

ON THE *PROBLÈME DES MÉNAGES*

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Introduction. The classical *problème des ménages* asks for the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife.

An outline of the history of the problem to 1946 was given by Kaplansky and Riordan (11). They also presented a bibliography, which is augmented and brought up to date in the bibliography of the present paper.

The first explicit solution of the problem is due to Touchard (23) and the simplest derivation of Touchard's formula is due to Kaplansky (9). In the present paper a new explicit solution to the problem is obtained, via an exponential generating function for certain numbers closely related to the ménage numbers and introduced by Cayley (4). Although the new explicit expression is quite complicated, it does lead to some new and deep results concerning the ménage numbers. In particular, it is shown that the usual asymptotic formula for these numbers can actually be used to compute the numbers exactly.

Several other new explicit expressions for the ménage numbers are obtained and one of these suggests a strong conjecture concerning Latin rectangles for which some evidence is presented.

The most extensive published tables of the ménage numbers are those given by Lucas (13). These go up to $n = 25$. In the present paper we present tables which give the numbers up to $n = 65$. These were computed by F. L. Miksa, using a recursion formula of Cayley (4), and checked by means of congruences due to Riordan (20).

1. A Generating Function. Rather than deal directly with the ménage numbers M_n many authors introduce the number U_n defined by

$$(1.1) \quad M_n = 2 (n!) U_n.$$

Further, Cayley (4) introduced an auxiliary sequence q_n defined by

$$(1.2) \quad U_n = q_n - q_{n-2},$$

and showed that the q_n satisfy the recurrence relation

$$(1.3) \quad q_n = n q_{n-1} + q_{n-2} + (-1)^{n-1} (n-2).$$

If we introduce the generating function $F(t)$ by

$$(1.4) \quad F(t) = \sum_{n=0}^{\infty} q_n \frac{t^n}{n!},$$

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then it is easily shown that $F(t)$ is the solution of

$$(1.5) \quad (1 - t) \ddot{F} - 2\dot{F} - F = t e^{-t}, \\ F(0) = \dot{F}(0) = 0,$$

where the "dot" means differentiation with respect to t .

The substitution

$$(1.6) \quad F = (1 - t)^{\frac{1}{2}}y, \quad x = 2(1 - t)^{\frac{1}{2}}$$

makes (1.5) take the form

$$(1.7) \quad y'' + x^{-1}y' - (1 + x^{-2})y = \frac{1}{2}x(1 - \frac{1}{4}x^2) e^{(x^2/4-1)}, \\ y(2) = y'(2) = 0,$$

where the prime denotes differentiation with respect to x . The homogeneous equation is well known and the complementary function can be expressed in terms of the modified Bessel functions as

$$(1.8) \quad A I_1(x) + B K_1(x),$$

where A, B are constants.

In order to determine a particular integral $P(x)$ of (1.7), we assume a series solution of the form

$$(1.9) \quad P(x) = \sum_{n=0}^{\infty} a_n x^{n+3}.$$

Substituting into (1.7) we immediately are led to

$$(1.10) \quad a_0 = e^{-1}/16, \quad a_{2n+1} = 0, \\ 4a_{2n}(n+1)(n+2) - a_{2n-2} = e^{-1}(1-n)/2^{2n+1}n!$$

This recurrence relation is easily solved and our particular solution can be put into the form

$$(1.11) \quad P(x) = e^{-1} \left[I_1(x) - \frac{1}{2}x e^{x^2/4} + 2 \sum_{n=1}^{\infty} b_n \left(\frac{1}{2}x\right)^{2n+1} \right],$$

where

$$b_n = \left(\sum_{s=1}^n s! \right) / n!(n+1)!.$$

Replacing $s!$ by

$$\int_0^{\infty} e^{-z} z^s dz,$$

we find

$$(1.12) \quad P(x) = e^{-1} \left[I_1(x) - \frac{1}{2}x e^{x^2/4} + 2 \int_0^{\infty} F(x, z) dz \right],$$

where $F(x, z) = z e^{-z} (I_1(x) - z^{\frac{1}{2}} I_1(x z^{-\frac{1}{2}})) / (1 - z)$.

