1,\ f(t))\,,\ q_n(x)\sim(1,\ g(t))\,,$ it is well known that

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x)\,,\ (n\geq 1)\,,\ (\text{see [14, 15]})\,. \tag{1.24}$$

For $s_n(x)\sim(g(t)\,,\ f(t))\,,\ r_n(x)\sim(h(t)\,,\ l(t))\,,$ let us assume that

$$s_n(x)=\sum_{m=0}^{\infty}C_{n,m}r_n(x)\,,\ (n\geq 0)\,.$$

Then we have

$$C_{n,m}=\frac{1}{m!}\left\langle \frac{h\left(\overline{f}(t)\right)}{g\left(\overline{f}(t)\right)}l\left(\overline{f}(t)\right)^m\middle|\,x^n\right\rangle,\quad (\text{see [8, 10, 14]})\,. \tag{1.25}$$

In this paper, we investigate some identities of Poisson-Charlier and poly-Cauchy mixed type polynomials arising from umbral calculus. That is, we give various identities of the Poisson-Charlier and poly-Cauchy polynomials of the first and second kind mixed type polynomials which are derived from umbral calculus.

2. Poisson-Charlier and poly-Cauchy mixed type polynomials

From (1.6), (1.7) and (1.20), we note that

$$PC_n^{(k)}(x:a)\sim\left(e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)},\ a\left(e^{-t}-1\right)\right), \tag{2.1}$$

and

$$P\hat{C}_n^{(k)}(x:a)\sim\left(e^{a\left(e^{t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)},\ a\left(e^{t}-1\right)\right). \tag{2.2}$$

Now, we observe that

$$\begin{aligned} & PC_n^{(k)}(y:a) \\ = & \left\langle \sum_{l=0}^{\infty} PC_l^{(k)}(y:a)\frac{t^l}{l!} \middle| x^n \right\rangle \\ = & \left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y} \middle| x^n \right\rangle \\ = & \left\langle e^{-t} \middle| \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y} x^n \right\rangle \\ = & \sum_{l=0}^{n} C_l^{(k)}(y)\frac{1}{a^l l!}(n)_l \left\langle e^{-t} \middle| x^{n-l}\right\rangle = \sum_{l=0}^{n} C_l^{(k)}(y)\binom{n}{l}\frac{(-1)^{n-l}}{a^l}. \end{aligned} \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 1. *For $n \geq 0$, we have*

$$PC_n^{(k)}(x:a) = \sum_{l=0}^{n} C_l^{(k)}(x)\binom{n}{l}\frac{(-1)^{n-l}}{a^l},$$

where $a \neq 0$.

Alternatively,

$$\begin{aligned} PC_n^{(k)}(y:a) &= \left\langle \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \middle| e^{-t}\left(1+\frac{t}{a}\right)^{-y} x^n \right\rangle \\ &= \sum_{l=0}^{n} C_l(-y:a)\frac{1}{l!}(n)_l \left\langle \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \middle| x^{n-l}\right\rangle \\ &= \sum_{l=0}^{n} C_l(-y:a)\binom{n}{l} C_{n-l}^{(k)}\frac{1}{a^{n-l}} \\ &= \sum_{l=0}^{n} \frac{\binom{n}{l} C_{n-l}^{(k)}}{a^{n-l}} C_l(-y:a). \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following proposition.

Proposition 2. *For $n \geq 0$, $a \neq 0$, we have*

$$PC_n^{(k)}(x:a) = \sum_{l=0}^{n} \frac{\binom{n}{l} C_{n-l}^{(k)}}{a^{n-l}} C_l(-x:a).$$

Remark. By the same method as (2.3) and (2.4), we get

$$P\hat{C}_n^{(k)}(x:a)=\sum_{l=0}^{n}\frac{(-1)^{n-l}\binom{n}{l}}{a^l}\hat{C}_l^{(k)}(x), \tag{2.5}$$

and

$$P\hat{C}_n^{(k)}(x:a)=\sum_{l=0}^{n}\frac{\binom{n}{l}\hat{C}_l^{(k)}}{a^l}C_{n-l}(x:a). \tag{2.6}$$

It is not difficult to show that

$$\left(-\frac{1}{a}\right)^n x^{(n)}=a^{-n}\sum_{l=0}^{n}(-1)^k S_1(n,k)\,x^k\sim\left(1,\,a\left(e^{-t}-1\right)\right), \tag{2.7}$$

and

$$a^{-n}(x)_n=a^{-n}\sum_{k=0}^{n}S_1(n,k)\,x^k\sim\left(1,\,a\left(e^{t}-1\right)\right). \tag{2.8}$$

By (2.1), we get

$$e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}PC_n^{(k)}(x:a)\sim\left(1,\,a\left(e^{-t}-1\right)\right). \tag{2.9}$$

