Bol. Soc. Mat. Mexicana (3) Vol. 11, 2005

THE LINEAR ALGEBRA OF THE PELL MATRIX

EMRAH KILIC AND DURSUN TASCI

ABSTRACT. In this paper we consider the construction of the Pell and symmetric Pell matrices. Also we discuss the linear algebra of these matrices. As applications, we derive some interesting relations involving the Pell numbers by using the properties of these Pell matrices.

1. Introduction

The Pell sequence $\{P_n\}$ is defined recursively by the equation

$$P_{n+1} = 2P_n + P_{n-1} \tag{1.1}$$

for $n \geq 2$, where $P_1 = 1$, $P_2 = 2$. The Pell sequence is

$$1, 2, 5, 12, 29, 70, 169, 408, \dots.$$

Matrix methods are major tools in solving many problems stemming from linear recurrence relations. As is well-known (see, e.g., [1]) the numbers of this sequence are also generated by the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

since by taking successive positive powers of M one can easily establish that

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}.$$

In [4] and [3], the authors gave several basic Pell identities as follows, for arbitrary integers a and b,

$$P_{n+a}P_{n+b} - P_n P_{n+a+b} = P_a P_b (-1)^n, \tag{1.2}$$

$$P_{2n+1} = P_n^2 + P_{n+1}^2, \tag{1.3}$$

$$P_n = \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} 2^r. \tag{1.4}$$

These identities occur as Problems B-136 [8], B-155 [11] and B-161 [5], respectively.

Now we define a new matrix. The $n \times n$ *Pell matrix* $H_n = [h_{ij}]$ is defined as

$$H_n = [h_{ij}] = \begin{cases} P_{i-j+1}, & i - j + 1 \geq 0, \\ 0, & i - j + 1 < 0. \end{cases}$$

2000 *Mathematics Subject Classification*: 05A19, 11B37, 15A18, 15A42.
Keywords and phrases: Pell matrix, symmetric Pell matrix, majorization.

For example,

$$H_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 & 0 \\ 12 & 5 & 2 & 1 & 0 & 0 \\ 29 & 12 & 5 & 2 & 1 & 0 \\ 70 & 29 & 12 & 5 & 2 & 1 \end{bmatrix},$$

and the first column of H_6 is the vector $(1, 2, 5, 12, 29, 70)^T$. Thus, the matrix H_n is useful to find the consecutive Pell numbers from the first to the nth Pell number.

The set of all n-square matrices is denoted by A_n. Any matrix $B \in A_n$ of the form $B = C^t \cdot C$, $C \in A_n$, may be written as $B = L \cdot L^t$, where $L \in A_n$ is a lower triangular matrix with nonnegative diagonal entries. This factorization is unique if C is nonsingular. This is called the *Cholesky factorization* of B. In particular, a matrix B is positive definite if and only if there exists a nonsingular lower triangular matrix $L \in A_n$ with positive diagonal entries such that $B = L \cdot L^t$. If B is a real matrix, L may be taken to be real.

A matrix $D \in A_n$ of the form

$$D = \begin{bmatrix} D_{11} & 0 & \dots & 0 \\ 0 & D_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{kk} \end{bmatrix}$$

in which $D_{ii} \in A_{n_i}$, $i = 1, 2, \dots, k$, and $\sum_{i=1}^{k} n_i = n$, is called a *block diagonal*. Notationally, such a matrix is often indicated as $D = D_{11} \oplus D_{22} \oplus \dots \oplus D_{kk}$ or more briefly, $\oplus \sum_{i=1}^{k} D_{ii}$; this is called the *direct sum* of the matrices $D_{11}, D_{22}, \dots, D_{kk}$.

2. Pell Identities

In this section we give some identities of the Pell numbers. We start with the following Lemma.