If we introduce the principal value of the integral at $z = 1$ we can rearrange the terms so that

$$(1.13) \quad P(x) = e^{-1} \left[L I_1(x) - \frac{1}{2} x e^{x^2/4} + 2 \int_0^\infty G(x, z) dz \right],$$

where

$$(1.14) \quad L = 2 \int_0^\infty \frac{e^{-z}}{1-z} dz - 1, \quad G(x, z) = \frac{z^{\frac{1}{2}} e^{-z} I_1(xz^{\frac{1}{2}})}{z-1}.$$

Thus the general solution of (1.7) must be of the form

$$(1.15) \quad y = A I_1(x) + B K_1(x) + P(x),$$

where the constants A, B must be chosen to satisfy $y(2) = y'(2) = 0$.

The analysis so far is straight-forward and it seems likely that it has been carried thus far before. The major difficulty is in the evaluation of the constants A and B . In view of the complexity of the functions involved it is, indeed, remarkable that these constants can be evaluated in a tractable form. The evaluation of the constants is given in the next section.

2. Evaluation of the constants. If $f_1(x), f_2(x)$ denote two functions of x we introduce the usual Wronskian notation $W(f_1, f_2)$ by

$$(2.1) \quad W(f_1, f_2) = f_1 f_2' - f_2 f_1'.$$

In order to satisfy the boundary conditions $y(2) = y'(2) = 0$ we have

$$(2.2) \quad \begin{aligned} A I_1(2) + B K_1(2) + P(2) &= 0 \\ A I_1'(2) + B K_1'(2) + P'(2) &= 0. \end{aligned}$$

Since it is well known that $W(I_1(2), K_1(2)) = -\frac{1}{2}$ we have

$$(2.3) \quad A = 2 W(P(2), K_1(2)), \quad B = 2 W(I_1(2), P(2)).$$

We evaluate these Wronskians, by the usual procedure, from the differential equations satisfied by $P(x)$ and $I_1(x)$. These differential equations are

$$(2.4) \quad x P'' + P' - (x + x^{-1}) P = \frac{1}{2} x^2 (1 - \frac{1}{4} x^2) \exp(\frac{1}{4} x^2 - 1),$$

$$(2.5) \quad x I_1'' + I_1' - (x + x^{-1}) I_1 = 0.$$

We multiply (2.4) by I_1 and (2.5) by P . By subtraction of the resulting equations and integration from $x = 0$ to $x = 2$ we obtain

$$(2.6) \quad 2 W(I_1(2), P(2)) = \frac{1}{2} e^{-1} \int_0^2 x^2 (1 - \frac{1}{4} x^2) e^{x^2/4} I_1(x) dx.$$

Hence

$$(2.7) \quad B = \frac{1}{2} e^{-1} \int_0^2 x^2 (1 - \frac{1}{4} x^2) e^{x^2/4} I_1(x) dx,$$

and similarly

$$(2.8) \quad A = -\frac{1}{2} e^{-1} \int_0^2 x^2 (1 - \frac{1}{4} x^2) e^{x^2/4} K_1(x) dx.$$

In order to evaluate (2.7) we write (2.5) in the form

$$(2.9) \quad I_1'' + (x^{-1} I_1)' - I_1 = 0.$$

Multiplying (2.9) by $\exp(x^2/4)$ and integrating from 0 to 2 we can show, by integrating by parts, that

$$(2.10) \quad \int_0^2 e^{x^2/4} (\frac{1}{4}x^2 - 1) I_1(x) dx = 1 - e I_1'(2) + \frac{1}{2} e I_1(2).$$

Similarly by multiplying the differential equation by $x^2 \exp(x^2/4)$ and repeating the process we find

$$(2.11) \quad \int_0^2 e^{x^2/4} (x^2 + \frac{1}{4}x^4) I_1(x) dx = 6 e I_1(2) - 4 e I_1'(2).$$