From (2.7), (2.9), we have

$$e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}PC_n^{(k)}(x:a)=\left(-\frac{1}{a}\right)^n x^{(n)}. \tag{2.10}$$

Thus, by (2.10) we get

$$\begin{aligned} PC_n^{(k)}(x:a) &= \mathrm{Lif}_k(-t)\,e^{-a\left(e^{-t}-1\right)}\left(-\frac{1}{a}\right)^n x^{(n)} \\ &= \left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t)\sum_{l=0}^{n}\frac{a^l}{l!}\left(1-e^{-t}\right)^l x^{(n)}. \end{aligned} \tag{2.11}$$

By (1.13), we see that $x^{(n)}\sim(1,\,1-e^{-t})$. From (1.21) and (2.11), we have

$$\left(1-e^{-t}\right)^l x^{(n)}=(n)_l\,x^{(n-l)}.$$

and

$$\begin{aligned} & PC_n^{(k)}(x:a) \qquad (2.12) \\ =& \left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t) \sum_{l=0}^{n} \frac{a^l}{l!} \left(1-e^{-t}\right)^l x^{(n)} \\ =& \left(-\frac{1}{a}\right)^n \mathrm{Lif}_k(-t) \sum_{l=0}^{n} \binom{n}{l} a^l x^{(n-l)} \\ =& \left(-\frac{1}{a}\right)^n \sum_{l=0}^{n} a^l \binom{n}{l} \sum_{m=0}^{n-l} (-1)^{n-l-m} S_1(n-l,\, m) \sum_{r=0}^{m} \frac{(-1)^r}{r!\,(r+1)^k} t^r x^m \\ =& a^{-n} \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{r=0}^{m} (-1)^{l+m+r} \binom{n}{l} \binom{m}{r} \frac{a^l}{(r+1)^k} S_1(n-l,\, m)\, x^{m-r} \\ =& a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} S_1(n-l,\, m) \right\} x^j. \end{aligned}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 3. *For $n \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned} & PC_n^{(k)}(x:a) \\ =& a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} S_1(n-l,\, m) \right\} x^j. \end{aligned}$$

Remark. By (2.2) and (2.8), we get

$$\begin{aligned} P\hat{C}_n^{(k)}(x:a) &= \mathrm{Lif}_k(-t)\, e^{-a\left(e^t-1\right)} a^{-n} (x)_n \qquad (2.13) \\ &= a^{-n} \mathrm{Lif}_k(-t) \sum_{l=0}^{n} \frac{(-a)^l}{l!} \left(e^t-1\right)^l (x)_n. \end{aligned}$$

By the same method as (2.12), we get

$$\begin{aligned} & P\hat{C}_n^{(k)}(x:a) \\ =& a^{-n} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \sum_{l=0}^{n-m} (-1)^{l+m+j} \binom{n}{l} \binom{m}{j} \frac{a^l}{(m-j+1)^k} S_1(n-l,\, m) \right\} x^j. \end{aligned}$$

From (1.22) and (2.1), we note that

$$
\begin{aligned}
&PC_n^{(k)}(x:a) \qquad (2.14)\\
&=\sum_{l=0}^{n}\frac{1}{l!}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(-\log\left(1+\frac{t}{a}\right)\right)^l\middle| x^n\right\rangle x^l\\
&=\sum_{l=0}^{n}\frac{(-1)^l}{l!}\sum_{r=0}^{n-l}\frac{l!}{(r+l)!a^{r+l}}S_1(r+l,\,l)\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\middle| t^{r+n}x^n\right\rangle x^l\\
&=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{r+l}}\binom{n}{r+l}S_1(r+l,\,l)\,PC_{n-r-l}^{(k)}(0:a)\right\}x^l\\
&=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{n-r}}\binom{n}{r}S_1(n-r,\,l)\,PC_r^{(k)}(0:a)\right\}x^l.
\end{aligned}
$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 4. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$
PC_n^{(k)}(x:a)=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{(-1)^l}{a^{n-r}}\binom{n}{r}S_1(n-r,\,l)\,PC_r^{(k)}(0:a)\right\}x^l.
$$

Remark. From (1.22) and (2.2), we can also derive the following equation.

$$
P\hat{C}_n^{(k)}(x:a)=\sum_{l=0}^{n}\left\{\sum_{r=0}^{n-l}\frac{\binom{n}{r}}{a^{n-r}}S_1(n-r,\,l)\,P\hat{C}_r^{(k)}(0:a)\right\}x^l. \qquad (2.15)
$$

By (2.1), we easily see that

$$
e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}PC_n^{(k)}(x:a)\sim\left(1,\,a\left(e^{-t}-1\right)\right),\quad x^n\sim(1,\,t). \qquad (2.16)
$$