LEMMA (2.1). *If P_n is the nth Pell number, then*

$$2P_nP_{n-1} + P_{n-1}^2 - P_n^2 = (-1)^n . \tag{2.2}$$

Proof. We will use the induction method. If $n = 1$, then we have

$$2P_1P_0 + P_0^2 - P_1^2 = -1.$$

We suppose that the equation holds for n. Now we show that the equation holds for $n + 1$. Thus

$$\begin{aligned} 2P_nP_{n-1} + P_{n-1}^2 - P_n^2 &= P_{n-1}\left(2P_n + P_{n-1}\right) - P_n^2 \\ &= \left(P_{n+1} - 2P_n\right)P_{n+1} - P_n^2 \end{aligned}$$

which, by definition of the Pell numbers, satisfy

$$\begin{aligned} 2P_nP_{n-1} + P_{n-1}^2 - P_n^2 &= -2P_nP_{n+1} - P_n^2 + P_{n+1}^2 \\ &= -\left(2P_nP_{n+1} + P_n^2 - P_{n+1}^2\right) \end{aligned}$$

which also, by induction hypothesis, satisfy

$$2P_nP_{n+1} + P_n^2 - P_{n+1}^2 = (-1)(-1)^n = (-1)^{n+1}.$$

Thus proof is complete. □

LEMMA (2.3). *Let P_n be the Pell number. Then*

$$2P_{n-1}P_n = P_{n+1}^2 - P_{n-1}^2 - 2P_nP_{n+1}.$$

Proof. By considering the proof of the previous Lemma, the proof is clear. □

LEMMA (2.4). *If P_n is the nth Pell number, then*

$$P_1^2 + P_2^2 + \ldots + P_n^2 = \frac{P_nP_{n+1}}{2}. \tag{2.5}$$

Proof. Let we take $a_i = \frac{P_iP_{i+1}}{2}$, now since

$$\begin{aligned} a_i - a_{i-1} &= \frac{P_iP_{i+1}}{2} - \frac{P_iP_{i-1}}{2} \\ &= \frac{P_i\left(P_{i+1} - P_{i-1}\right)}{2}, \end{aligned}$$

by definition of the Pell numbers, we have

$$a_i - a_{i-1} = \frac{P_i\,(2P_i)}{2} = P_i^2.$$

Now, using the idea of "creative telescoping" [13], we conclude

$$\sum_{i=2}^{n} P_i^2 = \sum_{i=2}^{n}\left(a_i - a_{i-1}\right) = a_n - a_1$$

or equivalently $(P_1 = 1)$,

$$\sum_{i=1}^{n} P_i^2 = a_n - a_1 + 1 = a_n = \frac{P_nP_{n+1}}{2}.$$

The proof is complete. □

LEMMA (2.6). *If P_n is the nth Pell number, then*

$$\begin{aligned} P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n &= \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{2} \\ &= \frac{P_{2n-1} + 2P_nP_{n-1} - 1}{2}. \end{aligned} \tag{2.7}$$

Proof. From Lemma (2.3) we write the following equations for $1, 2, \ldots, n$,

$$\begin{aligned} 2P_1P_2 &= P_3^2 - P_1^2 - 2P_2P_3 \\ 2P_2P_3 &= P_4^2 - P_2^2 - 2P_3P_4 \\ 2P_3P_4 &= P_5^2 - P_3^2 - 2P_4P_5 \\ &\vdots \\ 2P_{n-2}P_{n-1} &= P_n^2 - P_{n-2}^2 - 2P_{n-1}P_n \\ 2P_{n-1}P_n &= P_{n+1}^2 - P_{n-1}^2 - 2P_nP_{n+1}. \end{aligned}$$

By addition, we obtain

$$\begin{aligned} 2\left(P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n\right) &= P_{n+1}^2 - P_{n-1}^2 - P_1^2 - P_2^2 - 2P_{n+1}P_n \\ &\quad -2\left(P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n - P_1P_2\right). \end{aligned}$$

If we arrange this equation by $P_1 = 1,\ P_2 = 5$ and equation (1.3), then we have

$$P_1P_2 + P_2P_3 + \ldots + P_{n-1}P_n = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{2}.$$

The proof is complete. □

In [2], the authors gave the Cholesky factorization of the Pascal matrix. Also in [6], the authors consider the usual Fibonacci numbers and define the Fibonacci and symmetric Fibonacci matrices. Furthermore, the authors give the factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. In [7], the authors consider the generalized Fibobacci numbers and discuss the linear algebra of the k-Fibonacci matrix and the symmetric k-Fibonacci matrix.

3. Factorizations

In this section we consider construction and factorization of our Pell matrix of order n by using the $(0,1,2)-$matrix, where a matrix said to be a $(0,1,2)-$matrix if each of its entries are $0, 1$ or 2.

