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ON THE "PROBLÈME DES MÉNAGES"

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In this paper a generalization is given for the classical problem of "ménages".

1. Introduction

In this paper we are concerned with the following problem: At a dinner party n married couples are seated in $2n$ seats at a round table according to the following pattern: the women take alternate seats and the men choose the remaining seats at random. Let us suppose that all the $n!$ possible sitting arrangements of the n men are equally probable. Denote by ξ_n the number of men sitting next to their wives on their wives' right and η_n the number of men sitting next to their wives on their wives' left. The problem is to determine the joint distribution of ξ_n and η_n for every $n = 1, 2, \ldots$.

2. The joint distribution of ξ_n and η_n

The case of $n = 1$ is trivial. We have $\mathbf{P}\{\xi_1 = 1\} = \mathbf{P}\{\eta_1 = 1\} = 1$. For $n \geqslant 2$ we shall prove the following result.

Theorem 1. *If $n \geqslant 2$ and $j + k \leqslant n$, we have*

$$\mathbf{P}\{\xi_n = j, \eta_n = k\} = \frac{1}{j!\,k!\,(n-1)!} \sum_{0 \leqslant t \leqslant n-j-k-1} \frac{(-1)^{n-j-k-t-1}(n+t)!\,(j+t)!\,(k+t)!}{t!\,(n-j-k-t-1)!\,(j+k+2t+1)!} + \frac{(-1)^{n-j}\delta_{k,0}}{j!\,(n-j)!} + \frac{(-1)^{n-k}\delta_{j,0}}{k!\,(n-k)!} \tag{1}$$

where $\delta_{k,0} = 1$ for $k = 0$ and $\delta_{k,0} = 0$ for $k \neq 0$.

Proof. Let us define

$$B_{r,s}(n) = \mathbf{E}\left\{\binom{\xi_n}{r}\binom{\eta_n}{s}\right\} \tag{2}$$

for $r=0,1,2,\ldots$ and $s=0,1,2,\ldots$. Obviously, $B_{r,s}(n)=0$ if $r+s>n$. If $r+s\leqslant n$, then

$$B_{r,s}(n)=\sum_{j=r}^{n}\sum_{k=s}^{n}\binom{j}{r}\binom{k}{s}\mathbf{P}\{\xi_n=j,\ \eta_n=k\} \tag{3}$$

and by inversion we obtain that

$$\mathbf{P}\{\xi_n=j,\ \eta_n=k\}=\sum_{r=j}^{n}\sum_{s=k}^{n}(-1)^{r-j+s-k}\binom{r}{j}\binom{s}{k}B_{r,s}(n) \tag{4}$$

for $j+k\leqslant n$. The proof of (4) is easy. If in (4) we replace $B_{r,s}(n)$ by (3), then the right-hand side of (4) reduces to the left-hand side.

It remains to determine $B_{r,s}(n)$ for $r+s\leqslant n$. We can write that

$$B_{r,s}(n)=\sum_{\substack{1\leqslant i_1<i_2<\cdots<i_r\leqslant n\\ 1\leqslant j_1<j_2<\cdots<j_s\leqslant n}}\mathbf{P}\{A_{i_1}A_{i_2}\cdots A_{i_r}A^*_{j_1}A^*_{j_2}\cdots A^*_{j_s}\} \tag{5}$$

where A_i is the event that the ith husband is sitting next to his wife on his wife's right and A_i^* is the event that the ith husband is sitting next to his wife on his wife's left. Of course the simultaneous occurrence of A_i and A_i^* is the impossible event. To prove (5) let us define χ_i as 1 if A_i occurs and 0 otherwise, and χ_i^* as 1 if A_i^* occurs and 0 otherwise. Since $\binom{\xi_n}{r}\binom{\eta_n}{s}$ is the coefficient of x^ry^s in the expansion of