Thus, by (1.24) and (2.16), for $n \geq 1$ we get

$$\begin{aligned}
&e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k\left(-t\right)}PC_n^{(k)}\left(x:a\right) \qquad (2.17)\\
=&x\left(\frac{t}{a\left(e^{-t}-1\right)}\right)^n x^{-1}x^n=\left(-a^{-1}\right)^n x\left(\frac{-t}{e^{-t}-1}\right)^n x^{n-1}\\
=&\left(-a^{-1}\right)^n x\sum_{r=0}^{\infty}B_r^{(n)}\frac{(-t)^r}{r!}x^{n-1}\\
=&\left(-a^{-1}\right)^n x\sum_{r=0}^{n-1}B_r^{(n)}\left(n-1\right)_r\frac{(-1)^r}{r!}x^{n-r-1}\\
=&\left(-a^{-1}\right)^n\sum_{r=0}^{n-1}B_r^{(n)}\left(-1\right)^r\binom{n-1}{r}x^{n-r},
\end{aligned}$$

where $B_r^{(n)}=B_r^{(n)}(0)$ are called the Bernoulli numbers of order n.

From (2.17), we have

$$\begin{aligned}
&PC_n^{(k)}\left(x:a\right) \qquad (2.18)\\
=&\left(-a^{-1}\right)^n\sum_{r=0}^{n-1}\left(-1\right)^r\binom{n-1}{r}B_r^{(n)}\mathrm{Lif}_k\left(-t\right)e^{-a\left(e^{-t}-1\right)}x^{n-r}\\
=&\left(-a^{-1}\right)^n\sum_{r=0}^{n-1}\left(-1\right)^r\binom{n-1}{r}B_r^{(n)}\mathrm{Lif}_k\left(-t\right)\sum_{l=0}^{n-r}\frac{(-a)^l}{l!}\left(e^{-t}-1\right)^l x^{n-r}\\
=&\left(-a^{-1}\right)^n\sum_{r=0}^{n-1}\left(-1\right)^r\binom{n-1}{r}B_r^{(n)}\sum_{l=0}^{n-r}\left\{\frac{(-a)^l}{l!}\sum_{j=0}^{n-r-l}\frac{l!}{(j+l)!}S_2\left(j+l,\,l\right)\right.\\
&\qquad\left.\times\left(-1\right)^{j+l}\mathrm{Lif}_k\left(-t\right)t^{j+l}x^{n-r}\right\},
\end{aligned}$$

where $S_2(n,\,k)$ is the stirling number of the second kind.

Now, we observe that

$$\begin{aligned}
&\mathrm{Lif}_k\left(-t\right)t^{j+l}x^{n-r} \qquad (2.19)\\
=&\left(n-r\right)_{j+l}\mathrm{Lif}_k\left(-t\right)x^{n-r-j-l}\\
=&\left(n-r\right)_{j+l}\sum_{m=0}^{\infty}\frac{(-1)^m t^m}{m!\left(m+1\right)^k}x^{n-r-j-l}\\
=&\left(n-r\right)_{j+l}\sum_{m=0}^{n-r-j-l}\frac{(-1)^m}{m!\left(m+1\right)^k}\left(n-r-j-l\right)_m x^{n-r-j-l-m}.
\end{aligned}$$

Therefore, by (2.18) and (2.19), we obtain the following theorem.

Theorem 5. *For $n \geq 1$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned}
&PC_n^{(k)}(x:a)\\
=&a^{-n}\sum_{m=0}^{n}\left\{\sum_{r=0}^{n-m}\sum_{l=0}^{n-m-r}\sum_{j=0}^{n-m-r-l}(-1)^{l+m}\binom{n-1}{r}\binom{n-r}{j+l}\right.\\
&\left.\times\binom{n-r-j-l}{m}\frac{a^l S_2(j+l,l)}{(n-r-j-l-m+1)^k}B_r^{(n)}\right\}x^m.
\end{aligned}$$

Remark. We note that

$$e^{a\left(e^t-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}P\hat{C}_n^{(k)}(x:a)\sim\left(1,a\left(e^t-1\right)\right),\quad x^n\sim(1,t). \tag{2.20}$$

Thus, for $n\geq 1$ we have

$$e^{a\left(e^t-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}P\hat{C}_n^{(k)}(x:a)=a^{-n}\sum_{l=0}^{n-1}\binom{n-1}{l}B_l^{(n)}x^{n-l}. \tag{2.21}$$

From (2.21), for $n\geq 1$ we can derive

$$\begin{aligned}
&P\hat{C}_n^{(k)}(x:a) \qquad\qquad (2.22)\\
=&\left(-a^{-1}\right)^n\sum_{m=0}^{n}\left[\sum_{r=0}^{n-m}\sum_{l=0}^{n-m-r}\sum_{j=0}^{n-m-r-l}\left\{(-1)^{r+j+m}\frac{\binom{n-1}{r}\binom{n-r}{j+l}}{(n-r-j-l-m+1)^k}\right.\right.\\
&\left.\left.\times\binom{n-r-j-l}{m}a^l S_2(j+l,l)B_r^{(n)}\right\}\right]x^m.
\end{aligned}$$