Let I_n be the identity matrix of order n. Further, we define the $n \times n$ matrices L_n, $\overline{H_n}$ and A_k by

$$L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad L_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix},$$

and $L_k = L_0 \oplus I_k,\ k = 1, 2, \ldots,\ \overline{H_n} = [1] \oplus H_{n-1},\ A_1 = I_n,\ A_2 = I_{n-3} \oplus L_{-1}$, and, for $k \geq 3,\ A_k = I_{n-k} \oplus L_{k-3}$. Then we have the following Lemma.

Lemma (3.1). $\overline{H_k} \cdot L_{k-3} = H_k,\ k \geq 3.$

Proof. For $k = 3$, we have $\overline{H_3} \cdot L_0 = H_3$. From the definition of the matrix product and familiar Pell sequence, the conclusion follows. □

Considering the previous work on Pascal functional matrices, we can rewrite L_0, L_{-1} as follows:

$$L_{-1} = [1] \oplus P_{1,1}[1],\ L_0 = CP_{2,0}[1]\left([1] \oplus P_{1,0}[-1]\right)$$

in which $P_{n,k}[x]$ and $CP_{n,k}[x]$ are Pascal $k-$eliminated functional matrices [12].

From the definition of A_k, we know that $A_n = L_{n-3},\ A_1 = I_n$, and $A_2 = I_{n-3} \oplus L_{-1}$. The following Theorem is an immediate consequence of Lemma (3.1).

Theorem (3.2). *The Pell matrix H_n can be factored by the A_k's as follows:*

$$H_n = A_1 A_2 \ldots A_n.$$

For example

$$H_5 = A_1A_2A_3A_4A_5 = I_5(I_2 \oplus L_{-1})(I_2 \oplus L_0)([1] \oplus L_1)\, L_2$$
$$= \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&2&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&2&1&0\\0&0&1&0&1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&2&1&0&0\\0&1&0&1&0\\0&0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} 1&0&0&0&0\\2&1&0&0&0\\1&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1 \end{bmatrix}$$
$$= \begin{bmatrix} 1&0&0&0&0\\2&1&0&0&0\\5&2&1&0&0\\12&5&2&1&0\\29&12&5&2&1 \end{bmatrix}.$$

We give another factorization of H_n. Let $T_n = [t_{ij}]$ be $n \times n$ matrix as

$$t_{ij} = \begin{cases} P_i, & j = 1, \\ 1, & i = j, \\ 0, & otherwise \end{cases}, \quad i.e., \quad T_n = \begin{bmatrix} P_1 & 0 & \dots & 0 \\ P_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_n & 0 & \dots & 1 \end{bmatrix}.$$

The next Theorem follows by a simple calculation.

THEOREM (3.3). *For $n \geq 2$, $H_n = T_n\left(I_1 \oplus T_{n-1}\right)\left(I_2 \oplus T_{n-2}\right)\dots\left(I_{n-2} \oplus T_2\right)$.*

We can readily find the inverse of the Pell matrix H_n. We know that

$$L_0^{-1} = \begin{bmatrix} 1&0&0\\-2&1&0\\-1&0&1 \end{bmatrix}, \quad L_{-1}^{-1} = \begin{bmatrix} 1&0&0\\0&1&0\\0&-2&1 \end{bmatrix}, \quad \text{and} \quad L_k^{-1} = L_0^{-1} \oplus I_k.$$

Define $J_k = A_k^{-1}$. Then

$$J_1 = A_1^{-1} = I_n,\ J_2 = A_2^{-1} = I_{n-3} \oplus L_1^{-1} = I_{n-2} \oplus \begin{bmatrix} 1&0\\-2&1 \end{bmatrix}, \text{ and } J_n = L_{n-3}^{-1}.$$

Also, we know that

$$T_n^{-1} = \begin{bmatrix} P_1 & 0 & 0 & \dots & 0 \\ -P_2 & 1 & 0 & \dots & 0 \\ -P_3 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -P_n & 0 & 0 & \dots & 1 \end{bmatrix} \quad \text{and} \quad \left(I_k \oplus T_{n-k}\right)^{-1} = I_k \oplus T_{n-k}^{-1}.$$

Thus the following Corollary holds.