$$(1+x)^{\xi_n}(1+y)^{\eta_n}=\prod_{i=1}^{n}(1+x)^{\chi_i}(1+y)^{\chi_i^*}=\prod_{i=1}^{n}(1+\chi_ix)(1+\chi_i^*y), \tag{6}$$

we have

$$\binom{\xi_n}{r}\binom{\eta_n}{s}=\sum_{\substack{1\leqslant i_1<i_2<\cdots<i_r\leqslant n\\ 1\leqslant j_1<j_2<\cdots<j_s\leqslant n}}\chi_{i_1}\chi_{i_2}\cdots\chi_{i_r}\chi^*_{j_1}\chi^*_{j_2}\cdots\chi^*_{j_s}. \tag{7}$$

By forming the expectation of (7) we get (5).

The case of $r+s=n$ is trivial. We have $B_{n,0}(n)=B_{0,n}(n)=1/n!$ and $B_{r,n-r}(n)=0$ for $0<r<n$. If $r+s<n$, then we use (5) in determining $B_{r,s}(n)$.

To find (5) let us suppose that all the n women are already seated. Let us select $w=r+s$ women. They form $n-w$ blocks (empty or nonempty) such that any two adjacent blocks are separated by one of the remaining $n-w$ women. Let r_i+s_i $(i=1,2,\ldots,n-w)$ be the size of the ith block. Suppose that in the ith block r_i husbands will be seated next to their wives on their wives' right and s_i husbands will be seated next to their wives on their wives' left. For each block $(i=1,2,\ldots,n-w)$ this can be done in only one way. The remaining $n-w$ husbands can be seated in $(n-w)!$ ways in the remaining $n-w$ seats. Now let us form n cyclic permutations of each sitting arrangement described above in such a way that a given seat will be occupied by all the n women in succession. Then all the distinct sitting arrangements of the n men will appear precisely $n-w$ times. Thus we obtain that

$$B_{r,s}(n)=\frac{n}{(n-w)}\frac{(n-w)!}{n!}\sum_{\substack{r_1+\cdots+r_{n-w}=r\\ s_1+\cdots+s_{n-w}=s}} 1$$
$$=\frac{(n-r-s-1)!}{(n-1)!}\binom{n-r-1}{s}\binom{n-s-1}{r} \tag{8}$$

for $r+s<n$. If we take into consideration that

$$\sum_{r+s=n-t-1}\binom{n-r-1}{t+k}\binom{n-s-1}{t+j}=\binom{n+t}{2t+j+k+1} \tag{9}$$

for $t=0,1,\ldots,n-j-k-1$, then by (4) we get (1) which was to be proved.

From Theorem 1 we can deduce the following limit theorem.

Theorem 2. *We have*

$$\lim_{n\to\infty}\mathbf{P}\{\xi_n=j,\eta_n=k\}=\mathrm{e}^{-2}/j!\,k! \tag{10}$$

for $j=0,1,2,\ldots$ *and* $k=0,1,2,\ldots$.

Proof. We shall prove that there exist two independent random variables ξ and η each having a Poisson distribution with expectation 1, that is,

$$\mathbf{P}\{\xi=j\}=\mathbf{P}\{\eta=j\}=\mathrm{e}^{-1}/j! \tag{11}$$

for $j=0,1,2,\ldots$, such that

$$\lim_{n\to\infty}\mathbf{P}\{\xi_n=j,\eta_n=k\}=\mathbf{P}\{\xi=j\}\mathbf{P}\{\eta=k\} \tag{12}$$

for $j=0,1,2,\ldots$ and $k=0,1,2,\ldots$.

The rth binominal moments of ξ and η are

$$\mathbf{E}\left\{\binom{\xi}{r}\right\}=\mathbf{E}\left\{\binom{\eta}{r}\right\}=1/r! \tag{13}$$

for $r=0,1,2,\ldots$. Since the Poisson distribution is uniquely determined by its binomial moments, therefore (12) holds if

$$\lim_{n\to\infty}B_{r,s}(n)=1/r!\,s! \tag{14}$$

for $r=0,1,2,\ldots$ and $s=0,1,2,\ldots$. From (8) we can conclude that (14) is indeed true and consequently (12) also holds. This completes the proof of the theorem.