By (1.23), (2.1) and (2.2), we get

$$\begin{aligned}
PC_n^{(k)}(x+y:a) &= \sum_{j=0}^{n}\binom{n}{j}PC_j^{(k)}(x:a)\left(-a^{-1}\right)^{n-j}y^{(n-j)} \qquad (2.23)\\
&= \sum_{j=0}^{n}\binom{n}{j}PC_{n-j}^{(k)}(x:a)\left(-a^{-1}\right)^{j}y^{(j)}
\end{aligned}$$

and

$$\begin{aligned}
P\hat{C}_n^{(k)}(x+y:a) &= \sum_{j=0}^{n}\binom{n}{j}P\hat{C}_j^{(k)}(x:a)a^{-(n-j)}(y)_{n-j} \qquad (2.24)\\
&= \sum_{j=0}^{n}\binom{n}{j}P\hat{C}_{n-j}^{(k)}(x:a)a^{-j}(y)_{j}.
\end{aligned}$$

From (1.21), (2.1) and (2.2), we have

$$PC_n^{(k)}(x-1:a) - PC_n^{(k)}(x:a) = a^{-1}nPC_{n-1}^{(k)}(x:a), \tag{2.25}$$

and

$$P\hat{C}_n^{(k)}(x+1:a) - PC_n^{(k)}(x:a) = a^{-1}nP\hat{C}_{n-1}^{(k)}(x:a). \tag{2.26}$$

For $s_n(x) \sim (g(t), f(t))$, we note that recurrence formula for $s_n(x)$ is given by

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x). \tag{2.27}$$

Thus, by (2.1), (2.2) and (2.27), we get

$$\begin{aligned} &PC_{n+1}^{(k)}(x:a) \\ =& -\frac{1}{a}xPC_n^{(k)}(x+1:a) - PC_n^{(k)}(x:a) \\ &+ a^{-(n+1)}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+j}\binom{n}{l}\binom{m}{j}\frac{a^l}{(m-j+2)^k}S_1(n-l,m)\right\}(x+1)^j, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &P\hat{C}_{n+1}^{(k)}(x:a) \\ =&\frac{1}{a}xP\hat{C}_n^{(k)}(x-1:a) - P\hat{C}_n^{(k)}(x:a) \\ &- a^{-(n+1)}\sum_{j=0}^{n}\left\{\sum_{m=j}^{n}\sum_{l=0}^{n-m}(-1)^{l+m+j}\binom{n}{l}\binom{m}{j}\right. \\ &\left.\times\frac{a^l}{(m-j+2)^k}S_1(n-l,m)\right\}(x-1)^j. \end{aligned} \tag{2.29}$$

Note that

$$
\begin{aligned}
&PC_n^{(k)}(y:a) \qquad (2.30)\\
=&\left\langle \sum_{l=0}^{\infty} PC_l^{(k)}(y:a)\frac{t^l}{l!}\middle| x^n\right\rangle = \left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^n\right\rangle\\
=&\left\langle \partial_t\left(e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\right)\middle| x^{n-1}\right\rangle\\
=&\left\langle \left(\partial_t e^{-t}\right)\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle\\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle\\
&+\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(\partial_t\left(1+\frac{t}{a}\right)^{-y}\right)\middle| x^{n-1}\right\rangle\\
=&-\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle\\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle\\
&-\frac{y}{a}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\middle| x^{n-1}\right\rangle\\
=&-PC_{n-1}^{(k)}(y:a)-\frac{1}{a}yPC_{n-1}^{(k)}(y+1:a)\\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle.
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
&\partial_t\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right) \qquad (2.31)\\
=&\frac{1}{a\left(1+\frac{t}{a}\right)\log\left(1+\frac{t}{a}\right)}\left\{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right\}.
\end{aligned}
$$

From (2.31), we have

$$\begin{aligned}
&\left\langle e^{-t}\left(\partial_t \mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle \qquad (2.32)\\
=&\frac{1}{a}\left\langle e^{-t}\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)}{\left(1+\frac{t}{a}\right)\log\left(1+\frac{t}{a}\right)}\left(1+\frac{t}{a}\right)^{-y}\middle| x^{n-1}\right\rangle\\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{\hat{C}_l}{a^l}\left\{PC_{n-l}^{(k-1)}(y:a)-PC_{n-l}^{(k)}(y:a)\right\}.
\end{aligned}$$

Therefore, by (2.30) and (2.32), we obtain the following theorem.