COROLLARY (3.4).

$$H_n^{-1} = A_n^{-1}A_{n-1}^{-1}\dots A_2^{-1}A_1^{-1} = J_nJ_{n-1}\dots J_2J_1$$
$$= (I_{n-2}\oplus T_2)^{-1}\dots(I_1\oplus T_{n-1})^{-1}T_n^{-1}.$$

From Corollary (3.4), we have

$$H_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -2 & 1 & 0 & 0 & \dots & 0 \\ -1 & -2 & 1 & 0 & \dots & 0 \\ 0 & -1 & -2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \\ 0 & \dots & \dots & -1 & -2 & 1 \end{bmatrix}. \tag{3.5}$$

We define a symmetric Pell matrix $Q_n = [q_{ij}]$ as, for $i, j = 1, 2, \dots, n$,

$$q_{ij} = q_{ji} = \begin{cases} \sum\limits_{k=1}^{i} P_k^2, & i = j, \\ q_{i,j-2} + 2q_{i,j-1}, & i+1 \le j, \end{cases}$$

in which $q_{1,0} = 0$. Then we have $q_{1j} = q_{j1} = P_j$ and $q_{2j} = q_{j2} = P_{j+1}$.

For example,

$$Q_7 = \begin{bmatrix} 1 & 2 & 5 & 12 & 29 & 70 & 169 \\ 2 & 5 & 12 & 29 & 70 & 169 & 408 \\ 5 & 12 & 30 & 72 & 174 & 420 & 1014 \\ 12 & 29 & 72 & 174 & 420 & 1014 & 2448 \\ 29 & 70 & 174 & 420 & 1015 & 2450 & 5915 \\ 70 & 169 & 420 & 1014 & 2450 & 5915 & 14280 \\ 169 & 408 & 1014 & 2448 & 5915 & 14280 & 34476 \end{bmatrix}.$$

From the definition of Q_n, we arrive at the following Lemma.

LEMMA (3.6). *For* $j \ge 3,\ q_{3j} = P_4\left(P_{j-3} + \frac{P_{j-2}P_3}{2}\right)$.

Proof. By Lemma (2.4), we have that $q_{3,3} = P_1^2 + P_2^2 + P_3^2 = \frac{P_3P_4}{2}$; hence $q_{3,3} = \frac{P_3P_4}{2} = P_4\left(P_0 + \frac{P_1P_3}{2}\right)$ for $P_0 = 0$.

By induction, $q_{3,j} = P_4\left(P_{j-3} + \frac{P_{j-2}P_3}{2}\right)$. □

We know that $q_{3,1} = q_{1,3} = P_3$ and $q_{3,2} = q_{2,3} = P_4$. Also we have that $q_{4,1} = q_{1,4}$, $q_{4,2} = q_{2,4}$ and $q_{4,3} = q_{3,4}$. By similar argument, we have the following Lemma.

LEMMA (3.7). *For* $j \ge 4$, $q_{4,j} = P_4\left(P_{j-4} + P_{j-4}P_3 + \frac{P_{j-3}P_5}{2}\right)$.

From Lemmas (3.6) and (3.7), we obtain $q_{5,1},\ q_{5,2},\ q_{5,3}$ and $q_{5,4}$. From these results and the definition of Q_n, we arrive at the following Lemma.

LEMMA (3.8). *For $j \geq 5$, $q_{5,j} = P_{j-5}P_4(1 + P_3 + P_5) + \dfrac{P_{j-4}P_5P_6}{2}$.*

Proof. Since $q_{5,5} = \dfrac{P_5P_6}{2}$ we have, by induction, $q_{5j} = P_{j-5}P_4(1 + P_3 + P_5) + \dfrac{P_{j-4}P_5P_6}{2}$. □

From the definition of Q_n together with Lemmas (3.6), (3.7) and (3.8) we have the following Lemma by induction on i.

LEMMA (3.9). *For $j \geq i \geq 6$,*

$$q_{ij} = P_{j-i}P_4(1+P_3+P_5)+P_{j-i}P_5P_6+P_{j-i}P_6P_7+\ldots+P_{j-i}P_{i-1}P_i+\frac{P_{j-i+1}P_iP_{i+1}}{2}.$$

Considering the above lemmas, we obtain the following result.