Multiplying (2.10) by eight and subtracting (2.11) we obtain

$$(2.12) \quad \int_0^2 e^{x^2/4} (x^2 - \frac{1}{4}x^4) I_1(x) dx = 8 - 4e I_1'(2) - 2e I_1(2) + 8 \int_0^2 e^{x^2/4} I_1(x) dx$$

From the known recurrence relations of the modified Bessel functions we have

$$(2.13) \quad 2 I_1'(2) + I_1(2) = 2 I_0(2).$$

Hence

$$(2.14) \quad \int_0^2 e^{x^2/4} (x^2 - \frac{1}{4}x^4) I_1(x) dx = 8 - 4 e I_0(2) + 8 \int_0^2 e^{x^2/4} I_1(x) dx.$$

Let us now consider the integral

$$J = \int_0^2 e^{x^2/4} I_1(x) dx.$$

The substitution $x = 2u^{1/2}$ transforms J into

$$\begin{aligned} (2.15) \quad J &= \int_0^1 e^u I_1(2 u^{1/2}) u^{-1/2} du \\ &= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} \int_0^1 e^u u^n du \\ &= \sum_{n=0}^{\infty} \frac{(1-n+n(n-1)\dots(-1)^n n!)e + (-1)^{n+1} n!}{n!(n+1)!} \\ &= e[I_1(2) - I_2(2) + I_3(2) \dots] + e^{-1} - 1 \\ &= e \sum_{n=1}^{\infty} (-1)^{n+1} I_n(2) + e^{-1} - 1. \end{aligned}$$

However, from the generating function for $I_n(x)$ we can prove that

$$(2.16) \quad e^{-2} = I_0(2) + 2 \sum_{n=1}^{\infty} (-1)^n I_n(2).$$

Thus

$$(2.17) \quad J = \frac{1}{2}e^{-1} + \frac{1}{2}eI_0(2) - 1$$

and, from (2.14),

$$(2.18) \quad \int_0^2 e^{x^2/4} (x^2 - \frac{1}{4}x^4) I_1(x) dx = 4e^{-1}.$$

Finally from (2.7), (2.18) we have that the constant B is given by

$$(2.19) \quad B = 2e^{-2}.$$

The evaluation of the constant A can also be carried out with the help of the integral representation.

$$(2.20) \quad 2 K_1 (2 u^{\frac{1}{2}}) u^{-\frac{1}{2}} = \int_0^\infty \exp(-uz - z^{-1}) dz.$$

The final result is that

$$(2.21) \quad A = e^{-1} + 2e^{-1} \int_0^\infty e^{-z}/(z - 1) dz.$$

These results imply that the desired solution of (1.7) is

$$(2.22) \quad y = 2e^{-2}K_1(x) - \frac{1}{2}e^{-1}xe^{\frac{1}{2}x^4} - 2e^{-1} \int_0^\infty \frac{z^{\frac{1}{2}}e^{-z}I_1(x(z)^{\frac{1}{2}})dz}{1 - z}$$

and that the generating function $F(t)$, for q_n is given by

$$(2.23) \quad F(t) = 2e^{-2}(1 - t)^{-\frac{1}{2}}K_1(2(1 - t)^{\frac{1}{2}}) - e^{-t} - 2e^{-1} \int_0^\infty H(z, t) dz$$

where

$$H(z, t) = z^{\frac{1}{2}} e^{-z} I_1(2(z - zt)^{\frac{1}{2}})/(1 - z)(1 - t)^{\frac{1}{2}}.$$

The modified Bessel functions satisfy the well known differentiation formulae

$$(2.24) \quad \left(\frac{d}{zdz}\right)^m z^{-\alpha} I_\alpha(z) = z^{-\alpha-m} I_{\alpha+m}(z),$$

$$(2.25) \quad \left(\frac{d}{zdz}\right)^m z^{-\alpha} K_\alpha(z) = (-1)^m z^{-\alpha-m} K_{\alpha+m}(z).$$