By (12) it follows that

$$\lim_{n\to\infty}\mathbf{P}\{\xi_n=j\}=\mathbf{P}\{\xi=j\}=\mathrm{e}^{-1}/j! \tag{15}$$

for $j=0,1,2,\ldots$ and

$$\lim_{n\to\infty} \mathbf{P}\{\eta_n=k\}=\mathbf{P}\{\eta=k\}=e^{-1}/k! \tag{16}$$

for $k=0,1,2,\ldots$. Thus

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n=j, \eta_n=k\}=\lim_{n\to\infty} \mathbf{P}\{\xi_n=j\} \lim_{n\to\infty} \mathbf{P}\{\eta_n=k\} \tag{17}$$

for $j=0,1,2,\ldots$, and $k=0,1,2,\ldots$, that is ξ_n and η_n are asymptotically independent if $n\to\infty$.

3. The distribution of $\xi_n+\eta_n$

The random variable $\xi_n+\eta_n$ denotes the number of husbands sitting next to their wives in the sitting arrangement described in the Introduction. The problem of finding the probability $\mathbf{P}\{\xi_n+\eta_n=0\}$ was proposed in 1891 by E. Lucas [22, p. 215] and solved by C.A. Laisant and C. Moreau [22, pp. 491–495]. This problem is known as the "problème des ménages". An equivalent problem was proposed in 1877 by P.G. Tait [34, p. 159], [35, p. 287] and solved by Th. Muir [26], [27] and A. Cayley [4], [5]. By the results of Th. Muir, A Cayley, C.A. Laisant and C. Moreau we have

$$\mathbf{P}\{\xi_n+\eta_n=0\}=M_n/n! \tag{18}$$

where $M_1=M_2=0$, $M_3=1$ and for $n\geqslant 4$ we can determine M_n by the recurrence formula

$$(n-1)M_{n+1}=(n^2-1)M_n+(n+1)M_n+4(-1)^n \tag{19}$$

which is valid for $n\geqslant 3$. The explicit form of M_n was found in 1934 by J. Touchard [38], in 1942 by W. Schöbe [33] and in 1943 by I. Kaplansky [15]. See also J. Riordan [32, pp. 195–201].

For $n\geqslant 2$ we have

$$M_n=n!\sum_{r=0}^{n}(-1)^r C_r(n) \tag{20}$$

where

$$C_r(n)=\mathbf{E}\left\{\binom{\xi_n+\eta_n}{r}\right\}=\frac{2n}{(2n-r)}\binom{2n-r}{r}\frac{(n-r)!}{n!} \tag{21}$$

is the rth binomial moment of $\xi_n+\eta_n$. Since

$$C_r(n)=\sum_{k=r}^{n}\binom{k}{r}\mathbf{P}\{\xi_n+\eta_n=k\} \tag{22}$$

for $0\leqslant r\leqslant n$, we obtain by inversion that

$$\mathbf{P}\{\xi_n+\eta_n=k\}=\sum_{r=k}^{n}(-1)^{r-k}\binom{r}{k}C_r(n) \tag{23}$$

for $k=0,1,\ldots,n$ where $C_r(n)$ is given by (21). This result was found by I. Kaplansky [17] in 1945. See also J. Riordan [32, pp. 195–201] and L. Takács [36].