THEOREM (3.10). *For $n \geq 1$ a positive integer, $J_nJ_{n-1}\ldots J_2J_1Q_n = H_n^T$ and the Cholesky factorization of Q_n is given by $Q_n = H_nH_n^T$.*

Proof. By Corollary (3.4), $J_nJ_{n-1}\ldots J_2J_1 = H_n^{-1}$. So, if we have $H_n^{-1}Q_n = H_n^T$, then the proof is immediately seen.

Let $V = [v_{ij}] = H_n^{-1}Q_n$. Then, by (3.5), we have following:

$$v_{ij} = \begin{cases} P_j, & \text{if } i = 1, \\ P_{j-1}, & \text{if } i = 2, \\ -q_{i-2,j} - 2q_{i-1,j} + q_{ij}, & \text{otherwise.} \end{cases}$$

Now we consider the case $i \geq 3$. Since Q_n is a symmetric matrix, $-q_{i-2,j} - 2q_{i-1,j} + q_{ij} = -q_{j,i-2} - 2q_{j,i-1} + q_{ji}$. Hence, by the definition of Q_n, $v_{ij} = 0$ for $j + 1 \leq i$. Thus, we will prove that $-q_{i-2,j} - 2q_{i-1,j} + q_{ij} = P_{j-i+1}$ for $j \geq i$. In the case in which $i \leq 5$, we have $v_{ij} = P_{j-i+1}$ by Lemmas (3.6), (3.7) and (3.8). Now we suppose that $j \geq i \geq 6$. Then by Lemma (3.9) we have

$$\begin{aligned} v_{ij} &= -q_{i-2,j} - 2q_{i-1,j} + q_{ij} \\ &= (P_{j-i}-2P_{j-i+1} - P_{j-i+2})P_4(1 + P_3 + P_5) + (P_{j-i} - 2P_{j-i+1} - P_{j-i+2})P_5P_6 \\ &\quad + \cdots + (P_{j-i} - 2P_{j-i+1} - P_{j-i+2})P_{i-3}P_{i-2} \\ &\quad + \left(P_{j-i} - 2P_{j-i+1} - \frac{P_{j-i+3}}{2}\right)P_{i-2}P_{i-1} + \left(P_{j-i} - P_{j-i+2}\right)P_{i-1}P_i \\ &\quad + P_{j-i+1}\frac{P_iP_{i+1}}{2}. \end{aligned}$$

Since $P_{j-i} - 2P_{j-i+1} - P_{j-i+2} = -4P_{j-i+1}$, $P_{j-i} - 2P_{j-i+1} - \dfrac{P_{j-i+3}}{2} = -\frac{9}{2}P_{j-i+1}$ and $P_{j-i} - P_{j-i+2} = -2P_{j-i+1}$, we obtain

$$v_{ij} = P_{j-i+1}\left[\begin{array}{l} -4P_4 - 4\left(P_3P_4 + P_4P_5 + \ldots + P_{i-3}P_{i-2}\right) - \\ \\ \frac{1}{2}P_{i-2}P_{i-1} - 2P_{i-1}P_i + \frac{P_iP_{i+1}}{2}. \end{array}\right].$$

Since $P_4 = 12$, using Lemma (2.6) we get

$$\begin{aligned} v_{ij} &= P_{j-i+1}\left[\begin{array}{l} -48-4\left(\frac{P_{2(i-1)+1}-2P_{i-1}P_i-1}{4}\right)-12- \\ \\ \frac{P_{i-2}P_{i-1}}{2}-2P_{i-1}P_i+\frac{P_iP_{i+1}}{2} \end{array}\right] \\ &= P_{j-i+1}\left(-P_{2i-1}+1-\frac{P_{i-2}P_{i-1}}{2}+\frac{P_iP_{i+1}}{2}\right). \end{aligned}$$

Using equation (1.3) and the definition of the Pell numbers we obtain

$$\begin{aligned} v_{ij} &= P_{j-i+1}\left[-2P_{i-1}^2-2P_i^2+2-P_{i-2}P_{i-1}+P_i\left(2P_i+P_{i-1}\right)\right] \\ &= P_{j-i+1}. \end{aligned}$$

Therefore, $H_n^{-1}Q_n = H_n^T$, *i.e.*, the Cholesky factorizaton of Q_n is given by $Q_n = H_nH_n^T$. The proof is complete. $\square$