Hence

$$(2.26) \quad q_n = F^{(n)}(0) = 2e^{-2}K_{n+1}(2) + (-1)^{n+1} + 2(-1)^{n+1}e^{-1} \int_0^\infty M_{n+1}(z) dz,$$

where

$$M_{n+1}(z) = z^{\frac{1}{2}(n+1)} e^{-z} I_{n+1} (2z^{\frac{1}{2}})/(1 - z).$$

Since the ménage numbers U_n are given by $U_n = q_n - q_{n-2}$ we find that

$$(2.27) \quad U_n = 2e^{-2}nK_n(2) + 2(-1)^n + 2n(-1)^n e^{-1} \int_0^\infty M_n(z) dz.$$

If we replace $K_n(2)$, $I_n(2z^{\frac{1}{2}})$ by their known series expansions we can obtain an explicit series expression for U_n in terms of n . This expression is very complicated. However (2.27) is a useful expression in that one can derive many of

the known results directly without resorting to the series expression. For example, it is readily shown from (2.27) that

$$(2.28) \quad \sum_{n=2}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t) - I_0(2t) + I_1(2t).$$

Hence, by redefining U_0, U_1 , to be 1 and -1 respectively we obtain Touchard's result (24):

$$(2.29) \quad \sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t).$$

In the next section we shall use (2.27) to derive some new results for the ménage numbers.

3. New results. It has been shown (11) that an asymptotic expansion for U_n is given by

$$(3.1) \quad U_n \sim e^{-2} n! \left[1 - \frac{1}{(n-1)} + \frac{1}{2!(n-1)(n-2)} \cdots \right].$$

By means of (2.27) we shall prove a much deeper result.

To prove this result we write (2.27) in the form

$$(3.2) \quad U_n = 2e^{-2} n K_n(2) + J_n,$$

where

$$(3.3) \quad J_n = 2(-1)^n \left\{ 1 + n e^{-1} \int_0^{\infty} \frac{z^{n/2} e^{-z} I_n(2z^{\frac{1}{2}}) dz}{1-z} \right\}.$$

In (3.3) we replace the first term of the bracket by means of

$$(3.4) \quad 1 = e^{-1} \sum_{m=0}^{\infty} 1/m!$$

and $I_n(2z^{\frac{1}{2}})$ by its series expression

$$(3.5) \quad I_n(2z^{\frac{1}{2}}) = z^{\frac{1}{2}n} \sum_{m=0}^{\infty} \frac{z^m}{m!(m+n)!}.$$

Hence J_n takes the form

$$(3.6) \quad J_n = 2(-1)^n e^{-1} \left[\sum_{m=0}^{\infty} \left\{ (1/m!) + n \int_0^{\infty} \frac{e^{-z}}{1-z} \sum_{n=0}^{\infty} \frac{z^{m+n}}{m!(m+n)!} dz \right\} \right].$$

This can be put in the form

$$(3.7) \quad J_n = 2(-1)^n e^{-1} \left\{ C_n I_n(2) + \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m+n)!} \right\},$$

where

$$(3.8) \quad \begin{aligned} C &= \int_0^{\infty} \frac{e^{-z}}{1-z} dz, \\ b_{mn} &= (m+n)! - n\{(m+n-1)! + (m+n-2)! + \dots + 1\} \\ &= (m+n-1)!m - n\{(m+n-2)! + (m+n-3)! + \dots + 1\}. \end{aligned}$$

It is trivial to show

$$(3.9) \quad |C| < 4e^{-1},$$

and

$$(3.10) \quad |nI_n(2)| \leq e/(n-1)!.$$

Hence

$$(3.11) \quad |CnI_n(2)| \leq 4/(n-1)!.$$

Let us consider the series term of (3.7) and write

$$\begin{aligned} (3.12) \quad H_n &= \sum_{m=0}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= \frac{n! - n\{(n-1)! + \dots + 1\}}{n!} \\ &\quad + \frac{(n+1)! - n(n! + (n-1)! + \dots + 1)}{(n+1)!} \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!} \\ &= -\frac{(n-2)! + (n-3)! + \dots + 1}{(n-1)!} \left(1 + \frac{1}{n+1}\right) \\ &\quad + \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!}. \end{aligned}$$