In 1942 W. Schöbe [33] and in 1943 I. Kaplansky [15] proved that

$$\lim_{n\to\infty} M_n/n! = \mathrm{e}^{-2}. \tag{24}$$

In 1944 J. Wolfowitz [42] proved that

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n+\eta_n=k\} = \mathrm{e}^{-2}2^k/k! \tag{25}$$

for $k=0,1,2,\ldots$. This result can also be derived from (23). If we add less than $n-k$ terms in (23), then the error is of the same sign as, and has absolute value smaller (or equal) than the first term neglected. From (23) it follows that

$$\sum_{i=0}^{2m+1}(-1)^i\binom{k+i}{i}C_{k+i}(n) \leqslant \mathbf{P}\{\xi_n+\eta_n=k\} \leqslant \sum_{i=0}^{2m}(-1)^i\binom{k+i}{i}C_{k+i}(n) \tag{26}$$

if $1\leqslant 2m+1\leqslant n$. If we let $n\to\infty$ in (21) we obtain that

$$\begin{aligned}\frac{2^k}{k!}\sum_{i=0}^{2m+1}\frac{(-1)^i2^i}{i!} &\leqslant \lim_{n\to\infty}\inf \mathbf{P}\{\xi_n+\eta_n=k\}\\ &\leqslant \lim_{n\to\infty}\sup \mathbf{P}\{\xi_n+\eta_n=k\} \leqslant \frac{2^k}{k!}\sum_{i=0}^{2m}\frac{(-1)^i2^i}{i!}\end{aligned} \tag{27}$$

for $m=0,1,2,\ldots$. If $m\to\infty$ in the above inequalities, we get (25). The limit theorem (25) can also be deduced from (12). By (12) we have

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n+\eta_n=k\} = \mathbf{P}\{\xi+\eta=k\} = e^{-2}2^k/k! \tag{28}$$

for $k=0,1,2,\ldots$.

In 1945 I. Kaplansky [17] refined (25) and demonstrated that

$$\mathbf{P}\{\xi_n+\eta_n=k\} = \frac{\mathrm{e}^{-2}2^k}{k!}\left(1-\frac{k^2-3k}{2n}+\frac{k^4-8k^3+9k^2+22k-16}{8n(n-1)}\right)+\mathrm{O}(n^{-3}) \tag{29}$$

as $n\to\infty$.

The "problème des ménages" is the subject of several papers and is studied in several books. In 1903 H.M. Taylor [37] derived (19) again. In 1946 I. Kaplansky and J. Riordan [18] gave a historical account of the problem. The problem is discussed also by W.A. Whitworth [40, p. 271], E. Netto [28, pp. 75–78], [29, pp. 75-78], P.A. MacMahon [23, pp. 253–254], J. Wolfowitz [41], I. Kaplansky [16], J. Touchard [39], and others.

The related topic of three-line Latin rectangles is discussed by S.M. Jacob [13], S.M. Kerawala [19], L. Dulmage [9], J. Riordan [30], [31] and F.W. Light, Jr. [21]. See also C.W. Baur [1], M. Cantor [2] and C.A. Laisant [20].

The probabilities (23) are given in Table 1 for $n\leqslant 10$.

Table 1. $n!\,\mathbf{P}\{\xi_n+\eta_n=k\}$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
1	0	1									
2	0	0	2								
3	1	0	3	2							
4	2	8	4	8	2						
5	13	30	40	20	15	2					
6	80	192	210	152	60	24	2				
7	579	1344	1477	994	469	140	35	2			
8	4738	10800	11672	7888	3660	1232	280	48	2		
9	43387	97434	104256	70152	32958	11268	2856	504	63	2	
10	439792	976000	1036050	695760	328920	115056	30300	6000	840	80	2

