A New Version of the Ménages Problem

Ahmad Mahmood Qureshi

School of Mathematical Sciences,
GC University Lahore, Pakistan
sirahmad@gmail.com

Abstract. The *problème des ménages* (*married couples problem*) introduced by E.Lucas in 1891 is a classical problem that asks the number of ways to arrange n married couples around a circular table, so that husbands and wives are in alternate places but no couple is seated together. In this paper we present a new version of ménages problem that carries the constraints consistent with several cultures.

The following problem, introduced by Lucas in 1891, is known as the *problème des ménages*: in how many ways can one seat n couples at a circular table so that men and women are in alternate places and no husband will sit on either side of his wife? (see [1], [2]). In this paper we consider the following related problem:

Suppose there are n_k k-tuples (one husband and $k-1$ wife/wives), $2 \leq k \leq r$, m single men and w single women. In how many ways we can seat them around a circular table such that:
1. All members of a family sit together;
2. No man sits adjacent to a woman except his own wife/wives;
3. A husband having more than one wife will be surrounded by his wives.

For ease of notation, we restrict ourselves to the case when $r = 5$ (a real situation in muslim culture). In this case there are:
n_2 couples (husband and wife),
n_3 triples (husband and two wives),
n_4 quadruples (husband and three wives),
n_5 pentuples (husband and four wives),
m single men and w single women.

Theorem 1.1. *The solution to this problem for $r = 5$ and n_2 even is*

$$(1{\cdot}2!)^{n_3}(2{\cdot}3!)^{n_4}(3{\cdot}4!)^{n_5}\cdot\frac{n_2!}{\left(\frac{n_2}{2}\right)!}\cdot\frac{\left(\frac{n_2}{2}+m-1\right)!}{\left(\frac{n_2}{2}-1\right)!}\cdot\left(\frac{n_2}{2}+n_3+n_4+n_5+w-1\right)!$$

Proof:
Let us code a man by 1 and a woman by 0. We shall first find the number of ways to arrange the married persons around the circular table. Since all members of a family have to sit together, the possible coding arrangements of the given tuples (without distinguishing the wives) could be:

Couple	01	10			
Triple	001	010	100		
Quadruple	0001	0010	0100	1000	
Pentuple	00001	00010	00100	01000	10000

To fulfill constraints 1 and 2, we have a unique possibility to combine the arrangements having codes 01 and 10 from the couples to form the block having code 0110. For the others, if we merge consecutive zeros as one zero, then constraint 3 would only allow blocks having codes 010 with 3, 4 or 5 members representing a single family.

Let i denote the number of 0110 blocks, and j denote the number of 010 blocks in a given arrangement.
Counting the number of men in all the blocks, we find

$$2i + j = n_2 + n_3 + n_4 + n_5$$

From the conditions of the problem it follows that the number of solutions to this problem is different from zero if n_2 is even and in this case $i = \frac{n_2}{2}$, which implies $j = n_3 + n_4 + n_5$.
The number of circular arrangements of these $i + j$ blocks is

$$(i + j - 1)! = \left(\frac{n_2}{2} + n_3 + n_4 + n_5 - 1\right)!$$

Besides the circular arrangements of blocks, there are several ways to arrange the persons within a block.

Number of ways of choosing 010 blocks:
For a k-tuple (1 husband and $k-1$ wives) the husband may occupy any of the $(k-2)$ intermediate positions, while the wives may be positioned in the block in $(k-1)!$ ways. This gives a total of $(k-2)\cdot(k-1)!$ internal arrangements of a 010 block for a k-tuple.

Accordingly, a triple may be coded as 010 in $1 \cdot 2!$ ways;
a quadruple may be coded as 010 in $2 \cdot 3!$ ways;
a pentuple may be coded as 010 in $3 \cdot 4!$ ways.
Since there are n_3 triples, n_4 quadruples and n_5 pentuples, the total number of ways of choosing 010 blocks is

$$(1 \cdot 2!)^{n_3}(2 \cdot 3!)^{n_4}(3 \cdot 4!)^{n_5}$$

Number of ways of choosing 0110 blocks:
Any 0110 block consists of two couples, so there are $\frac{n_2}{2} = i$ such blocks.
The number of ways of pairing couples to generate these $\frac{n_2}{2}$ blocks is

$$\frac{\binom{n_2}{2}\binom{n_2-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{\left(\frac{n_2}{2}\right)!} = \frac{n_2!}{\left(\frac{n_2}{2}\right)! \cdot 2^{\frac{n_2}{2}}}$$