If $n \geq 7$ it is easily shown that

$$(3.13) \quad \frac{(n-2)! + (n-3)! + \dots + 1}{(n-1)!} \left(1 + \frac{1}{n+1}\right) \leq \frac{2}{n+1}$$

and

$$(3.14) \quad \left| \sum_{m=2}^{\infty} \frac{b_{mn}}{m!(m+n)!} \right| \leq \frac{2(e-1)}{n+1}.$$

Hence for $n \geq 7$,

$$(3.15) \quad |H_n| \leq \frac{2e}{n+1}.$$

Actually (3.15) is a very crude inequality. It is, however, sufficient for our purposes.

Combining these results we have from (3.7)

$$(3.16) \quad |J_n| \leq \frac{4}{n+1} + \frac{8}{e(n-1)!}$$

if $n \geq 7$.

Hence for $n \geq 8$ we have

$$(3.17) \quad |J_n| \leq 0.45.$$

Let us now return to (3.2) and examine the series expression for $K_n(2)$. This is given by

$$(3.18) \quad K_n(2) = \frac{1}{2} \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + \frac{1}{2} (-1)^n \sum_{m=0}^{\infty} \frac{\Psi(n+m+1) + \Psi(m+1)}{m!(n+m)!},$$

where

$$(3.19) \quad \Psi(k+1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \gamma, \Psi(1) = -\gamma$$

and γ is Euler's constant.

It is easily shown that

$$(3.20) \quad \left| \sum_{m=0}^{\infty} \frac{\Psi(n+m+1) + \Psi(m+1)}{m!(n+m)!} \right| \leq \frac{e}{2(n-1)!}.$$

This implies

$$(3.21) \quad 2n K_n(2) = n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + R_n,$$

where the remainder satisfies $|R_n| \leq n e/(n-1)!$

Combining the results of (3.2), (3.17) and (3.21) we obtain

$$(3.22) \quad U_n = e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} + R'_n$$

where for $n \geq 8$ the remainder R'_n is definitely less than $\frac{1}{2}$.

Using the notation $\{x\}$ to denote the closest integer to x , we have shown that, for $n \geq 8$

$$(3.23) \quad U_n = \left\{ e^{-2} n \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!}{m!} \right\}.$$

It is easy to verify that (3.23) remains valid for $0 \leq n \leq 7$. Hence we have proved the following theorem:

THEOREM. *For all values of n the ménage numbers U_n are given by (3.23).*

It is thus seen that the asymptotic expansion obtained in (11) is much more than an asymptotic expansion.

In concluding this section we might remark that about half of the terms in (3.23) are redundant in that their sum adds up to less than $\frac{1}{2}$. Further our analysis also implies that

$$(3.24) \quad U_n = \{2e^{-2} n K_n(2)\}.$$

We shall make use of (3.24) in the next section to make an interesting conjecture.

4. A Conjecture. The modified Bessel function $K_n(2)$ has the integral representation

$$(4.1) \quad K_n(2) = \frac{1}{2} \int_0^{\infty} t^{n-1} e^{-t-t^{-1}} dt.$$

Hence (3.24) may be written

$$(4.2) \quad U_n = \left\{ e^{-2} n \int_0^\infty t^{n-1} e^{-t-t^{-1}} dt \right\}.$$

The discovery of (4.2) led us to re-examine some of the known results in Latin rectangles. The simplest problem in this class is the so-called “problème des rencontres.” This asks for the number of ways R_n of writing a second line of integers 1, 2, . . . n which is discordant with a first line of integers written in their normal order. It is well known that

$$(4.3) \quad R_n = \{e^{-1} n!\} = \left\{ e^{-1} \int_0^\infty x^n e^{-x} dx \right\}.$$

Next in simplicity, in this class of problems, is the so-called reduced three line Latin rectangle problem. This asks for the number of ways P_n of having two lines of integers each of which is discordant with the first line of integers, written in normal order. For this case it was shown by Yamamoto (26) that

$$(4.4) \quad P_n \sim e^{-3} (n!)^2 \left[1 + \frac{H_1(-\frac{1}{2})}{n} + \frac{H_2(-\frac{1}{2})}{n(n-1)} + \dots \right],$$

where $H_n(x)$ is a Hermite polynomial.