In particular, since $Q_n^{-1} = \left(H_n^T\right)^{-1}H_n^{-1} = \left(H_n^{-1}\right)^T H_n^{-1}$, we have

$$Q_n^{-1} = \begin{bmatrix} 6 & 0 & -1 & 0 & \cdots & & \cdots & 0 \\ 0 & 6 & 0 & -1 & & & & \vdots \\ -1 & 0 & 6 & 0 & & \vdots & & \\ 0 & -1 & 0 & 6 & \cdots & & \cdots & 0 \\ \vdots & & & \vdots & & & & \vdots \\ & & & & \ddots & 6 & 0 & -1 \\ & & & & & 0 & 5 & -2 \\ 0 & \cdots & & 0 & \cdots & -1 & -2 & 1 \end{bmatrix}. \tag{3.11}$$

From Theorem (3.10), we have the following Corollary.

COROLLARY (3.12). *If P_n is the nth Pell number and k is an odd number, then*

$$P_nP_{n-k}+\ldots+P_{k+1}P_1 = \begin{cases} \left(P_nP_{n-(k-1)}-P_k\right)/2, & \text{if } n \text{ is odd,} \\ \left(P_nP_{n-(k-1)}\right)/2, & \text{if } n \text{ is even.} \end{cases}$$

If k is an even number, then

$$P_nP_{n-k}+\ldots+P_{k+1}P_1 = \begin{cases} \left(P_nP_{n-(k-1)}\right)/2, & \text{if } n \text{ is odd,} \\ \left(P_nP_{n-(k-1)}-P_k\right)/2, & \text{if } n \text{ is even.} \end{cases}$$

For the case when we multiply the ith row of H_n and the ith column of H_n^T, we obtain the formula (2.5). Also, formula (2.5) is the case when $k = 0$ in Corollary (3.12).

4. Eigenvalues of Q_n

In this section we consider the eigenvalues of Q_n.

Let $\mathfrak{B} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n;\ x_1 \geq x_2 \geq \ldots \geq x_n\}$. For $x, y \in \mathfrak{B}$, $x \prec y$ if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, $k = 1, 2, \ldots, n$ and if $k = n$, then equality holds. When $x \prec y$,

x is said to be *majorized* by y, or y is said to be *majorize* x. The condition for majorization can be written as follows: for $x, y \in \mathfrak{B}$, $x \prec y$ if $\sum_{i=0}^{k} x_{n-i} \geq \sum_{i=0}^{k} y_{n-i}$, $k = 0, 1, \ldots n-2$, and if $k = n-1$, then equality holds.

The following is an interesting simple fact:

$$(\overline{x}, \overline{x}, \ldots, \overline{x}) \prec (x_1, x_2, \ldots, x_n), \text{ where } \overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

More interesting facts about majorizations can be found in [9] and [10].

An $n \times n$ matrix $P = \left[p_{ij}\right]$ is *doubly stochastic* if $p_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$, $\sum_{i=1}^{n} P_{ij} = 1$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} P_{ij} = 1$, $i = 1, 2, \ldots, n$. In 1929, Hardy, Littlewood and Polya proved that a necessary and sufficient condition that $x \prec y$ is that there exist a doubly stochastic matrix P such that $x = yP$.

We know that both the eigenvalues and the main diagonal elements of real symmetric matrix are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetric matrix is majorized by the diagonal elements of the matrix.

Note that $\det H_n = 1$ and $\det Q_n = 1$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of Q_n. Since $Q_n = H_n \cdot H_n^T$ and $\sum_{i=1}^{k} P_i^2 = \dfrac{P_{k+1}P_k}{2}$, the eigenvalues of Q_n are all positive and

$$\left(\frac{P_{n+1}P_n}{2}, \frac{P_nP_{n-1}}{2}, \ldots, \frac{P_2P_1}{2}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

In [4], we find the combinatorial property, $P_n = \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} 2^r$. Therefore we have following Corollaries.