Within each block, there are two ways of arranging the two couples ($W_1H_1H_2W_2$ or $W_2H_2H_1W_1$). Therefore, the number of ways of internal arrangements in all $\frac{n_2}{2}$ blocks is $2^{\frac{n_2}{2}}$ and so the total number of ways of choosing 0110 blocks is $\frac{n_2!}{(\frac{n_2}{2})!}$.

Hence, the number of ways of arranging the married persons around the circular table as per the constraints is:

$$\left(\frac{n_2}{2} + n_3 + n_4 + n_5 - 1\right)! \cdot (1 \cdot 2!)^{n_3}(2 \cdot 3!)^{n_4}(3 \cdot 4!)^{n_5} \cdot \frac{n_2!}{\left(\frac{n_2}{2}\right)!}$$

Finally, the m single men may only be placed in-between two male persons of the i 0110 blocks. If $r_1, r_2, \cdots, r_i$ men are placed in these allowed i slots, then

$$r_1 + r_2 + \cdots + r_i = m,$$

where $r_k \geq 0, \ \ 1 \leq k \leq i$.
The number of integer solutions of this equation is

$$\binom{i+m-1}{m} = \frac{(i+m-1)!}{m! \cdot (i-1)!}$$

Any of these solutions represents a way to distribute in the i slots each of the $m!$ permutations of the single men. Hence, the number of all possible arrangements in which m men, respecting the constraints, can be seated around the table for any fixed arrangement of married people is:

$$\frac{(i+m-1)!}{m!\cdot(i-1)!}\cdot m! = \frac{(i+m-1)!}{(i-1)!} = \frac{(\frac{n_2}{2}+m-1)!}{(\frac{n_2}{2}-1)!}$$

Similarly, the w single women may only be placed in-between blocks. Accordingly, there are $i+j$ slots available for them. Hence the number of arrangements in which w single women, obeying the constraints, can be seated around the table for any fixed arrangement of married people and single men is:

$$\frac{(i+j+w-1)!}{(i+j-1)!} = \frac{(\frac{n_2}{2}+n_3+n_4+n_5+w-1)!}{(\frac{n_2}{2}+n_3+n_4+n_5-1)!}$$

Hence the number of ways of seating all persons around the circular table considering opposite orientations different and respecting the given constraints with n_2 even is:

$$(\frac{n_2}{2}+n_3+n_4+n_5-1)!\cdot(1\cdot 2!)^{n_3}(2\cdot 3!)^{n_4}(3\cdot 4!)^{n_5}\cdot\frac{n_2!}{(\frac{n_2}{2})!}\cdot\frac{(\frac{n_2}{2}+m-1)!}{(\frac{n_2}{2}-1)!}\cdot\frac{(\frac{n_2}{2}+n_3+n_4+n_5+w-1)!}{(\frac{n_2}{2}+n_3+n_4+n_5-1)!}$$

$$= (1\cdot 2!)^{n_3}(2\cdot 3!)^{n_4}(3\cdot 4!)^{n_5}\cdot\frac{n_2!}{(\frac{n_2}{2})!}\cdot\frac{(\frac{n_2}{2}+m-1)!}{(\frac{n_2}{2}-1)!}\cdot\Big(\frac{n_2}{2}+n_3+n_4+n_5+w-1\Big)! \quad \blacksquare$$

Corollary 1.2. *The solution to this problem for general r and n_2 even is*

$$\prod_{k=3}^{r}((k-2)\cdot(k-1)!)^{n_k}\cdot\frac{n_2!}{(\frac{n_2}{2})!}\cdot\frac{(\frac{n_2}{2}+m-1)!}{(\frac{n_2}{2}-1)!}\cdot\left(\sum_{k=3}^{r}n_k+\frac{n_2}{2}+w-1\right)!$$

References

1. J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, second edition, Cambridge Univ. Press, 2001.
2. F. E. A. Lucas, *Théorie des nombres*, Gauthier-Villars, Paris, 1891.