We have been able to prove an equivalent formula, namely

$$(4.5) \quad P_n \sim e^{-3} (n!) \int_0^\infty x^n e^{-x-x^{-1}-x^{-2}} dx.$$

Finally Erdős and Kaplansky (7) have shown that the number P_n^k of reduced (n by $(k + 1)$), Latin rectangles is given asymptotically by

$$(4.6) \quad P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-1} \left[1 - \binom{k}{3} n^{-1} + \left(\frac{1}{2} \binom{k}{3}^2 + \frac{1}{2} \binom{k}{3} (k-5) \right) n^{-2} + \dots \right]$$

for $K \leq (\log n)^{3/2-\epsilon}$. The validity of the same formula was proved by Yamamoto (26) for $k < n^{1/3-\delta}$. The structure of the formula suggests an integral representation of the type

$$(4.7) \quad P_n^k \sim e^{-\frac{1}{2}k(k-1)} (n!)^{k-2} \int_0^\infty x^n \exp\left(-x - \binom{k}{3} x^{-1} + \frac{1}{2} \binom{k}{3} (k-5) x^{-2} + \dots\right) dx.$$

Formula (4.7) is, as we have seen, true for $k = 2, 3$. If it were possible to prove an integral relation of this type then the asymptotic behavior of P_n^k could be determined for all values of k .

5. An exact expression for the ménage numbers. The usual explicit expression given for the ménage numbers U_n is

$$(5.1) \quad U_n = \sum_{k=0}^n (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.$$

In this section we shall derive a second expression from Touchard's generating function (2.9)

$$(5.2) \quad \sum_{n=0}^{\infty} U_n I_n(2t) = e^{-2t}/(1-t).$$

Touchard has remarked that (5.2) constitutes a Neumann expansion for the function $e^{-2t}/(1-t)$ in terms of the modified Bessel functions $I_n(2t)$. However as far as we are aware, (5.2) has never been inverted to give an explicit expression for the U_n .

If we expand $e^{-2t}/(1-t)$ into a Maclaurin expansion of the form

$$(5.3) \quad \frac{e^{-2t}}{1-t} = \sum_{r=0}^{\infty} \frac{k_r t^r}{r!}.$$

then

$$(5.4) \quad k_r = \left[\frac{d^r}{dt^r} \frac{e^{-2t}}{1-t} \right]_{t=0} = r! \sum_{s=0}^r \frac{(-2)^s}{s!}.$$

Further from the well formulae for the coefficients of a Neumann expansion, (5.2) gives

$$(5.5) \quad U_n = \frac{2(i^n)}{\pi} \int_C \frac{e^{-2t} O_n(2it)}{1-t} dt,$$

where C is any closed contour, enclosing $t = 0$, such that $|t| < 1$. $O_n(z)$ are the so-called Neumann polynomials given explicitly by

$$(5.6) \quad O_n(z) = \frac{1}{4} \sum_{m=0}^{[\frac{1}{2}n]} \frac{n(n-m-1)! (\frac{1}{2}z)^{2m-n-1}}{m!}.$$

It follows immediately from (5.4), (5.5) and (5.6) that

$$(5.7) \quad U_n = \sum_{m=0}^{[\frac{1}{2}n]} \frac{(-1)^m n(n-m-1)! k_{n-2m}}{m!(n-2m)!}$$

If we use the umbral convention of replacing k_r by k^r we obtain the neat, mnemonic, formula

$$(5.8) \quad U_n = 2 T_n(\frac{1}{2}k).$$

where $T_n(k)$ is the Chebyshev polynomial.