COROLLARY (4.1). *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of Q_n. Then*

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \begin{cases} \left[\left(\left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2 - 1\right)/4\right], & \textit{if } n \textit{ is odd,} \\ \left[\left(\left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2\right)/4\right], & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Since $\left(\frac{P_{n+1}P_n}{2}, \frac{P_nP_{n-1}}{2}, \ldots, \frac{P_2P_1}{2}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and from Corollary (3.12),

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = \begin{cases} \frac{(P_{n+1})^2 - P_1}{4}, & \text{if } n \text{ is odd,} \\ \frac{P_{n+1}^2}{4}, & \text{if } n \text{ is even.} \end{cases}$$

By formula 1.4, the proof is immediately seen. $\square$

COROLLARY (4.2). *If n is an odd number, then*

$$4n\lambda_n \leq \left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2 - 1 \leq 4n\lambda_1.$$

If n is an even number, then

$$4n\lambda_n \leq \left(\sum_{r=0}^{[n/2]} \binom{n+1}{2r+1} 2^r\right)^2 \leq 4n\lambda_1.$$

Proof. Let $S_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. Since

$$\left(\frac{S_n}{n}, \frac{S_n}{n}, \ldots, \frac{S_n}{n}\right) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

we have $\lambda_n \leq \frac{S_n}{n} \leq \lambda_1$. Therefore, the proof is readily seen. □

From equation (3.11), we have

$$(6, 6, \ldots, 6, 5, 1) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1}\right). \tag{4.3}$$

Thus there exists a doubly stochastic matrix $G = [g_{ij}]$ such that

$$(6, 6, \ldots, 6, 5, 1) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \ldots, \frac{1}{\lambda_1}\right) \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{bmatrix}.$$

That is, we obtain $\frac{1}{\lambda_n} g_{1n} + \frac{1}{\lambda_{n-1}} g_{2n} + \ldots + \frac{1}{\lambda_1} g_{nn} = 1$ and $g_{1n} + g_{2n} + \ldots + g_{nn} = 1$.

LEMMA (4.4). *For each* $i = 1, 2, \ldots, n,$ $g_{n-(i-1),n} \leq \dfrac{\lambda_i}{n-1}$.

Proof. Suppose that $g_{n-(i-1),n} > \dfrac{\lambda_i}{n-1}$. Then

$$\begin{aligned} g_{1n} + g_{2n} + \ldots + g_{nn} &> \frac{\lambda_1}{n-1} + \frac{\lambda_2}{n-1} + \ldots + \frac{\lambda_n}{n-1} \\ &= \frac{1}{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n). \end{aligned}$$

Since $g_{1n} + g_{2n} + \ldots + g_{nn} = 1$ and $\sum_{i=1}^{n} \lambda_i \geq n$, this yields a contradiction, so $g_{n-(i-1),n} \leq \dfrac{\lambda_i}{n-1}$. □

From Lemma (4.4), we have $1-(n-1)\frac{1}{\lambda_i} g_{n-(i-1),n} \geq 0$. Let $\gamma = S_n - (n-1)$. Therefore, we have the following Theorem.

THEOREM (4.5). *For* $(\gamma, 1, 1, \ldots, 1) \in \mathfrak{B}$, $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof. A necessary and sufficient condition that $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is that there exist a doubly stochastic matrix C such that $(\gamma, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\, C$.

We define an $n \times n$ matrix $C = [c_{ij}]$ as follows:

$$C = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{12} \\ c_{21} & c_{22} & \ldots & c_{22} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{n2} \end{bmatrix},$$

where $c_{i2} = \frac{1}{\lambda_i} g_{n-(i-1),n}$ and $c_{i1} = 1 - (n-1)\, c_{i2},\ i = 1, 2, \ldots, n$. Since G is doubly stochastic and $\lambda_i > 0$ and $c_{i2} \geq 0,\ i = 1, 2, \ldots, n$. By Lemma (4.4), $c_{i1} \geq 0,\ i = 1, 2, \ldots, n$. Then

$$c_{12} + c_{22} + \ldots + c_{n2} = \frac{g_{nn}}{\lambda_1} + \frac{g_{n-1,n}}{\lambda_2} + \ldots + \frac{g_{1n}}{\lambda_n} = 1$$

$$c_{i1} + (n-1)\, c_{i2} = 1 - (n-1)\, c_{i2} + (n-1)\, c_{i2} = 1,$$

and

$$\begin{aligned} c_{11} + c_{21} + \ldots + c_{n1} &= 1 - (n-1)\, c_{12} + 1 - (n-1)\, c_{22} + \ldots + 1 - (n-1)\, c_{n2} \\ &= n - n\,(c_{12} + c_{22} + \ldots + c_{n2}) + c_{12} + c_{22} + \ldots + c_{n2} = 1. \end{aligned}$$