Theorem 6. *For $n \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned}
PC_n^{(k)}(x:a) &= -PC_{n-1}^{(k)}(x:a)-\frac{1}{a}xPC_{n-1}^{(k)}(x+1:a)\\
&+\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{\hat{C}_l}{a^l}\left\{PC_{n-l}^{(k-1)}(x:a)-PC_{n-l}^{(k)}(x:a)\right\}.
\end{aligned}$$

Remark. Note that

$$\begin{aligned}
&\frac{1}{a}\left\langle e^{-t}\left(\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_{k}\left(\log\left(1+\frac{t}{a}\right)\right)}{\frac{t}{a}}\right)\right. \\
&\qquad\qquad \times\left.\left(1+\frac{t}{a}\right)^{-y-1}\right|\left.\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}x^{n-1}\right\rangle \qquad (2.33)\\
=&\sum_{l=0}^{n-1}\frac{C_l}{a^l}\binom{n-1}{l} \\
&\times\left\langle e^{-t}\left(\frac{\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_{k}\left(\log\left(1+\frac{t}{a}\right)\right)}{t}\right)\left(1+\frac{t}{a}\right)^{-y-1}\right|\left. t\frac{x^{n-l}}{n-l}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{1}{n-l}\frac{C_l}{a^l}\left\langle e^{-t}\left(\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)\right.\right. \\
&\qquad\left.\left.-\mathrm{Lif}_{k}\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\right|\left. x^{n-l}\right\rangle \\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l} \\
&\times\left\langle e^{-t}\left(\mathrm{Lif}_{k-1}\left(\log\left(1+\frac{t}{a}\right)\right)-\mathrm{Lif}_{k}\left(\log\left(1+\frac{t}{a}\right)\right)\right)\left(1+\frac{t}{a}\right)^{-y-1}\right|\left. x^{n-l}\right\rangle \\
=&\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{PC_{n-l}^{(k-1)}\left(y+1:a\right)-PC_{n-l}^{(k)}\left(y+1:a\right)\right\}.
\end{aligned}$$

By (2.30) and (2.33), we also get the following equation :

$$\begin{aligned}
&PC_n^{(k)}\left(x:a\right) \qquad (2.34)\\
=&-PC_{n-1}^{(k)}\left(x:a\right)-\frac{1}{a}xPC_{n-1}^{(k)}\left(x+1:a\right) \\
&+\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{PC_{n-l}^{(k-1)}\left(x+1:a\right)-PC_{n-l}^{(k)}\left(x+1:a\right)\right\}.
\end{aligned}$$

By the same method as Theorem 6, we see that

$$\begin{aligned}
&P\hat{C}_n^{(k)}\left(x:a\right) \qquad (2.35)\\
=&-P\hat{C}_{n-1}^{(k)}\left(x:a\right)+\frac{1}{a}xP\hat{C}_{n-1}^{(k)}\left(x-1:a\right) \\
&+\frac{1}{n}\sum_{l=0}^{n-1}\frac{\hat{C}_l}{a^l}\binom{n}{l}\left\{P\hat{C}_{n-l}^{(k-1)}\left(x:a\right)-P\hat{C}_{n-l}^{(k)}\left(x:a\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
&P\hat{C}_{n}^{(k)}(x:a) \qquad (2.36)\\
=&-P\hat{C}_{n-1}^{(k)}(x:a)+\frac{1}{a}xP\hat{C}_{n-1}^{(k)}(x-1:a)\\
&+\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}\frac{C_l}{a^l}\left\{P\hat{C}_{n-1}^{(k-1)}(x-1:a)-P\hat{C}_{n-l}^{(k)}(x-1:a)\right\}.
\end{aligned}$$

Here, we compute

$$\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m\middle| x^n\right\rangle$$

in two different ways.

On the one hand,

$$\begin{aligned}
&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)^m\right)\middle| x^n\right\rangle \qquad (2.37)\\
=&\sum_{l=0}^{n-m}\frac{m!}{a^{l+m}}\binom{n}{l+m}S_1(l+m,m)\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\middle| x^{n-l-m}\right\rangle\\
=&\sum_{l=0}^{n-m}\frac{m!}{a^{l+m}}\binom{n}{l+m}S_1(l+m,m)P\hat{C}_{n-l-m}^{(k)}(0:a)\\
=&\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1(n-l,m)P\hat{C}_{l}^{(k)}(0:a).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \Big| x^n\right\rangle \qquad (2.38)\\
=&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \Big| x\cdot x^{n-1}\right\rangle\\
=&\left\langle \partial_t\left\{e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m\right\} \Big| x^{n-1}\right\rangle\\
=&\left\langle \left(\partial_t e^{-t}\right)\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \Big| x^{n-1}\right\rangle\\
&+\left\langle e^{-t}\left(\partial_t\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \Big| x^{n-1}\right\rangle\\
&+\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\partial_t\left(\log\left(1+\frac{t}{a}\right)\right)^m\right) \Big| x^{n-1}\right\rangle\\
=&-\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \Big| x^{n-1}\right\rangle\\
&+\frac{m-1}{a}\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \Big| \left(\log\left(1+\frac{t}{a}\right)\right)^{m-1} x^{n-1}\right\rangle\\
&+\frac{1}{a}\left\langle e^{-t}\mathrm{Lif}_{k-1}\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \Big| \left(\log\left(1+\frac{t}{a}\right)\right)^{m-1} x^{n-1}\right\rangle.
\end{aligned}$$