Table of Ménage Numbers, U_n

n									
0									1
1									-1
2									0
3									1
4									2
5									13
6									80
7									579
8									4738
9									43387
10								4	39792
11								48	90741
12								592	16642
13								7755	96313
14								1	09274 34464
15								16	48064 35783
16								264	93914 69058
17								4522	64356 01207
18								81705	64062 24416
19								15	57461 89109 94665
20								312	40021 86712 53762
21								6577	61864 45769 02053
22								1	45051 25042 12302 24304
23								33	43382 81820 37841 46955
24								803	99425 36462 30706 80706
25								20136	19745 87449 39236 99123
26								5	24412 12770 21518 36760 81296
27								141	80874 54121 35441 26917 90045
28								3976	29238 67612 00144 54828 24194
29								1	15464 79231 29989 49665 85597 50193
30								34	68204 08266 14983 47273 40955 31712
31								1076	37754 44394 44821 25463 33529 40175
32								34481	07559 89439 56929 18585 03293 19426
33								11	39021 31602 21345 03795 43638 02432 51567
34								387	63360 88757 64510 83282 09689 42454 55168
35								13579	25683 97610 83548 12838 24806 55155 91633
36								4	89263 68181 72674 64273 50357 97412 89388 39554
37								181	17111 44161 23578 95013 36816 90501 14249 74653
38								6889	66679 77874 33823 33907 79975 80757 02511 45232
39								2	68887 96926 13377 25044 79310 17322 96268 42696 37331
40								107	62771 05129 32852 47921 55467 77103 56797 10498 56642

Table of Ménage Numbers, U_n .

n											
41		4415	56290	19891	48194	39830	83196	99970	42707	08660	48747
42	1	85566	65097	95828	03659	83212	57515	14716	68334	59763	96848
43	79	83996	94833	63418	59137	63816	96992	08396	12446	35031	15589
44	3514	90268	88496	81285	48747	33216	99334	22942	19228	03980	73090
45										1	58254
		17445	46717	35843	13657	70852	22706	45836	45728	90212	00713
46										72	83366
		69590	77881	51946	38308	62111	11982	87007	50904	26641	27392
47										3424	83522
		40098	53669	54471	72547	68779	40243	86115	44706	26361	30391
48										1	64468
		06041	87840	60152	09219	74918	57252	67810	16409	45316	09090
49										80	62507
		27218	60142	82965	59317	42716	23933	33754	57054	84141	72839
50										4032	96672
		58890	36142	10938	08808	59685	47971	75030	78168	58457	34752
51										2	05765
		33778	81355	42153	61997	39439	63306	20885	84756	93390	60409
52										107	03985
		61744	28069	71363	85025	44933	81813	48096	08655	89107	05410
53										5675	25075
		27158	19299	47659	39404	93066	88177	80601	40960	60903	31861
54										3	06574
		42199	94397	97715	89238	83812	11946	29345	13169	39005	76112
55										168	67497
		92576	17576	87982	88650	64735	97608	01398	80030	29273	13563
56										9448	97804
		94695	41458	72821	08832	08427	22881	97653	69427	61227	49570
57										5	38766
		19035	75909	00166	11666	74714	78157	33164	78496	20304	86819
58										312	58246
		52151	42090	06485	17961	70756	96110	13984	56829	80382	26128
59										18447	94228
		14181	37324	89815	08658	95856	70696	19067	11401	86377	75277
60										11	07199
		94487	31311	19751	93291	00461	34299	17114	32556	45129	57442
61										675	58267
		00422	09590	14676	72585	11993	77206	51822	90952	23565	78401
62										41897	56666
		40473	50277	33700	66678	14090	58913	27371	86013	61045	83552
63										26	40244
		61742	01966	97316	33064	88618	71601	99260	68529	43996	56223
64										1690	18892
		12623	19720	95291	01110	79691	33663	80938	68220	78795	03874
65										1	09889
		74686	09288	18130	98379	85654	60435	38029	33627	62308	89183

NOTE: $U_{45} = 15825417445 \dots etc.$

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