Thus, G is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1 c_{12} + \lambda_2 c_{22} + \ldots + \lambda_n c_{n2} &= \lambda_1 \frac{g_{nn}}{\lambda_1} + \lambda_2 \frac{g_{n-1,n}}{\lambda_2} + \ldots + \lambda_n \frac{g_{1n}}{\lambda_n} \\ &= g_{nn} + g_{n-1,n} + \ldots + g_{1n} = 1 \end{aligned}$$

and

$$\begin{aligned} \lambda_1 c_{11} + \lambda_2 c_{21} + \ldots + \lambda_n c_{n1} &= \lambda_1 (1 - (n-1)\, c_{12}) + \ldots + \lambda_n (1 - (n-1)\, c_{n2}) \\ &= \lambda_1 + \lambda_2 + \ldots + \lambda_n - \\ &\quad (n-1)(\lambda_1 c_{12} + \lambda_2 c_{22} + \ldots + \lambda_n c_{n2}) \\ &= \lambda_1 + \lambda_2 + \ldots + \lambda_n - (n-1) = \gamma. \end{aligned}$$

Thus, $(\gamma, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)\, C$, so $(\gamma, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$. □

From equation (4.3), we arrive at the following Lemma.

LEMMA (4.6). *For* $k = 2, 3, \ldots, n,\ \lambda_k \geq \dfrac{1}{6\,(k-1)}$.

Proof. From equation (4.3), for $k \geq 2$,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_k} \leq \underbrace{1 + 5 + 6 + + \ldots + 6}_{k} = 6(k-1).$$

Thus,

$$\frac{1}{\lambda_k} \leq 6\,(k-1) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \ldots + \frac{1}{\lambda_{k-1}}\right) \leq 6(k-1).$$

Therefore, for $k = 2, 3, \ldots, n,\ \lambda_k \geq \dfrac{1}{6\,(k-1)}$. So the proof is complete. □

Acknowledgement

We wish to thank the annonymous reviewer for his/her many valuable comments, suggestions and constructive critism.

Received November 19, 2004

Final version received July 18, 2005

DEPARTMENT OF MATHEMATICS
GAZI UNIVERSITY
06500 TEKNIKOKULLAR
ANKARA,
TURKEY
emkilic@gazi.edu.tr (corresponding author)
dtasci@gazi.edu.tr

REFERENCES

[1] N. BICKNELL, *A primer on the Pell sequence and related sequence*, Fibonacci Quart. **13** (4), (1975), 345–349.
[2] R. BRAWER & M. PIROVINO, *The linear algebra of the Pascal matrix*, Linear Algebra Appl. **174** (1992), 13–23.
[3] J. ERCOLANO, *Matrix generator of Pell sequence*, Fibonacci Quart. **17** (1), (1979), 71–77.
[4] A. F. HORADAM, *Pell identities*, Fibonacci Quart. **9** (3), (1971), 245–252, 263.
[5] J. IVIE, *Problem B-161*, Fibonacci Quart. **8** (1), (1970), 107–108.
[6] G. Y. LEE, J. S. KIM, AND S. G. LEE, *Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices*, Fibonacci Quart. **40** (3), (2002), 203–211.
[7] G. Y. LEE AND J. S. KIM, *The linear algebra of the k-Fibonacci matrix*, Linear Algebra Appl. **373**, (2003), 75–87.
[8] P. MANA, *Problems B-136, B-137*, Fibonacci Quart. **7** (1), (1969), 108–109.
[9] A. W. MARSHALL & I. OLKIN, İnequalities: Theory of Majorization and Its Applications, Academic Press, New York, 1979.
[10] D. S. MITRINOVIC, Analytic Inequalities, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
[11] M. N. S. SWAMY AND C. A. VESPE, *Problem B-155*, Fibonacci Quart. **7** (5), (1969), 547.
[12] H. TEIMOORI AND M. BAYAT, *Pascal k-eliminated functional matrix and Eulerian numbers*, Linear Multilinear Algebra **49** (3), (2001), 183–194.
[13] D. ZEILBERGER, *The method of creative telescoping*, J. Symbolic Comput. Vol. **11**, (1991), 195–204.