It is easy to show that

$$\begin{aligned}
&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-1} \Big| \log\left(1+\frac{t}{a}\right)^{m-1} x^{n-1}\right\rangle \qquad (2.39)\\
=&\sum_{l=0}^{n-m}\frac{(m-1)!}{a^{l+m-1}}\binom{n-1}{l+m-1}S_1\left(l+m-1,m-1\right)P\hat{C}_{n-l-m}^{(k)}\left(-1:a\right)\\
=&\sum_{l=0}^{n-m}\frac{(m-1)!}{a^{n-l-1}}\binom{n-1}{l}S_1\left(n-1-l,m-1\right)P\hat{C}_{l}^{(k)}\left(-1:a\right).
\end{aligned}$$

Thus, by (2.38) and (2.39), we get

$$\begin{aligned}
&\left\langle e^{-t}\mathrm{Lif}_k\left(-\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n\right\rangle \qquad (2.40)\\
&=-\sum_{l=0}^{n-m-1}\frac{m!}{a^{n-l-1}}\binom{n-1}{l}S_1(n-1-l,m)\,P\hat{C}_l^{(k)}(0:a)\\
&+\left(\frac{m-1}{m}\right)\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n-1}{l}S_1(n-l-1,m-1)\,P\hat{C}_l^{(k)}(-1:a)\\
&+\frac{1}{m}\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n-1}{l}S_1(n-l-1,m-1)\,P\hat{C}_l^{(k-1)}(-1:a).
\end{aligned}$$

Therefore, by (2.37) and (2.40), we obtain the following theorem.

Theorem 7. *For $n, m > 0$ with $n-m \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1(n-l,m)\,P\hat{C}_l^{(k)}(0:a)\\
&+\sum_{l=0}^{n-1-m}\frac{\binom{n-1}{l}}{a^{n-1-l}}S_1(n-1-l,m)\,P\hat{C}_l^{(k)}(0:a)\\
&=\left(1-\frac{1}{m}\right)\sum_{l=0}^{n-m}\frac{\binom{n-1}{l}}{a^{n-l}}S_1(n-1-l,m-1)\,P\hat{C}_l^{(k)}(-1:a)\\
&+\frac{1}{m}\sum_{l=0}^{n-m}\frac{\binom{n-1}{l}}{a^{n-l}}S_1(n-1-l,m-1)\,P\hat{C}_l^{(k)}(-1:a).
\end{aligned}$$

Remark. From the computation of

$$\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(\log\left(1+\frac{t}{a}\right)\right)^m \middle| x^n\right\rangle$$

in two different ways, we can also derive the following equation :

$$\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1(n-l,m)\,PC_l^{(k)}(0:a) \tag{2.41}$$
$$+\sum_{l=0}^{n-1-m}\frac{\binom{n-1}{l}}{a^{n-l-1}}S_1(n-l-1,m)\,PC_l^{(k)}(0:a)$$
$$=\left(1-\frac{1}{m}\right)\sum_{l=0}^{n-m}\frac{\binom{n-1}{l}}{a^{n-l}}S_1(n-1-l,m-1)\,PC_l^{(k)}(1:a)$$
$$+\frac{1}{m}\sum_{l=0}^{n-m}\frac{\binom{n-1}{l}}{a^{n-l}}S_1(n-1-l,m-1)\,PC_l^{(k-1)}(1:a).$$

By (1.21), (2.1) and (2.2), we easily see that

$$\frac{d}{dx}PC_n^{(k)}(x:a)=(-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^l}{(n-l)\,l!a^{n-l}}PC_l^{(k)}(x:a), \tag{2.42}$$

and

$$\frac{d}{dx}P\hat{C}_n^{(k)}(x:a)=(-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^{l-1}}{(n-l)\,l!a^{n-l}}P\hat{C}_l^{(k)}(x:a). \tag{2.43}$$