4. The distributions of ξ_n and η_n

By symmetry the random variables ξ_n and η_n are identically distributed. We have

$$\mathbf{P}\{\xi_n=k\}=\mathbf{P}\{\eta_n=k\}=\frac{1}{k!}\sum_{i=0}^{n-k}\frac{(-1)^i}{i!} \tag{30}$$

for $k=0,1,\ldots,n$. Obviously,

$$\mathbf{P}\{\xi_n=k\}=\mathbf{P}\{\eta_n=k\}=\binom{n}{k}Q(n-k)/n! \tag{31}$$

for $0\leqslant k\leqslant n$ where $Q(0)=1$ and $Q(n)$ $(n=1,2,\ldots)$ denotes the number of permutations of $1,2,\ldots,n$ in which no coincidence occurs. If in a permutation of $1,2,\ldots,n$ the ith element is i, we say that a coincidence occurs at the ith place. In 1708 P.R. Montmort [24, pp. 58–59] stated without proof that

$$Q(n)=(n-1)[Q(n-1)+Q(n-2)] \tag{32}$$

for $n\geqslant 2$ where $Q(0)=1$ and $Q(1)=0$, and concluded that

$$Q(n)=n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} \tag{33}$$

for $n\geqslant 0$. In a letter to Montmort, dated March 17, 1710, Nikolaus Bernoulli [25, pp. 299–303] gave two proofs for (33). In the first proof he used the method of inclusion and exclusion and proved (33) directly. In the second proof he demonstrated (32) and indicated how (32) implies (33). If we write (32) in the form of

$$\frac{Q(n)}{n!}-\frac{Q(n-1)}{(n-1)!}=-\frac{1}{n}\left[\frac{Q(n-1)}{(n-1)!}-\frac{Q(n-2)}{(n-2)!}\right] \tag{34}$$

where $n \geqslant 2$, and apply (34) repeatedly, then we obtain that

$$\frac{Q(n)}{n!} - \frac{Q(n-1)}{(n-1)!} = \frac{(-1)^n}{n!} \tag{35}$$

for $n \geqslant 1$ which proves (33).

Equation (32) was proved also by L. Euler [11] in 1779, E. Catalan [3] in 1837, and I.B. Haáz [12] in 1942.

By using the method of inclusion and exclusion, in 1718 A. De Moivre [6, pp. 59–66], [7, pp. 95–103], [8, pp. 109–117] proved (30) directly.

We note that it is easy to prove that

$$B_r = \mathbf{E}\left\{\binom{\xi_n}{r}\right\} = \mathbf{E}\left\{\binom{\eta_n}{r}\right\} = 1/r! \tag{36}$$

for $r = 0, 1, \ldots, n$, and since

$$B_r = \sum_{k=r}^{n} \binom{k}{r} \mathbf{P}\{\xi_n = k\} = \sum_{k=r}^{n} \binom{k}{r} \mathbf{P}\{\eta_n = k\} \tag{37}$$

for $r = 0, 1, \ldots, n$ we obtain (30) from (37) by inversion. See also C. Jordan [14], and L. Takács [36].

From (33) it follows immediately that

$$\lim_{n\to\infty} \frac{Q(n)}{n!} = e^{-1}. \tag{38}$$

This result was found by P.R. Montmort [24, pp. 58–59] in 1708 and was known to L. Euler [10] in 1751 and to A. De Moivre [8, pp. 116–117] in 1754. From (31) and (38) it follows that

$$\lim_{n\to\infty} \mathbf{P}\{\xi_n = k\} = \lim_{n\to\infty} \mathbf{P}\{\eta_n = k\} = e^{-1}/k! \tag{39}$$

for $k = 0, 1, 2, \ldots$. This is the first appearance of the Poisson distribution as a limit distribution.

The probabilities (30) are given in Table 2 for $n \leqslant 10$.

Table 2. $n!\,\mathbf{P}\{\xi_n = k\}$

n \ k	0	1	2	3	4	5	6	7	8	9	10
1	0	1									
2	1	0	1								
3	2	3	0	1							
4	9	8	6	0	1						
5	44	45	20	10	0	1					
6	265	264	135	40	15	0	1				
7	1854	1855	924	315	70	21	0	1			
8	14833	14832	7420	2464	630	112	28	0	1		
9	133496	133497	66744	22260	5544	1134	168	36	0	1	
10	1334961	1334960	667485	222480	55650	11088	1890	240	45	0	1

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