For

$$PC_n^{(k)}(x:a)\sim\left(e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)},a\left(e^{-t}-1\right)\right),\ (a\neq 0)$$

and

$$B_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right),\ (s\in\mathbb{Z}_{\geq 0}),$$

let us assume that

$$PC_n^{(k)}(x:a)=\sum_{m=0}^{n}C_{n,m}B_m^{(s)}(x). \tag{2.44}$$

From (1.25), we note that

$$C_{n,m} \tag{2.45}$$
$$=\frac{(-1)^m}{m!}$$
$$\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\left(\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}\right)^s\middle|\left(\log\left(1+\frac{t}{a}\right)\right)^m x^n\right\rangle$$

Now, we observe that

$$\left(\log\left(1+\frac{t}{a}\right)\right)^{m}x^{n}=\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1\left(n-l,m\right)x^{l}. \tag{2.46}$$

By (2.45) and (2.46), we get

$$\begin{aligned} &C_{n,m} \qquad (2.47)\\ =&\frac{(-1)^m}{m!}\sum_{l=0}^{n-m}\frac{m!}{a^{n-l}}\binom{n}{l}S_1\left(n-l,m\right)\\ &\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\middle|\left(\frac{\frac{t}{a}}{\log\left(1+\frac{t}{a}\right)}\right)^{s}x^{l}\right\rangle\\ =&(-1)^m\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1\left(n-l,m\right)\sum_{i=0}^{l}\frac{\binom{l}{i}\mathbb{C}_i^{(s)}}{a^i}\\ &\times\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\middle|x^{l-i}\right\rangle\\ =&(-1)^m\sum_{l=0}^{n-m}\frac{\binom{n}{l}}{a^{n-l}}S_1\left(n-l,m\right)\sum_{i=0}^{l}\frac{\binom{l}{i}\mathbb{C}_i^{(s)}}{a^i}PC_{l-i}^{(k)}\left(s:a\right)\\ =&(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1\left(n-l,m\right)\mathbb{C}_i^{(s)}PC_{l-i}^{(k)}\left(s:a\right). \end{aligned}$$

Therefore, by (2.44) and (2.47), we obtain the following theorem.

Theorem 8. *For $n \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned} &PC_n^{(k)}\left(x:a\right)\\ =&\sum_{m=0}^{n}\left\{(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1\left(n-l,m\right)\mathbb{C}_i^{(s)}PC_{l-i}^{(k)}\left(s:a\right)\right\}B_m^{(s)}\left(x\right). \end{aligned}$$

Remark. By the same method as Theorem 8, we get

$$\begin{aligned} &P\hat{C}_n^{(k)}\left(x:a\right) \qquad (2.48)\\ =&\sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m}\sum_{i=0}^{l}\frac{\binom{n}{l}\binom{l}{i}}{a^{n-l+i}}S_1\left(n-l,m\right)\hat{\mathbb{C}}_i^{(s)}P\hat{C}_{l-i}^{(k)}\left(s:a\right)\right\}B_m^{(s)}\left(x\right). \end{aligned}$$

For

$$PC_n^{(k)}(x:a) \sim \left(e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}, a\left(e^{-t}-1\right)\right), \ (a \neq 0),$$

and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \ (s \in \mathbb{Z}_{\geq 0}),$$

let us assume that

$$PC_n^{(k)}(x:a) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda). \tag{2.49}$$

From (1.25), we have

$$\begin{aligned} &C_{n,m} \\ =&\frac{(-1)^m}{m!(1-\lambda)^s} \\ &\times \left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\left(1+\frac{t}{a}\right)^{-s}\left(1-\lambda-\frac{\lambda t}{a}\right)^{-s} \middle| \left(\log\left(1+\frac{t}{a}\right)\right)^m x^n\right\rangle \\ =&\frac{(-1)^m}{(1-\lambda)^s}\sum_{l=0}^{n-m}\sum_{i=0}^{s}\frac{\binom{n}{l}\binom{s}{i}(l)_i}{a^{n-l+i}}(1-\lambda)^{s-i}(-\lambda)^i S_1(n-l,m) PC_{l-i}^{(k)}(s:a). \end{aligned} \tag{2.50}$$

Therefore, by (2.49) and (2.50), we obtain the following theorem.

Theorem 9. *For $n \geq 0$, $k \in \mathbb{Z}$ and $a \neq 0$, we have*

$$\begin{aligned} &PC_n^{(k)}(x:a) \\ =&\frac{1}{(1-\lambda)^s}\sum_{m=0}^{n}\left\{(-1)^m\sum_{l=0}^{n-m}\sum_{i=0}^{s}\frac{\binom{n}{l}\binom{l}{i}(l)_i}{a^{n-l+i}}\right. \\ &\left.\times(1-\lambda)^{s-i}(-\lambda)^i S_1(n-l,\,m) PC_{l-i}^{(k)}(s:a)\right\} H_m^{(s)}(x \mid \lambda). \end{aligned}$$

By the same method as Theorem 9, we get

$$\begin{aligned} &P\hat{C}_n^{(k)}(x:a) \\ =&\frac{1}{(1-\lambda)^s}\sum_{m=0}^{n}\left\{\sum_{l=0}^{n-m}\sum_{i=0}^{s}\frac{\binom{n}{l}\binom{s}{i}}{a^{n-l}}(-\lambda)^{s-i}S_1(n-l,m)P\hat{C}_l^{(k)}(i:a)\right\}H_m^{(s)}(x \mid \lambda). \end{aligned} \tag{2.51}$$

For

$$\begin{aligned} PC_n^{(k)}(x:a) &\sim \left(e^{a\left(e^{-t}-1\right)}\frac{1}{\mathrm{Lif}_k(-t)}, a\left(e^{-t}-1\right)\right), \\ x^{(n)} &= x(x+1)\cdots(x+n-1)\sim\left(1,1-e^{-t}\right), \end{aligned}$$

let us assume that

$$PC_n^{(k)}(x:a)=\sum_{m=0}^{n}C_{n,m}x^{(m)}. \tag{2.52}$$

From (1.25), we have

$$\begin{aligned} C_{n,m} &= \frac{1}{m!(-a)^m}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\middle| t^m x^n\right\rangle \qquad (2.53)\\ &= \frac{1}{(-a)^m}\binom{n}{m}\left\langle e^{-t}\mathrm{Lif}_k\left(\log\left(1+\frac{t}{a}\right)\right)\middle| x^{n-m}\right\rangle \\ &= \frac{1}{(-a)^m}\binom{n}{m}PC_{n-m}^{(k)}(0:a). \end{aligned}$$

Therefore, by (2.52) and (2.53), we obtain the following theorem.

Theorem 10. *For $n\geq 0$, $k\in\mathbb{Z}$ and $a\neq 0$, we have*

$$PC_n^{(k)}(x:a)=\sum_{m=0}^{n}\frac{\binom{n}{m}}{(-a)^m}PC_{n-m}^{(k)}(0:a)\,x^{(m)},$$

where $x^{(m)}=x(x+1)\cdots(x+m-1)$.

Remark. By the same method as Theorem 10, we get

$$P\hat{C}_n^{(k)}(x:a)=\sum_{m=0}^{n}\frac{\binom{n}{m}}{a^m}P\hat{C}_{n-m}^{(k)}(0:a)\,(x)_m,$$

where $(x)_m=x(x-1)\cdots(x-m+1)$.

ACKNOWLEDGEMENTS. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MOE) (No.2012R1A1A2003786).

References

[1] M. Can, M. Cenkci, V. Kurt, Y.Simsek, *Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-funcdtions,* Adv. Stud. Contemp. Math. 18 (2009), no. 2, 135-160.

[2] L. Carlitz, *A note on Bernoulli and Euler polynomials of the second kind*, Scripta Math. 25 (1961), 323-330.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[4] R. Dere, Y.Simsek, *Applications of umbral algebra to some special polynomials,* Adv. Stud. Contemp. Math. 22 (2012), no. 3, 433-438.

[5] H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly 79(1972), 44-51.

[6] D. S. Kim, T. Kim, *Some identities of Bernoulli and Euler polynomials arising from umbral calculus*, Adv. Stud. Contemp. Math. 23 (2013), no. 1, 159-171.

[7] D. S. Kim, T. Kim, Y. H. Kim, D. V. Dolgy, *A note on Eulerian polynomials associated with Bernoulli and Euler numbers and polynomials,* Adv. Stud. Contemp. Math. 22 (2012), no. 3, 379-389.

[8] T. Kim, D. S. Kim, T. Mansour, S.-H. Rim, M. Schork, *Umbral calculus and Sheffer sequences of polynomials,* J. Math. Phys. 54 (2013), 083504.

[9] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermonic p-adic integral on* $\mathbb{Z}_p$*,* Russ. J. Math. Phys. 16 (2009), no. 4, 484-491.

[10] D. S. Kim, T. Kim, S.-H. Lee, *Poly-Cauchy Numbers and polynomials with umbral calculus viewpoint***,** Int. Journal of Math. Analysis, Vol. 7, 2013, no. 45, 2235-2253.

[11] D. S. Kim, T. Kim, *Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials,* Adv. Stud. Contemp. Math. 23(2013), 621-636.

[12] D. Merlini, R. Sprugnoli and M. C. Verri, *The Cauchy numbers,* Discrete Math. 306(2006), 1906-1920.

[13] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q- Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. 18 (2009), no. 1, 41-48.

[14] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, 111,Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984, x+193 pp. ISBN: 0-12-594380-6.

[15] S. Roman, G.-C. Rota, *The umbral calculus,* Advances in Math. 27 (1978), no. 2, 95-188.

Received: April 3, 2014